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Langlands Program and Mirror Symmetry

60s Langlands Program : Number Theory $\longleftrightarrow$ Harmonic Analysis

$^L G$ $\qquad\qquad$ G

70s Nonabelian (analogues) generalization of electro-magnetic duality

$$e \longleftrightarrow \frac{1}{e}$$ gauge theory with abelian gauge group $U(1)$.

G - nonabelian compact Lie group.

$$({}^L G, e) \overset{?}{\longleftrightarrow} (G, \frac{1}{e})$$

S-duality

Kapustin-Witten $\quad N=4$ Super-Yang-Mills (topological)

$M_4 = \Sigma \times X \qquad \Sigma, X$ - two Riemann surfaces

Look at the limit when X becomes very small!

Effectively, can describe this as a two-dimensional model on Σ which describes maps $\Sigma \to \mathcal{M}_H^{(G)}$ = Hitchin moduli space of G Higgs bundles on X

Sigma model for $M_H({}^L G)$ $\xleftrightarrow{\text{mirror symmetry}}$ Sigma model for $M_H(G)$

Look at the duality for D-branes

Physics

Number Theory

$\mathbb{Q}$ - rational numbers

$\left\{\frac{p}{q}\right\} = \{p,q \in \mathbb{Z}\}$ $\quad x^2+1=0$

$\overline{\mathbb{Q}}$ - adjoin the roots of polynomial eqns to $\mathbb{Q}$

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ - group of automorphisms of $\overline{\mathbb{Q}}$ which preserve $\mathbb{Q}$

$\{a+bi \mid a,b \in \mathbb{Q}\}$ ↺ complex conjugation

$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow GL_n(\mathbb{C})$

Algebraic Curves defined over a finite field

p - prime

$\{0,1,\ldots,p-1\} = \mathbb{F}_p$

$q = p^n$, $\mathbb{F}_q$ = field with q elements

$y^2 = x^3+ax+b$

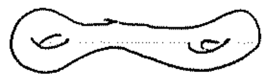
$a,b \in \mathbb{F}_q$

$(x,y) \in \mathbb{F}_{q'}^2$, $q' = q^m$

$\mathbb{P}^1_{\mathbb{F}_q} = \mathbb{A}^1_{\mathbb{F}_q} \amalg \{\infty\}$ ($q+1$ $\mathbb{F}_q$-points)

function field $= \left\{ \frac{P(t)}{Q(t)} \;\middle|\; P, Q \in \mathbb{F}_q[t] \right\}$

Riemann Surfaces



Think of them as algebraic varieties.

$y^2 = x^3+ax+b$

$a,b \in \mathbb{C}$.

look at complex solutions

Classical Langlands Program

$$\{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})\} \longleftrightarrow \{\text{automorphic representation of } GL_n(\mathbb{A}_\mathbb{Q})\}$$

adèles

$$\mathbb{A}_\mathbb{Q} = \prod{}' \mathbb{Q}_\ell \times \mathbb{R}$$

X – algebraic variety over $\mathbb{Q}$

étale cohomology $H^i_{ét}(X, \mathbb{Q}_\ell)$

X – elliptic curve 

$H^0 = \mathbb{C}$
$H^1 = \mathbb{C}^2$
$H^2 = \mathbb{C}$

$H^1_{ét}(X, \mathbb{Q}_\ell)$ = 2-dimensional
$\circlearrowright$
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ $(n=2)$

$$a_p = \# X(\mathbb{F}_p)$$

$$q = e^{2\pi i \tau}$$

$$f(q) = \sum_{n=1}^{\infty} f_n q^n$$

$$f_1 = 1.$$

$$a_p + p - 1 = f_p$$

Shimura-Taniyama-Weil Conjecture

	$G(\mathbb{A}_\mathbb{Q})$
	GL_n
	SL_n

(cont.)

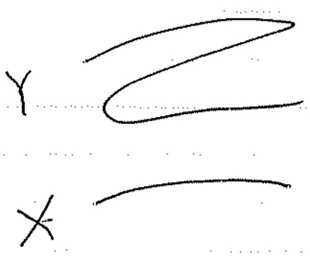
$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(\mathbb{C})$ $\qquad$ $G(\mathbb{A}_{\mathbb{Q}})$

	${}^L G$	G
	GL_n	GL_n
$SL_n/\mathbb{Z}_n =$	PGL_n	SL_n
	Sp_{2n}	SO_{2n+1}
	SO_{2n}	SO_{2n}
	E_8	E_8

$\mathrm{Gal}(\bar{F}/F) \longrightarrow {}^L G$

$F = \mathbb{F}_q(X) =$ field of rational function on a curve $X/\mathbb{F}_q$.

Geometric Langlands Program



$F' = k(Y) \circlearrowleft \mathrm{Gal}(F'/F) =$ deck transformations of Y/X

$\uparrow$

$F = k(X)$

$k = \mathbb{C}$ or $\mathbb{F}_q$ or ...

$$\{\pi_1(X) \to {}^L G\}$$
$$\| $$
$$\text{LHS}$$

RHS = some objects on moduli space of G-bundles on X