Quantum Hydrodynamics from Large-N Gauge Theories

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Talk at UCSB
November 19th 2015

Large-N Gauge Theories

Gauge theories are known to have effective descriptions when the number of colors is large $U(N)$ $N \to \infty$

For supersymmetric gauge theories we expect to compute the effective large-N theory exactly

There are plenty of examples in the literature

Mathematically we will be studying stable limits of some spaces

N=2 Gauge Theories

We focus on theories in four and five dimensions with ***eight*** supercharges which famously have Seiberg-Witten description in IR

At the moment we have plethora of exact results for those theories thanks to Nekrasov's computation of instanton partition functions

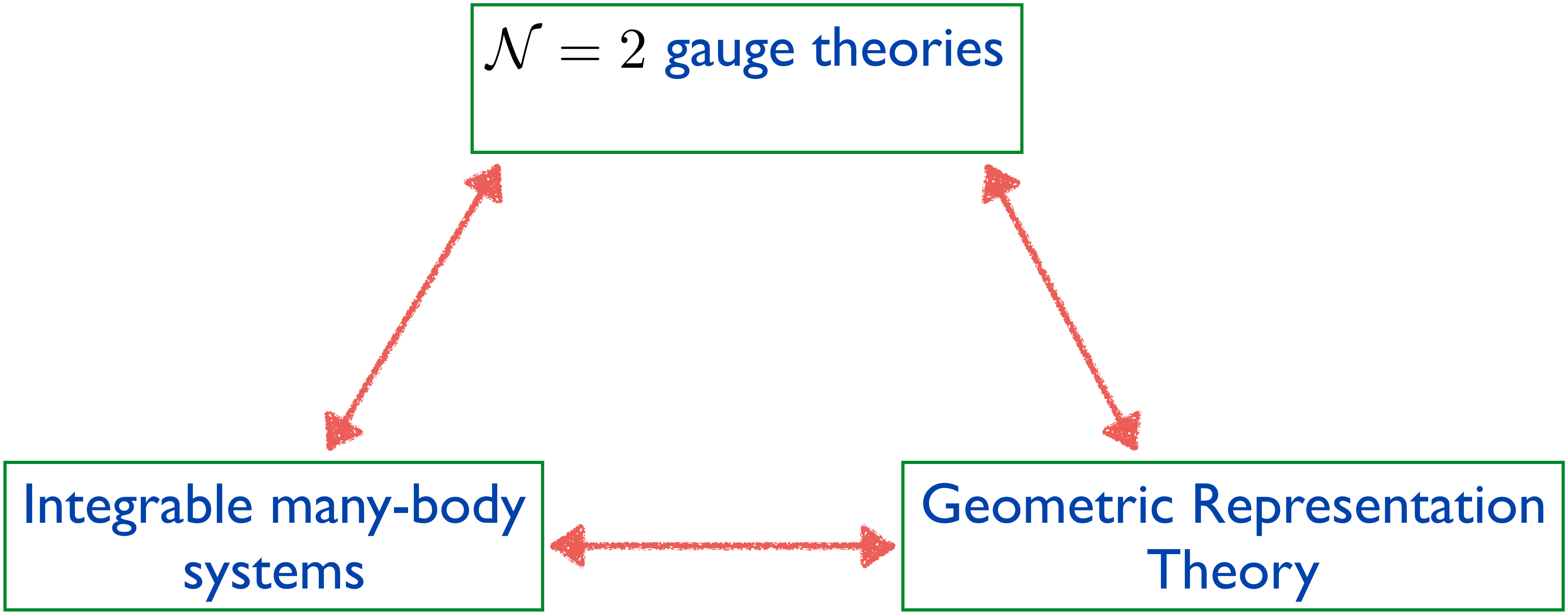
Nekrasov's original woks has been greatly extended in to:

- various supergravity backgrounds (e.g. spheres)
- quiver gauge theories
- five and six-dimensional theories on $X_D = \mathbb{R}^4 \times \Sigma$
- low dimensional theories

We shall study theories with adjoint matter on

$$X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma \qquad X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$$

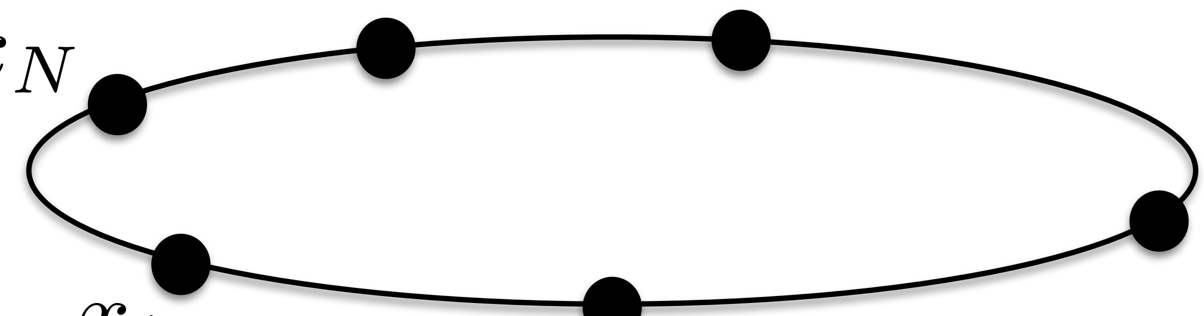
Three-way approach



Large-N limits are manifest in each description!

Integrable Models

N particles on a line or circle with near-neighbor interaction



Admits N integrals of motion (hamiltonians) $\mathcal{H}_1, \ldots, \mathcal{H}_N$

Integrability $[\mathcal{H}_i, \mathcal{H}_j] = 0$

Example: Trigonometric Ruijsenaars-Schneider model

$$D^{(1)}_{n,\vec{\tau}}(q,t) = \sum_{i=1}^{n} \prod_{j\neq i}^{n} \frac{t\tau_i - \tau_j}{\tau_i - \tau_j} T_{q,i} \qquad \tau_i = e^{x_i} \qquad q = e^{\hbar}$$

$$T_{q,i} f(\tau_1, \ldots, \tau_i, \ldots, \tau_n) = f(\tau_1, \ldots, q\tau_i, \ldots, \tau_n)$$

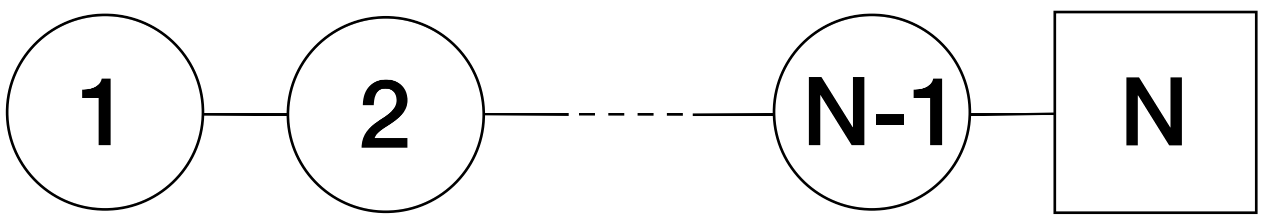
Integrable Models

[ITEP group 90']

momentum \ coordinate	rational	trigonometric	elliptic
rational	rational Calogero	trigonometric Calogero	elliptic Calogero
trigonometric	rational Ruijsenaars	trigonometric Ruijsenaars	elliptic Ruijsenaars
elliptic	dual Calogero	dual Ruijsenaars	Dell system

3d Theory

$\mathcal{N}=2^*$ quiver gauge theory on $X_3=\mathbb{C}_{\epsilon_1}\times S^1_\gamma$



target space $T^*\mathbb{F}_N$

Theory depends on twisted masses μ_i and FI parameters τ_i

and N=2* mass $t=e^m$

The partition function computed by localization for N=2 $q=e^{\epsilon_1}$

$$\mathcal{B}(\tau_1,\tau_2;\mu_1,\mu_2,t,q)=\frac{\Theta_q(t^{-1/2}\,\tau_1)\Theta_q(t^{1/2}\,\tau_2)}{\Theta_q(\mu_1\tau_1)\Theta_q(\mu_2\tau_2)}{}_2F_1\left(t,t\frac{\mu_1}{\mu_2};q\frac{\mu_1}{\mu_2};q;qt^{-1}\frac{\tau_1}{\tau_2}\right)$$

is the eigenstate of the trigonometric Ruijsenaars system!

$$D_q^{(1)}\mathcal{B}=(\mu_1+\mu_2)\mathcal{B}$$ for $T^*\mathbb{P}^1$

Generic 3d quiver

For a generic T[U(N)] quiver $$T_k \mathcal{Z} = \left\langle W_k^{U(n)} \right\rangle \mathcal{Z}$$

In other words, the eigenvalue of tRS Hamiltonian is a VEV of background Wilson loop around the compact circle

The Hamiltonians themselves are certain quantizations of the 't-Hooft-vortex loops in the corresponding representation

[Ito Okuda Taki]

The eigenvalue problem itself can be realized via S-duality wall in 4d N=2* theory

[Gaiotto Witten] [Bullimore Kim PK]
[Gaiotto PK]

We have just constructed a (complex) representation of the double affine Hecke algebra (DAHA)

[PK Gukov in prog]
[Cherednik]
[Oblomkov]

Gauge/Integrability duality

quantum tRS model	**3d** $\mathcal{N}=2^*$ $T[U(n)]$ **theory**
number of particles n	rank 3d flavor group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N}=2^*$ deformation parameter
shift parameter q	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
eigenvalue $E_{tRS}^{(\lambda;n)}$	$\langle W_{\square}^{U(n)}\rangle$ for flavour $U(n)$ at fixed μ_a
eigenfunctions $P_\lambda(\vec{\tau};q,t)$	holomorphic blocks B_l at fixed μ_a

Classification

[Bullimore, Kim, PK]

p \ q	rational	trigonometric	elliptic
r	rational CMS *2d N=(2,2)* *quiver theory*	trigonometric CMS *2d N=(2,2)** *quiver theory*	elliptic CMS *4d N=2** *2d defect*
t	rational RS (dual trig. CMS)	trigonometric RS *3d N=2** *quiver theory*	elliptic RS *5d N=1** *3d defect*
e	dual elliptic CMS	dual elliptic RS *'dual' 5d N=1** *'dual' 3d defect*	'Double periodic' model *6d (1,0)** *4d defect*

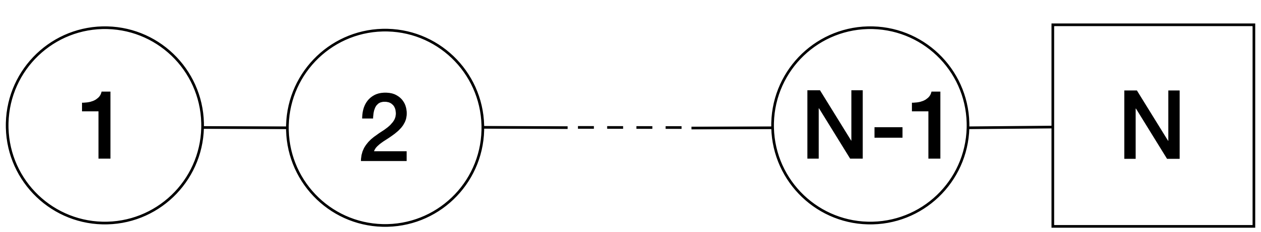
Elliptic Generalization

Seiberg-Witten curve of the 5d N=1* theory describes phase space of the elliptic Ruijsenaars-Schneider model

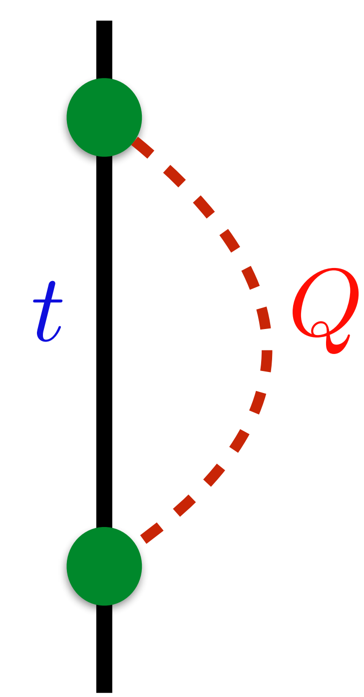
3d theory describes trigonometric model, so we need a continuous parameter which interpolates between two regimes

The way out-couple 3d theory to 5d theory by making its mass parameters dynamical

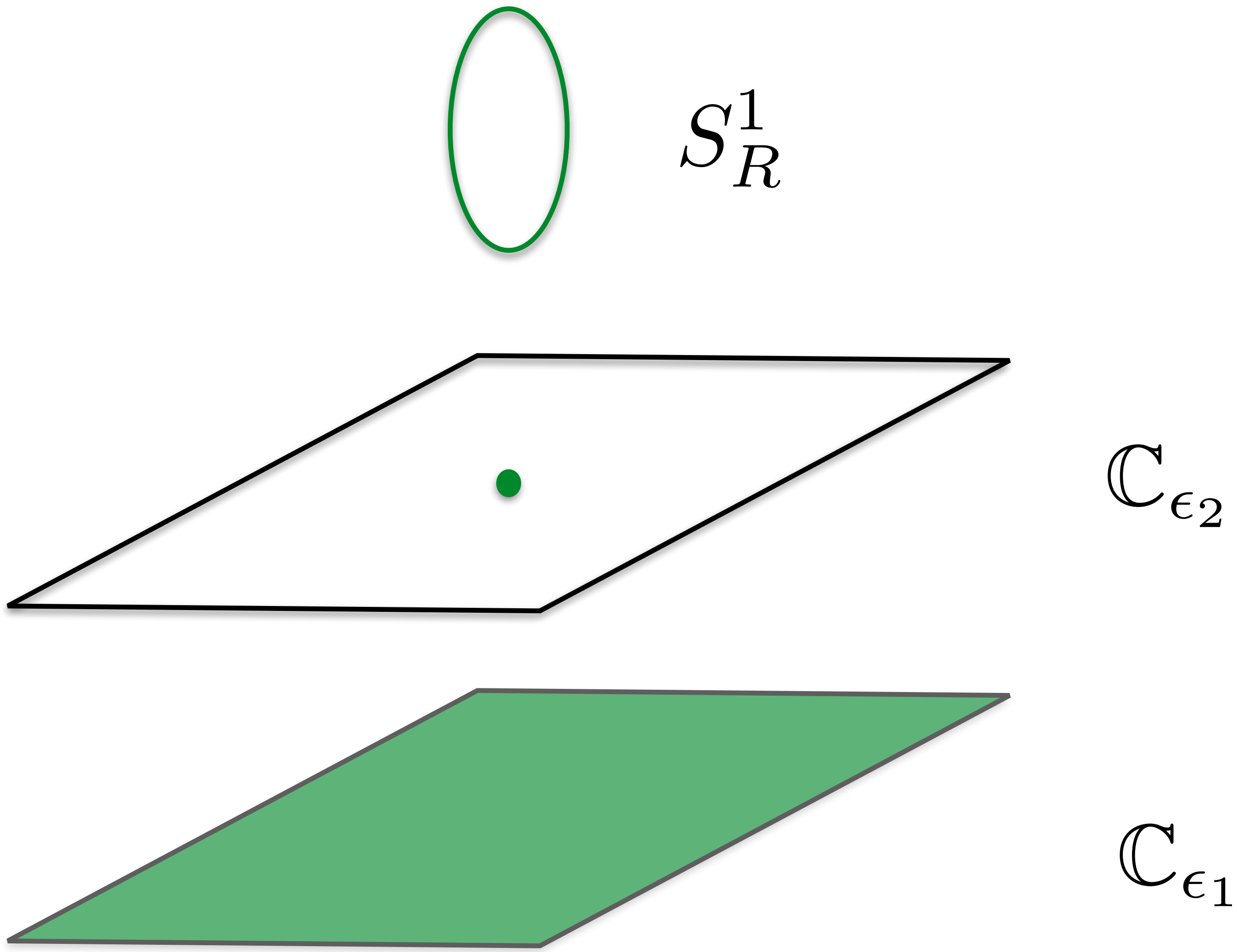
Gauging global symmetry of 3d theory
by the gauge group of the bulk 5d theory



5d U(N)
N=1*



Defects



5d N=1* U(N) theory

with monodromy defect

$$\oint_{|z_2|=\epsilon} A^a = 2\pi m^a\,, \quad a = 1, \ldots, N \qquad m^a = (\underbrace{m_1, \cdots, m_1}_{n_1}, \underbrace{m_2, \cdots, m_2}_{n_2}, \cdots, \underbrace{m_s \cdots, m_s}_{n_s})$$

Wilson loops

Wilson loop wrapping the circle

In fundamental representation

$$\langle W_{(1)} \rangle = \frac{\sum_{\vec{\lambda}} q^{|\vec{\lambda}|} \chi^{(\mathcal{E})}_{\vec{\lambda}} \prod_{\alpha} \left(2 \sinh\left(\frac{w_\alpha}{2}\right)\right)^{-n_\alpha}}{\sum_{\vec{\lambda}} q^{|\vec{\lambda}|} \prod_{\alpha} \left(2 \sinh\left(\frac{w_\alpha}{2}\right)\right)^{-n_\alpha}}$$

U(1) factor needs to be decoupled

$$\langle W^{SU(N)}_{(1)} \rangle = \frac{\langle W^{U(N)}_{(1)} \rangle}{\langle W^{U(1)}_{(1)} \rangle}$$

In the Nekrasov-Shatashvili limit

$$E_{(1)} = \lim_{\epsilon_2 \to 0} \langle W_{(1)} \rangle$$

$$E^{U(2)}_{(1)} = (\mu_1 + \mu_2) \left[1 - (1 - \eta^2)(q - \eta^2) \frac{\mu_1 \mu_2 \left(\eta^2 + q\left(\eta^4 + \eta^2 + q\right)\right) - (\mu_1 + \mu_2)^2 \eta^2 q}{\eta^4 q \left(\mu_1 q - \mu_2\right)\left(\mu_2 q - \mu_1\right)} Q + \mathcal{O}(Q^2) \right] \tag{4.37}$$

Difference Equations

Normalize the ramified partition function $$\mathcal{D}^{(\pm)}_{[1,1]} = \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}^{(\pm)}_{[1,1]}}{\mathcal{Z}}$$

U(2) theory $$\mathcal{D}^{(+)}_{[1,1]} = 1 + \frac{\left(\eta^2 - 1\right) q \left(\eta^2 \mu_2 - \mu_1\right)}{\eta^2 (q-1) \left(\mu_2 q - \mu_1\right)} z + \frac{\left(\eta^2 - 1\right) q \left(\eta^2 \mu_1 - \mu_2\right)}{\eta^2 (q-1) \left(\mu_1 q - \mu_2\right)} \frac{Q}{z} + \cdots$$

Obey the desired set of elliptic difference equations in the NS limit!!

$$\eta \left(\frac{\theta\left(\tau_2/\eta^2\tau_1, Q\right)}{\theta\left(\tau_2/\tau_1, Q\right)} p^1_\tau + \frac{\theta\left(\tau_1/\eta^2\tau_2, Q\right)}{\theta\left(\tau_1/\tau_2, Q\right)} p^2_\tau \right) \mathcal{D}^{(\pm)} = \mathcal{N}_{(1)} E_{(1)} \, \mathcal{D}^{(\pm)}$$

$$p^1_\tau \, p^2_\tau \, \mathcal{D}^{(\pm)} = \mu_1 \mu_2 \, \mathcal{D}^{(\pm)} \, .$$

When Q is sent to zero we get the trigonometric relation back

Gauge/Integrability duality

quantum eRS model	**5d/3d theory**
number of particles n	rank 3d flavor group / 5d gauge group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N}=2^*$ / 5d $\mathcal{N}=1^*$ deformation $e^{-i\gamma m}$
shift parameter q	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic deformation p	5d instanton parameter $Q = e^{-8\pi^2\gamma/g_{YM}^2}$
eigenvalues $E_{tRS}^{(\lambda;n)}$	$\langle W_{\Box}^{U(n)}\rangle$ for 5d $U(n)$ in NS limit at fixed μ_a
eigenfunctions	$Z_{\rm inst}^{5d/3d}$ in NS limit at fixed μ_a

Large-N Limit and Hydrodynamics

Collective Coordinates

Send the number of particles to infinity [Abanov Bettelheim Wiegmann]

The system can now be described using density functions or velocity fields

EOM become hydrodynamical equations

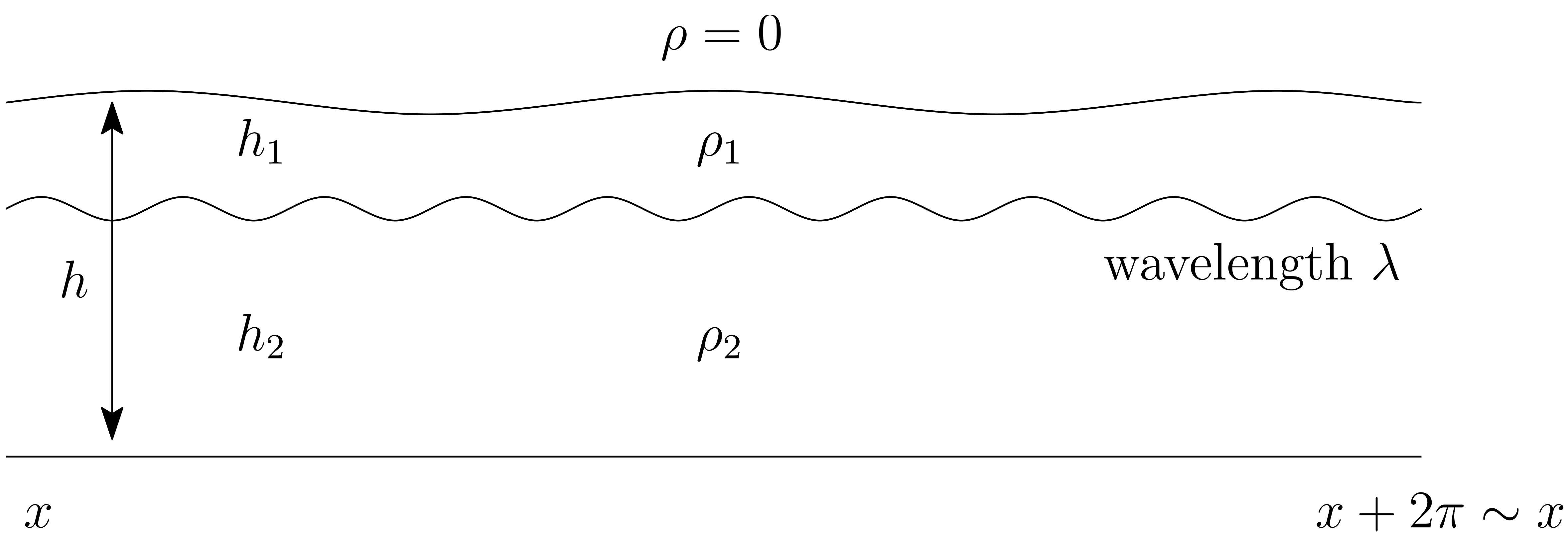
Classically Large-n elliptic Calogero model turns into intermediate long wave (ILW) system

Elliptic Ruijsenaars-Schneider model becomes finite-difference ILW

Our task it to understand quantum spectrum!

ILW model

Describes propagation of waves along the interface of two 1d fluids



- $h \ll \lambda$, long wave: Korteweg-de Vries (KdV) regime for $\delta \to 0$
- $h \gg \lambda$, short wave: Benjamin-Ono (BO) regime for $\delta \to \infty$
- $h \sim \lambda$, intermediate wave: Intermediate Long Wave (ILW) regime for $\delta \sim 1$

Integrable ILW equation

$$u_t = 2u_{xx} - i\beta\partial_x^2 u^H \qquad u^H = \frac{1}{2\pi} P.V. \int_0^{2\pi} \zeta(y - x; \widetilde{p}) u(y) dy$$

Kernel- Weserstrass function, simplifies in KdV and BO limits

KdV equation

$$u_t = 2uu_x + \frac{\beta}{3} u_{xxx}$$

Poisson bracket

$$\{u(x), u(y)\} = \delta'(x - y)$$

Rewrite ILW as evolution equation

$$u_t = \{u, I_2\}$$

Integrals of motion

$$I_1 = \int \left[\frac{1}{2}u^2\right] dx\,, \quad I_2 = \int \left[\frac{1}{3}u^3 + i\frac{\beta}{2}uu_x^H\right] dx\,,$$

$$\{I_l, I_m\} = 0$$

Soliton Solutions

n-Solitonic Ansatz

$$u(x,t) = \sum_{j=1}^{n} \left(\frac{i\beta}{x - a_j(t)} - \frac{i\beta}{x - a_j^*(t)} \right)$$

For non-periodic Benjamin-Ono we get equations of motion for Calogero

$$\ddot{a}_j = \sum_{l \neq j}^{n} \frac{2\beta^2}{(a_j - a_l)^3}$$

Poles describe propagation of solitons

Difference BO → Trigonometric RS

Difference ILW → Elliptic RS

Our challenge is to effectively describe the quantum spectrum
We need to see what happens with the *algebra* and the *states*

Quantization

Expand in Fourier modes $$u(x) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} a_m e^{imx}$$

Promote Poisson brackets to commutators $$[a_m, a_{-n}] = \hbar m \delta_{m,n}$$

Quantum Hamiltonians need to be corrected to ensure commutativity

$$\widehat{I}_l \;=\, : I_l : + o(\hbar) \quad \text{such that} \quad [\widehat{I}_l, \widehat{I}_m] = 0$$

$$\widehat{I}_2 = \sum_{m>0} a_{-m} a_m$$

$$\widehat{I}_3 = i\frac{\beta+\beta^{-1}}{2} \sum_{m>0} m \frac{1+(-\widetilde{p})^m}{1-(-\widetilde{p})^m} a_{-m}a_m + \frac{1}{2}\sum_{m,n>0} (a_{-m-n}a_m a_n + a_{-m}a_{-n}a_{m+n})$$

$$\widehat{I}_3(c_1 \overline{a}_{-1}^3 + c_2 \overline{a}_{-2}\overline{a}_{-1} + c_3\overline{a}_{-3})|0\rangle = E_3(c_1 \overline{a}_{-1}^3 + c_2 \overline{a}_{-2}\overline{a}_{-1} + c_3\overline{a}_{-3})|0\rangle$$

Finding quantum spectral is hard in general - use gauge theory for help

Gauge/ILW

Consider partition λ of $k < n$

Specify $\mu_a = q^{\lambda_a} t^{n-a} \quad , \quad a = 1, \ldots, n$ for T[U(n)] theory

Recall that $q = e^{\epsilon} = e^{\hbar}$ and $t = e^{m}$

Partition function series truncates to Macdonald polynomials!

$$D^{(1)}_{n,\vec{\tau}}(q,t) P_\lambda(\vec{\tau}; q, t) = E^{(\lambda;n)}_{tRS} P_\lambda(\vec{\tau}; q, t)$$

E.g. k=2

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2} q, t^{1/2} q) = P_{\square\square}(\tau_1, \tau_2; q, t)$$

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2}, t^{-1/2} q^2) = P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\tau_1, \tau_2 | q, t)\,.$$

Their exact form depends on n

$$P_{(2,0)}(\tau_1, \tau_2; q, t) = \tau_1 \tau_2 + \frac{1 - qt}{(1+q)(1-t)} (\tau_1^2 + \tau_2^2)$$

Change of Variables

However, after change of variables $p_m = \sum_{l=1}^{n} \tau_l^m$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

Starting for Fock vacuum $|0\rangle$

Construct Hilbert space $a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $a_{-\lambda}|0\rangle = a_{-\lambda_1}\cdots a_{-\lambda_l}|0\rangle$

Free boson realization

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$$

Vortex series encodes all states! Now need to describe eigenvalues

U(1) Instantons

Mathematicians know this space already. They found similar structure on quantum K-theory of the moduli space of U(1) (non-commutative) instantons

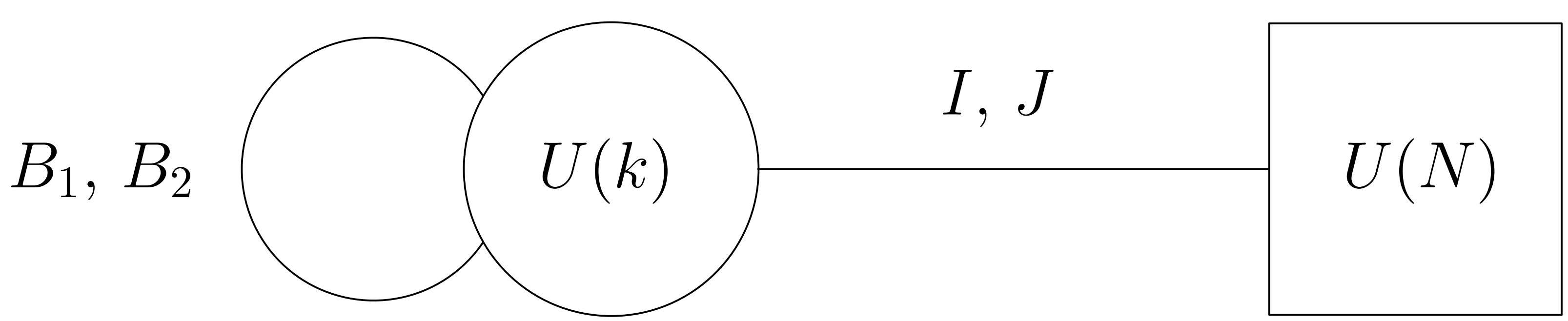
[Nakjima]
[Schiffmann Vaserot]

Physically 5d theory on $X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$

Instanton - KK monopole propagating along the compact circle

KK modes give different topological sectors

Higgs branch of the 3d N=4 ADHM quiver $\mathcal{M}_{k,N}$



superpotential $W = \mathrm{Tr}_k \left\{ \chi \left([B_1, B_2] + IJ \right) \right\}$

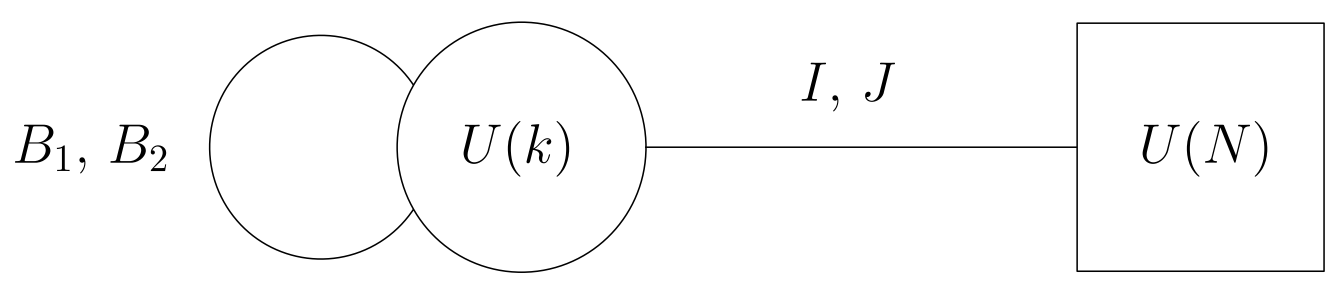
ADHM quiver

Using supersymmetry we can effectively describe K-theory of $\mathcal{M}_{k,N}$

According to Nekrasov and Shatashvili we need to find the twisted chiral ring of the ADHM gauge theory

$$(\sigma_s - 1)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\sigma_s - q\sigma_t)(\sigma_s - t^{-1}\sigma_t)}{(\sigma_s-\sigma_t)(\sigma_s - qt^{-1}\sigma_t)} = \frac{\widetilde{p}}{\sqrt{qt^{-1}}}(1-qt^{-1}\sigma_s)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\sigma_s - q^{-1}\sigma_t)(\sigma_s - t\sigma_t)}{(\sigma_s-\sigma_t)(\sigma_s - q^{-1}t\sigma_t)}$$

where $\sigma_s = e^{i\gamma\Sigma_s},\ q = e^{i\gamma\epsilon_1},\ t = e^{-i\gamma\epsilon_2}$ $\widetilde{p} = e^{-2\pi\xi}$ FI coupling



Elliptic deformation of Heisenberg algebra [Feigin et.al.]

$$[\lambda_m, \lambda_n] = -\frac{1}{m}\frac{(1-q^m)(1-t^{-m})(1-(pq^{-1}t)^m)}{1-p^m}\delta_{m+n,0}$$

The Duality

We claim that at large-n

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

Wilson line VEV becomes an equivariant Chern character for $\mathcal{M}_{k,N}$

elliptic RS	**3d ADHM theory**	**3d/5d coupled theory, $n \to \infty$**
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N}=1^*$ mass deformation $e^{-i\gamma m}$
quantum shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic parameter p	FI parameter $\widetilde{p} = -p/\sqrt{qt^{-1}}$	5d instanton parameter Q
eigenstates λ	ADHM Coulomb vacua	5d Coulomb branch parameters
eigenvalues	$\langle \mathrm{Tr}\, \sigma \rangle$	$\langle W_{\square}^{U(\infty)} \rangle$ in NS limit $\widetilde{\epsilon}_2 \to 0$

Mathematical Interpretation

Trigonometric RS to BO

$$\lim_{n\to\infty} K_T(T^*\mathbb{F}_n) \simeq K^{\mathrm{cl}}_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

$$\widetilde{\mathcal{M}_1} = \bigoplus_{k=0}^{\infty} \mathcal{M}_{1,k}$$ Instanton moduli space

No mathematical object is known to describe spectrum of elliptic RS

Our proposal

$$\mathcal{E}^Q_T(T^*\mathbb{F}_n) := \mathbb{C}[p_i^{\pm 1}, \tau_i^{\pm 1}, Q, t, \mu_i^{\pm 1}]/\mathcal{I}_{\mathrm{eRS}}$$

Large-n limit

$$\lim_{n\to\infty} \mathcal{E}^Q_T(T^*\mathbb{F}_n) \simeq K_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

Open questions

Nonabelian generalization of ILW

Quantum KdV

Knot homology

What happens for 6d theories at large n?   Holography?

Physics construction for elliptic cohomology