Elliptic Calabi-Yau Torsoes

Given E.C.Y variety, there is a Weierstrass model

\[ \frac{\mathbb{C}^3}{\mathbb{C}} \to \mathbb{B} \]

\[ y^2 = x^3 + f \times x + g \]

\[ \pi^{-1}(b) \to \mathbb{B} \text{ is an elliptic curve} \]

These define:

\[ \mathbb{L}_1 \oplus \mathbb{L}_2 \subset P(\mathbb{C} \otimes \mathbb{L}_1 \otimes \mathbb{L}_2) \]

Data of Weierstrass model used in F-theory for compactification

\[ 10 - 2d \text{ dimensions (so don't go over 2=5)} \]

\[ S^1 \times \mathbb{R}^{8-2d}, \quad \to \quad \text{"M-theory compactified at"} \]

Are there discrete gauge theories?

\[ \text{Are there multiple X's w/ some f}^g? \]

If so, is discrete symmetry of order k
Classify all $X$ with given $g$ in

Weierstrass model $\pi$, where $X \to B$, $\pi^{-1}(b) \to \text{a point}$

Birational geometry of $X$

$K_X \equiv 0$; $\omega_X \equiv 0$; $\exists$ nowhere vanishing $h_{\omega}(x)$

$h_{\omega}(x)$ $(d+1)$-form on $X$

$X$ should be a Mori minimal model
(only mild singularities are allowed)

Any algebraic curve $CX$, $\text{deg } \omega_x|_C \equiv 0$.

$X \xrightarrow{\pi} B$; it is a genus one fibration if

$g\left(\pi^{-1}(b)\right) = 1$ for general $b$

It is an elliptic fibration if there is a rational
section: $S \subset X$ (i.e. $\mathcal{O} \overset{\omega}{\to} \text{deg } \mathcal{O} \overset{\pi^{-1}(b)}{\to} (S) = 1$)

This means $S \to B$ is generally 1-1, onto
This determines an algebraic curve of genus 1 over $k = k(C)$, if $C$ is genus one.

We get an actual elliptic curve $E$.

e.g. If $k = \mathbb{Q}$, $y^2 = x^3 + 2$ (genus 1 curve)

$V(-y^2 + x^3 + 2z^2) \subseteq \mathbb{P}^1(1, 2, 1)$

this has no $\mathbb{Q}$-rational point

This curve has a Jacobian $J(C)$:

$y^2 = x^3 - 8x$

The Jacobian has no rational points.

The group scheme $J(C)$ acts on $C$:

$J(C) \times C \rightarrow C$

Thus

$\Rightarrow C$ is a principal homogeneous space on $\text{tangent}$ over $J(C)$. 
Elliptic curve

Weierstrass model

$p \in E \to \mathcal{O}_E(p)$ (line bundle, with $p$)

by Riemann Roch:

$\dim \Gamma (\mathcal{O}_E(nP)) = n - 1 \text{ if } \theta = \chi$

$\Gamma (\mathcal{O}_E(2P)) \ni l^2, x$

$\Gamma (\mathcal{O}_E(3P)) \ni l^3, l^2x, y$

$\Gamma (\mathcal{O}_E(4P)) \ni l^4, l^3x, l^2y, l^x$

$\Gamma (\mathcal{O}_E(5P)) \ni l^5, l^4x, l^3y, l^2x, l^x, xy$

$\Gamma (\mathcal{O}_E(6P)) \ni l^6, l^5x, l^4y, l^3x^2, l^2xy, l^x, y^2$
\[ \omega_E \geq 0 \]

\[ 0 = -y^2 - x^3 \]

\[ (y, 1) \quad \mu \]

\[ (\mu + 1, \mu \cdot 2^\mu \cdot y^2) \quad \nu \]

\[ 0 = -y^2 - a_1 xy - a_3 y + a_4 x^3 + a_5 x + a_6 \]

If \( \text{char } k \neq 2, 3 \) then complete square to get

\[ 0 = (\frac{1}{2} a_1 x - \frac{1}{2} a_3 y)^2 + a_2 \cdot \frac{1}{3} a_1^2 x^3 + \]

\[ (a_4 + \frac{1}{2} a_5, a_7) \quad \chi \]

\[ (a_4 + \frac{1}{2} a_5, a_7) \quad \chi \]

\[ \Rightarrow \text{complete case} \quad \text{case} \]

\[ \chi \cdot \frac{1}{2} a_1 \cdot \frac{1}{2} a_1 \]

\[ \left( y + \frac{1}{2} a_4 x + \frac{1}{2} a_7 \right)^2 + \left( \chi + \frac{1}{3} a_2 \cdot \frac{1}{3} a_1 \right)^3 \quad \chi \]

\[ \frac{1}{2} \left( \frac{1}{2} a_1 \right)^2 \]

\[ \int \]

\[ 0 = -y^2 + x^3 + f_4 x + g_6 \]

\( (x, y) \mapsto (u^2 x, u^3 y) \)

\[ u^{-6} (-u^3 y)^2 + (u^2 x)^3 + f_4 (u^4 x) + g_6 \]

\[ \mapsto -y^2 + x^3 + u^{-4} f_4 x + u^{-6} g_6 \]
$K \ni \theta \quad (\theta = \theta_u \text{ for some open } U \ni B)$

$f_u, g_0 \in K \implies \text{choosing some } \theta \in \theta, \text{ we can guarantee } u^{-\theta} f_u, u^{-\theta} g_0 \in \theta.$

\[ \therefore \ WLOG, \ f_u, g_0 \in \theta. \]

If \( \exists \theta \in \Theta \text{ s.t. } v^\theta \mid f_u, v^\theta \mid g_0 \)

we can change coordinates \( (f_u, g_0) \to (\frac{f_u}{v^\theta}, \frac{g_0}{v^\theta}) \)

\underline{Lemma}

\( \exists \ f_u, g_0 \text{ s.t. } v^\theta \mid f_u, v^\theta \mid g_0 \implies v^\theta g_0 \mid f_u \in \theta \)

\[ \implies \text{ (Weierstrass) minimal model} \]

\[ f_u, g_0 \in \Theta_u, U \ni B \]

\[ V(-y^2 + x^3 + f_u x + g_0) \leq \Theta_u \oplus \Theta_u \]

\[ \downarrow \]

U
\[
L \rightarrow \mathcal{L} \in \mathcal{L}(\mathbb{P}^2)
\]
\[
g \in \mathcal{L}(\mathbb{P}^2)
\]
\[
\delta \in \mathcal{L}(\mathbb{P}^2) \quad y \in \mathcal{L}(\mathbb{P}^2)
\]

Standard way of projecting:

\[
V(-y^2 + c_1 + s_1 x^2 + x_2 z) \leq \mathbb{P}(\mathbb{P}^2 \times \mathbb{P}^2) \quad \mathbb{P}(\mathbb{P}^2 \times \mathbb{P}^2) \quad \mathbb{P}(\mathbb{P}^2 \times \mathbb{P}^2)
\]

The point is \(W \leftarrow B\) is birational to the origin \(k\)

\[
\gamma \quad \text{genus 1 fibration} \quad J(\gamma)
\]

\[
C \quad \text{genus 1 curve over } k
\]

\[
\Rightarrow J(C) = \{ \text{line bundles of degree 0 on } C \}
\]

= Picard scheme of \(C\)

\[
\text{Loc} \in J(C) \quad \text{this contains the data of an elliptic curve}
\]
\[ J(C) \times C \rightarrow C \]
\[ L \quad \phi \quad \longrightarrow \quad \text{some complex} \]

Where \[ J(C) \otimes \mathbb{Q}(p) \cong \mathbb{Q}(p) \]

\[ \{L_1, [L_2] \in J(C) \Rightarrow [L_1 \otimes L_2] \text{ is product} \]

\[ Y \]
\[ genus \: 1 \: fibration \]

\[ B \]

Modify \( Y \) birationally, as well as \( B \), into a form in which the Jacobian of \( Y \) can be easily defined.

Also, in this form, we get a formula:

\[ \omega_Y = \pi^* (\omega_B (\Lambda)) \]
\[ \text{L has Q-coefficients} \]

\[ k_Y = \pi^* (k_B + \Lambda) \quad \text{and} \quad 12k_Y = \pi^* (12k_B + 12\Lambda) \]
After getting $\tilde{\mathcal{Y}}$ and $\mathcal{J}(\mathcal{Y})$ birationally, (i.e. blow up base and fiber), we want to blow back down.

\[ \mathcal{Y} \xrightarrow{\alpha} \mathcal{J}(\mathcal{Y}) \xrightarrow{\phi} \mathcal{J}'(\mathcal{Y}) \xrightarrow{\psi} \mathcal{Y} \]

\[ \beta \longrightarrow \tilde{\mathcal{B}} \leftarrow \mathcal{B} \]

$\tilde{\mathcal{Y}}$ = minimal model ; $\mathcal{Y}$ blow up

Given $\mathcal{J}'(\mathcal{Y})$ model, what we still $\mathcal{Y}$'s that can happen? If $\mathcal{Y}$ has dim. $\leq 4$, done. dim. $\geq 5$.

by Brave, Cao, Dolgachev, Gross
Example (PM + 6 ones)

\[ f'(Y) \text{ not a minimal model is the surprise} \]

Start \( u_1 \) \[ x \]
[0,0,0]

\[ -y_1^2 + x_1^3 + x_1^2 + 9x_1 + 2 \]

Choose \( \alpha \), \( \delta \) \( \alpha \) to be invariant under \[ (s,t,u) \rightarrow [c \cos c \sin c, c \sin c, c \cos c] \]

\( \alpha \) is (a first function of \( s^3 t^3 u^3 \))

\( \mathbb{C}^3 \) action on \( \mathbb{P}^2 \) lifts to action on \( \mathbb{A} \) fixes the 0-section

\( \mathbb{Z}/3 \) action \( \mathbb{Z}/3 \) \( \phi : \) \( \mathbb{A} \rightarrow \mathbb{A} \) then translation \( S_3 \) (zero)
\[ X/\beta \] has no section; genus 1 fibred over \\
\[ P^1/\mathbb{Z}_2, \]
\[ X/\chi = \overline{\mathcal{M}(X/\beta)} \]
\[ P^2/\mathbb{Z}_3, \]
\[ \pi_1(X/\beta) = \mathbb{Z}_3 \]

Singular over fixed points on \( P^2/\mathbb{Z}_3 \)

Blow up \( P^2/\mathbb{Z}_3 \to \overline{P^2/\mathbb{Z}_3}, \overline{\mathcal{M}(X/\beta)} \)
[0,1/10] = "0"

[0,0,1] = point of order 3