DRM 2/12/16
On Calabi-Yau Torsors

Big Plan

$$\begin{array}{c} X \\ \downarrow \\ B \end{array} \quad \text{elliptically fibered}$$

$$\begin{array}{c} \bar{X} \\ \downarrow \\ \tilde{B} \end{array} \qquad \begin{array}{c} J(X) \\ \downarrow \\ \tilde{B} \end{array} \qquad \begin{array}{c} W(X) \\ \downarrow \\ \tilde{B} \end{array} \qquad \begin{array}{c} W(X) \leftarrow \tilde{W}(X) \\ \downarrow \quad\quad \\ \tilde{B} \quad\quad \end{array}$$

Birational geometry and minimal model program.

$$B \leftarrow B_1 \leftarrow B_2 \leftarrow \cdots$$

$$\underbrace{K_B \cdot C \geq 0.}_{}$$ The requirement for minimal model.

$$X \leftarrow X_1 \leftarrow X_2 \cdots$$

$$K_X \cdot C \geq 0$$

$$K_X = \pi^*(K_B + \Lambda)$$

Λ ← divisor (with $\mathbb{Q}$-coeffs)

$K_X \cdot C < 0$, blow down.

$$(K_B + \Lambda) \cdot C < 0.$$

Key starting properties:

1) $X \xrightarrow{\pi} B$ is a flat family

i.e. all fibers $\pi^{-1}(b)$ have dim 1.

2) set of singular fibers $\Delta \subset B$ is a divisor with normal crossings.

$$\Delta = \sum \Delta_j$$

each $p \in \bigcap_{j \in J(p)} \Delta_j$

$\Delta_{j_1} \cap \cdots \cap \Delta_{j_k}$

meet normally. analytic ....

$$(z_j = 0) = \Delta_j$$

J-invariant of the elliptic fibration is well-defined.

$$B \to \mathbb{P}^1.$$

Kodaira Classification and its fake variant,

(how to resolve singularities in codim 1 in B).

Kodaira $\quad \begin{array}{c} X \\ \downarrow \\ B \end{array}$ $\quad$ dim $X = 2$, dim $B = 1$, $b \in B$ has small nbhd.

X smooth, $X \supset C$ s.t.

$\pi(C) = b$,

$C^2 = -1$, $K_X \cdot C = -1$

(any C could be blown down to nonsingular point)

$\pi^{-1}(b)$

$\pi^{-1}(b) = \bigcup E_j$

if $\pi^{-1}(b)$ has ≥ 2 components, $E_j^2 = -2$

$\{E_j\}$ forms an affine dynkin diagram.

$\pi^{-1}(b)$ can have mult. m.

$m I_n$, $m I_0$

Weierstrass model :

$$\sigma \left(\begin{array}{c} X \\ \downarrow \pi \\ B \end{array} \right.$$

genus 1 curve with point (section)

"all components of fibers of π which do not meet $\sigma(B)$ are contracted to points"

$$y^2 = x^3 + fx + g$$

$$f \in \Gamma(\mathcal{L}^{\otimes 4})$$

$$g \in \Gamma(\mathcal{L}^{\otimes 6})$$

$$\subseteq \underline{\mathrm{Spec}}(\mathcal{O} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3})$$

$$(f, g) \longmapsto (\lambda^4 g, \lambda^6 g)$$

$$\Delta = \{4f^3 + 27g^2\}$$

generic cusp singularity of Δ

$$y^2 = x^3 + ux^2 + vx + w.$$

$$\Delta = \{4u^3w - u^2v^2 - 18uvw + 27w^2 + 4v^3 = 0\}$$

$$u \in \Gamma(\mathcal{L}^{\otimes 2})$$

$$v \in \Gamma(\mathcal{L}^{\otimes 4})$$

$$w \in \Gamma(\mathcal{L}^{\otimes 6})$$

$C \subset B$ s.t. C is a component of Δ.

Analyze X near C (at generic point)

In local coordinates

$$C = \{z = 0\}$$

Case(1) C is not a component of Δ

X smooth near C, fibers nonsingular.

$n E_j$, $E_j^2 = -2$, $E_j \cdot E_{j+1} = 1$

	$\mathrm{ord}_C(\Delta)$	$\mathrm{ord}_C(f)$	$\mathrm{ord}_C(g)$
I_0	0	—	—
$n I_0$			
I_n	n	0	0
mI_n / II	2	≥ 1	1 m
III	3	1	≥ 2
IV	4	≥ 2	2
I_0^*	6	≥ 2 (not both strict)	≥ 3
I_n^*	$6+n$	2	3
IV^*	8	3	4
III^*	9	3	5
II^*	10	4	5

Case 2 $(z=0) \subset \Delta \Rightarrow z \mid \Delta$

$x \longmapsto x + a$ (put singular fiber at origin. $z \mid w$, $z \mid v$, $I_n \longleftrightarrow z \nmid u$.

$x^3 + fx + g$

$3x^2 + f$

$\frac{2}{3} f x + g \quad , \quad x = -3g/2f$

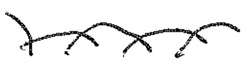
$3x^2 = \frac{27g^2}{4f^2}$

$\mathrm{ord}_c(\Delta) = n$

$\Rightarrow \; y^2 = x^3 - u x^2 + z^{[n/2]} v_n x + z^n w_n$

<u>Surface singularities</u> : $y^2 = u x^2 + z^n$

A_{n-1} singularity

 $n-1$ curves.

<u>Blow up singularity</u> : — few on the box

$y^2 = u\bar{x}^2 + \bar{z}^{n-2}$

 $\bar{y}^2 = u\bar{x}^2$

$(\bar{y} - \sqrt{u}\,\bar{x})(\bar{y} + \sqrt{u}\,\bar{x})$

If u is a square

$(u = -9g/2f \,|_c$

s : u is a square

ns : u not a square.

<u>Case 3</u> $\quad z|v, \; z|w, \; z|u$

return to $y^2 = x^3 + fx + g$

$z|f, \; z|g$

$y^2 = x^3 + z f_1 x + z g_1$

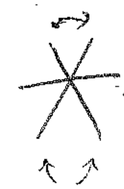
So...

II : $y^2 = x^3 + z f_1 x + z g_1$

III : $y^2 = x^3 + z f_1 x + z^2 g_1$

A_1 

IV $\quad y^2 = x^3 + z^2 f_2 x + z^2 g_2$

A_2  third curve

might be interchanged by monodromy.

<u>Case</u> 6,7 (I_0^*, I_n^*) $\quad z^2|f, \; z^3|g$

$\mathrm{ord}(\Delta) = n+6$

$I_n^* \quad D_{n+4}$

<u>Case 8</u> $\quad z^3|f, \; z^4|g$ $\quad E_6 \; E_7 \; E_8.$

(related to simply laced dynkin diagram).

Careful

Summary in detail in

Appendix B 1108.0042

Next time! Follow a strategy by Miranda to deal with flat families.

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