Brauer Group & Elliptic Torsors

2/19/2016 ⓪

Picard group describes lines bundles, i.e.

$$Pic(X) = \{\text{iso classes of line bundles}\}$$

In complex geometry, we can study $H^1(X, \mathcal{O}_X^\times)$

We can use the short exact sequence:

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O} \to \mathcal{O}_X^\times \to 0$$

There are also Kummer sequences in alg. geometry:

$$0 \to \mu_m \to \mathcal{O}_X^\times \to \mathcal{O}_X^\times \to 0$$

$$\check{H}^1(X, \mathcal{O}_X) \to \check{H}^1(X, \mathcal{O}_X^\times)$$

We identify $Pic(X)$ w/ $\check{H}^1(X, \mathcal{O}_X) / H^1(X, 2\pi i \mathbb{Z})$

In the long exact sequence:

$$\check{H}^1(X, \mathcal{O}_X) \to \check{H}^1(X, \mathcal{O}_X^\times) \to Pic(X) \xrightarrow{c_1} H^2(X, 2\pi i \mathbb{Z})$$

Brauer group

$$H^2(X, \mathcal{O}_X^*) \to H^3(X, 2\pi i\mathbb{Z})$$

$\parallel$

Cohomological Brauer group, which has a natural map to $H^3(X, 2\pi i\mathbb{Z})$

Instead, we could look at equivalence classes of Azumaya algebras; these are locally isomorphic to $n\times n$ matrix algebras.

This includes the theory of central simple algebras $/k$

It's related to the classification of $\mathbb{P}^n$-bundles over X

Dolgachev & Gross relate the Brauer group to Elliptic Fibrations

Dolgachev & Gross look at "good models" of genus 1 fibrations

$$\begin{array}{c} X \\ \downarrow f \\ S \end{array}$$

satisfying:

f flat, projective

X, S regular, integral (nonsingular)

We want the generic point X_η to be a torsor for some given E = elliptic curve over $K(S)$.

Over X_η, there is a free action:

$$\underset{p}{E} \times X_\eta \longrightarrow X_\eta$$

Given E, we want to classify all such fibrations.

has a section

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[d] & Y_\eta = E \\ S & S \arrow[u, bend right] & \end{tikzcd}$$

good model — has no good model unless S is blown up

D & G use a short exact sequence –

<u>Def</u> $\quad Br'(X) = H^2(X, \mathbb{G}_m)$

Then $\quad Pic(X) = H^1(X, \mathbb{G}_m)$

<u>Lemma (D&G)</u>

For a good model, $\quad R^i f_* \mathbb{G}_m = 0 \quad \forall\, i > 1$

D&G exact sequence is:

$$0 \to \mathrm{Pic}(S) \xrightarrow{\text{pullback}} \mathrm{Pic}(X) \to H^0(S, R^1 f_* \mathbb{G}_m)$$

$$\to \mathrm{Br}'(S) \to \mathrm{Br}'(X) \to H^1(S, R^1 f_* \mathbb{G}_m) \to H^3(S, \mathbb{G}_m) \to \cdots$$

Geometric Claim

$$\begin{array}{ccc} \overset{\text{divisor}}{D} & \subseteq & X \\ \downarrow & & \downarrow f \\ S & = & S \end{array}$$

D a div. on X that surjects onto S

δ_η = min degree of a divisor

δ_S = min deg of a divisor to S, that's finite under projection

This is related to the question - what is our torsor

Claim: $\mathrm{Ker}(H^i(S, \mathbb{G}_m) \to H^i(X, \mathbb{G}_m))$ is killed by δ_S

Illustrative Examples

$$\begin{array}{c} Y^{\text{w/ section}} \\ \downarrow \\ S \end{array}$$

This may have no model over S which is nonsingular, but after blowing S up, it may have a nonsingular model.

Resolving Weierstrass Model

Resolve all Kodaira types

We have Kodaira divisors $\Delta_j \subset \overset{\substack{X \\ \downarrow}}{S}$

If $\dim S = 2$, Miranda's idea is to produce a model by blow ups that gives a flat over the base, such that:

* Δ^{red} has simple normal crossings

Then we

* Check $J : S \dashrightarrow \mathbb{P}^1$ is well-defined (a priori only rational)

* We study all possible collisions

$J=0$ collisions include $\mathrm{II}, \mathrm{IV}, I_0^*, \mathrm{IV}^*, \mathrm{II}^*$;

The $J=1$ collisions include $\mathrm{III}, I_0^*, \mathrm{III}^*$

$J=\infty$ collisions include I_m, I_n^*

Miranda showed blowups change type of collision.

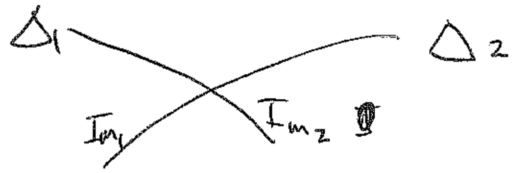
Miranda only allows collisions:

$$\underline{I_{M_1} + I_{M_2}}\ ,\ I_{M_1} + I_{M_2}^*\ ,\ II + IV,\ II + I_0^*,\ II + IV^*$$

M_1 or $M_2 \geq 2$ $\quad IV + I_0^*\ ,\ III + I_0^*$

If we see other collisions, we blow them; eventually, we'll get one of the allowable collisions.

Why do this? We're in dim 2;



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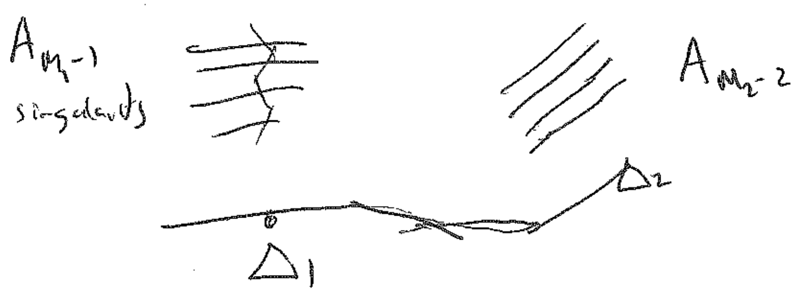
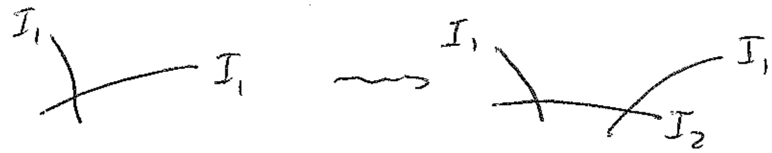


$$y^2 = x^2 + s^{m_1} t^{m_2}$$

We end up with terminal singularities, which can be resolved by blowing up Weil divisors on $\overline{Y}$.

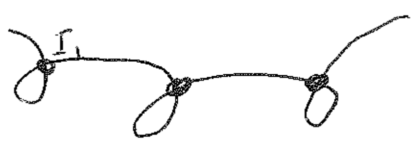
Miranda shows we can do this for an I_{M_1} & I_{M_2} collision if $M_1 \geq 2$ or $M_2 \geq 2$.

If we get $I_1 + I_1$, blow it up again

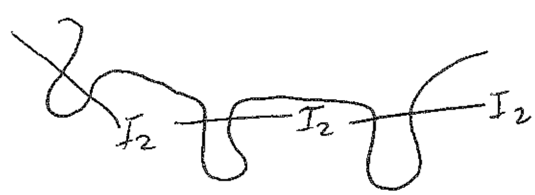


Back to examples of Torsion

<u>Phenomenon</u>: $\Delta\left(\begin{smallmatrix} Y \\ \downarrow \\ S \end{smallmatrix}\right)$ (discriminant locus) will have nodes if there are interesting torsion



Blow up at I_1's



$H_3(Y, 2\pi i \mathbb{Z})$

$\therefore \exists\ x \in B_r'(x)$ killed by δ_{S*}

Look at nonsingular cubics in $\mathbb{P}^2$

$$y^2 = x^3 + fx + g \quad \text{Weierstrass}$$

More general cubic, e.g. $x^3+y^3+z^3$ in a family has no section

Let S be a basepoint free net of cubics in $\mathbb{P}^2/k=\bar{k}$, char $k=0$
(vector space of dim 3)

$$S \cong \mathbb{P}^2$$

We look at $X \subseteq \mathbb{P}^2 \times S$
$\downarrow$ $\{(p, C) \mid p \in C\}$ (cubic)
S

This gives a genus 1 fibration over S

There's a divisor

$$\begin{array}{ccc} D & \subseteq & X \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

3-section given by a fixed line in $\mathbb{P}^2$

$\delta_S = 3$ (there are 3-sections but not 2-sections)

The Tate-Shafarevich group of this is $Ш_S(A) = \mathbb{Z}/\delta_S\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$.

The Jacobian of X/S is

$$\begin{array}{c} J \\ \downarrow \\ S \supset \Delta \end{array}$$

It's classically know that Δ is a curve in $\mathbb{P}^2$ of degree 12 with $\underset{4\cdot 6}{\overset{24}{\shortparallel}}$ cusps & 21 nodes.
$(S = \mathbb{P}^2)$.

$$\Delta = 4f^3 + 27g^2$$

[Note - if components of Δ are not distinct, we can't do Miranda's trick: in particular, the existence of nodes tells us we can't resolve the $I_1 + I_1$ collision by blowups].