GTP Seminar.
DRM.
02/18/16

Brauer Group & elliptic torsors:

Picard group:

line bundles $H^1(X, \mathcal{O}_X^*)$

complex geometry:

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

algebraic geometry:

$$0 \to \mu_n \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

$$\underline{\check{H}^1(X, \mathcal{O}_X) \to \check{H}^1(X, \mathcal{O}_X^*)}$$

$$H^1(X, \mathcal{O}_X)/H^1(X, 2\pi i\mathbb{Z}) \longrightarrow \mathrm{Pic}(X)$$

$$\xrightarrow{c_1} H^2(X, 2\pi i \mathbb{Z}).$$

Brauer Group:

$$H^2(X, \mathcal{O}_X^*) \longrightarrow H^3(X, 2\pi i \mathbb{Z})$$

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Cohomological Brauer Group

Brauer Group:

{ Azumaya algebras }/equivalence

locally isomorphic to $n \times n$ matrix algebras

$\supseteq$ central simple algebras/k

(k'/k)

Related to $\mathbb{P}^n$ bundles over X.

Dolgachev & Gross.

"Good model" of a genus 1 fibration

$$\begin{array}{ll} X & f.\ \text{flat, projective} \\ f \downarrow & \\ S & X, S \ \text{regular (nonsingular)} \\ & \quad \text{integral} \end{array}$$

X_η = torsor for some given $\underline{\underline{E}}$

elliptic fibration over $K(S)$

Over X_η: $E \times X_\eta \longrightarrow X_\eta$ free action

$$\begin{array}{ccc} \Downarrow & & \\ p & & id \end{array}$$

$\tilde{\eta}/\eta$
"
generic pt. of D.

$$\begin{tikzcd} D \arrow[d] & \subseteq & X \arrow[d, "f"'] \\ s & = & S \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d] \\ s \end{tikzcd} \quad Y_\eta = E$$

good model   has no good model unless s is blown up.

$$Br'(X) = H^2(X, \mathbb{G}_m)$$

$$Pic(X) = H^1(X, \mathbb{G}_m)$$

<u>Lemma</u>: For a good model,

$$R^i f_* \mathbb{G}_m = 0 \quad \forall i \geq 1.$$

$$0 \to Pic(S) \to Pic(X) \to \boxed{H^0(S, R^1 f_* \mathbb{G}_m)}$$

$$\to Br'(S) \to Br'(X) \to H^1(S, R^1 f_* \mathbb{G}_m)$$

$$\to H^3(S, \mathbb{G}_m) \to \cdots$$

<u>Geometric Claim</u>:

δ_η = min degree of divisor

δ_s = min degree of div. which is finite and flat over S.

$$Ker\left(H^i(S, \mathbb{G}_m) \to H^i(X, \mathbb{G}_m)\right)$$

is killed by δ_S

<u>Illustrative Examples</u>:

$$\begin{tikzcd} Y \arrow[d] \\ S \end{tikzcd}$$

This may have no model over over S which is nonsingular, but after blowing up S, it may have a nonsingular model.

<u>Rescaling Weierstrass model</u>:

resolve all Kodaira types:

$$\begin{tikzcd} X \arrow[d] & \supseteq & C_j \arrow[d] \\ S & \supset & \Delta_j \end{tikzcd}$$

<u>$\dim S = 2$</u>:

Miranda's idea:

to get a model flat over S

$\to \Delta^{red}$ has simple normal crossings

$\to$ ensure that

$$J : S \dashrightarrow \mathbb{P}^1$$

is well-defined.

$\to$ possible "collisions": $I_1 + I_1$: blow it up

$$\underbrace{I_{M_1} + I_{M_2}}_{\text{caveat}}, \quad I_{M_1} + I_{M_2}^*, \quad \mathrm{II} + \mathrm{IV},$$

$M_1 \geq 2$ or $M_2 \geq 2$

$$\mathrm{II} + I_0^*, \quad \mathrm{II} + \mathrm{IV}^*,$$
$$\mathrm{IV} + I_0^*, \quad \mathrm{III} + I_0^*$$

$J = 0$ Collisions:

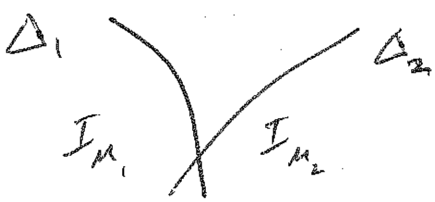
II, IV, I_0^*, IV^*, II^*

$J = 1$ Collisions:

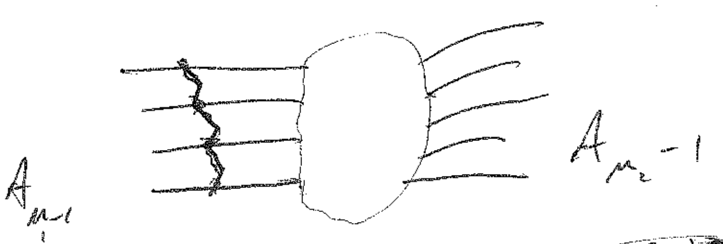
III, I_0^*, III^*

$J = \infty$ Collisions:

I_M, I_N^*



$xy = zt$



$$y^2 = x^2 + s^{M_1} t^{M_2}$$

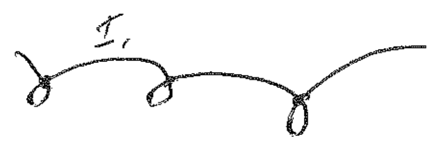
terminal singularities, can be resolved by blowing up additional

Weil divisors on $\overline{Y}$

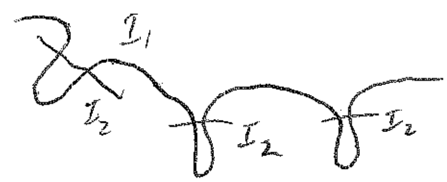
if $M_1 \geq 2$ or $M_2 \geq 2$.

Phenomenon:

$\Delta\begin{pmatrix} Y \\ \downarrow \\ S \end{pmatrix}$ will have nodes



(so $I_1 \pitchfork I_1$ collision,
blow up:



$\alpha \in Br'(X)$ or torsors

$\delta_s X = 0$

$H^3(Y, 2\pi i \mathbb{Z})$

Cubics in $\mathbb{P}^2$:

$y^2 = x^3 + fx + g$

$X^3 + Y^3 + Z^3$ (Fermat) in a family, no section.

Let S be a base point free ~~net~~ (pencil) of cubics in $\mathbb{P}^2/_{k = \bar{k}}$, char $k = 0$ (vector space of dim 3)

$S \cong \mathbb{P}^2$, $X \subseteq \mathbb{P}^2 \times S \leftarrow$ (genus 1 fibration)
$= \{(p, c) \mid p \in C\} \to S$

3

$$\begin{array}{ccc} D & \subseteq & X \\ \downarrow & & \downarrow \\ S & & \end{array}$$

<u>3-section</u>: given by a fixed line in $\mathbb{P}^2$

$\delta_S = 3$

Tate-Schafarevich group of fibration

$$\text{Ш}_S(A) \cong \mathbb{Z}/\delta_S\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$$

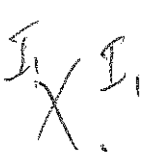
Jacobian of X/S

$$\begin{array}{c} J \\ \downarrow \\ S \supset \Delta \end{array}$$

<u>Classically known</u>:

Δ is a curve in $\mathbb{P}^2$ of degree 12 with 24 cusps and 21 nodes

$$\Delta = (4f^3 + 27g^2)$$



$4 \cdot 6 = 24$ cusps