Cremona, Fisher, O'Neil, Simon, Stoll

"Explicit n-descent on elliptic curves I, II, III"

E/K , K = f.o. fractions of coord. ring of B where B is the base

3 groups come up:

$\begin{array}{c} X \\ \downarrow \\ B \end{array}$

Mordell-Weil group $\cong E[K]$ of K-valued points

Tate-Shafarevich group of E ~ E-torsors (approx)
$Ш(E)$

n-Selmer group of E

The n-Selmer group is the one we study.

There's an exact sequence:

$$0 \to \underbrace{E[K]/_{n}E[k]}_{E[n] \text{ n-torsion points}} \to Sel^{(n)}(E/K) \to Ш(E/K)[n] \to 0$$

n-torsion

F-theory

$Ш(E/k)$ is closely related to the group of

Calabi-Yau E-torsors ~ $\pi_0(G)$

↳ gauge group of F-theory model

G = compact Lie group $\mathfrak{g}$ = easily computable from geometric data

$G/[G,G] \cong A$, max'l abelian quotient

Inside G, we have $T \subseteq G$, max'l ab. subgroup.

$T \to A$ has a finite kernel.

The kernel of $G \xrightarrow{\alpha} A$ is semisimple;

$\ker(\alpha) = \prod G_j$ & we have $|\pi_1(G_j)| < \infty$

$\Rightarrow$ $\pi_1(G)$ is finitely generated; the free part corresponds to A; the torsion part corresponds to $\prod \pi_1(G)$.

The data of $\mathfrak{g}$ & $\pi_1(G)$ determines a connected compact group.

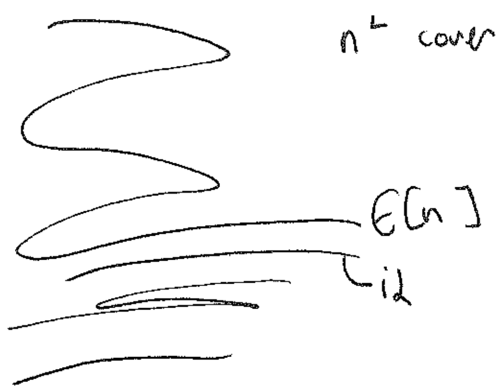
The $\mathfrak{g}_{ss}$ is determined by the geometry, by looking at the Kodaira type of singular fibers.

$\pi_1(G) \simeq MW(E/K)$

Now we study function field of $\begin{array}{c} E[n] \\ | \\ K \end{array}$

$$\begin{array}{ccc} E & \supseteq & E[n] \\ \downarrow & & \\ \operatorname{Spec} k & & \end{array}$$

Over an alg. closed field, we would have n^2 points.



n-torsion points on elliptic curves over *any* field

$k = \mathbb{C} \longrightarrow \quad H^1(E, \mathbb{Z}) \times H^1(E, \mathbb{Z}) \longrightarrow H^2(E, \mathbb{Z}) \cong \mathbb{Z}$

cup product pairing

$$\langle \alpha, \beta \rangle \in \mathbb{Z}$$

skew-symmetric

This induces a pairing on n-torsion:

$$\cancel{H^1(E, \mathbb{Z}/n) \times H^1(E, \mathbb{Z}/n) \to}$$

$$H^1(E, \mathbb{Z}/n\mathbb{Z}) \cong E[n] \times E[n] \longrightarrow \mu_n$$

k arbitrary $\Rightarrow$ Weil pairing

$$E[n] \times E[n] \longrightarrow \mu_n = \operatorname{Spec} k[x]/x^n - 1$$

The Weil pairing is at the heart of Gemora, et al's work.

R = étale algebra of $E[n]$

= $\prod$ frac. fields of coord. rings of irred. comp. of $E[n]$

$$W_1 : H^1(K, E[n]) \longrightarrow R^\times / (R^\times)^n$$

is injective; image is known.

The 2^{nd} map we need is:

$$w_2 : H^1(K, E[n]) \longrightarrow (R \otimes R)^\times / \partial R^\times$$

(also injective w/ known image)

$$\mathrm{Ob} : H^1(K, E[n]) \longrightarrow \mathrm{Br}(K)$$ ← This obstruction map would need to vanish

<u>Warning</u>

Ob is not a homomorphism & ker(Ob) need not be a group.

<u>Set:</u>

$$w : E[n] \longrightarrow R^\times = \mathrm{Map}(E[n], \bar{K}^\times]$$

where $w(S)(T) = \langle S, T \rangle_n$ (Weil pairing)

$$w(\xi_\sigma) = \frac{\sigma(\gamma)}{\gamma}$$

γ : $\alpha = \gamma^n \in R^\times$, $\rho = \partial \gamma \in R^\times \otimes R^\times$ where ∂ is defined below

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There's a short exact sequence:

$$E[n] \xrightarrow{w} R^\times \xrightarrow{\partial} (R \otimes R)^\times$$

<u>where</u> $$\partial \alpha(T_1, T_2) = \frac{\alpha(T_1)\,\alpha(T_2)}{\alpha(T_1 + T_2)}$$

Now define w_1, w_2:

$$w_1(\xi) = \alpha \cdot (R^\times)^n$$

$$w_2(\xi) = \rho \cdot \partial R^\times$$

The goal (of Cremona, et al) is to represent each element of $Sel^{(n)}(E/K)$ by an "elliptic normal curve" $C \subseteq \mathbb{P}^n$ which is an E-torsor.

$$|\mathcal{O}_E(nP)| : C \hookrightarrow \mathbb{P}^{n-1}$$ ($n > 2$; else $\mathcal{O}_E(nP)$ is a 2-1 map)

Obstruction map

Elements of $H^1(E, K[n])$ can be represented by

$$\begin{tikzcd} C \arrow[d] \\ S \end{tikzcd}$$

— (C is a twist of E)
(deg 2 if $n=2$; embedding otherwise)

where S is a Brauer-Severi variety$/K$ of dim $n-1$.

$$\left(\begin{tikzcd} \mathbb{P}^{n-1}_{\bar{K}} \cong \bar{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ \operatorname{Spec} \bar{K} \arrow[r] & \operatorname{Spec} K \end{tikzcd} \right)$$

dim 1

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ \operatorname{Spec} \bar{K} \arrow[r] & \operatorname{Spec} K \end{tikzcd}$$

Conics w/ no pts

$S \xhookrightarrow{|\omega_S^{-1}|} \mathbb{P}^2$

$K^\times/(K^\times)^2$ gives obst.

There's always:

$$S \times S \longrightarrow \mathbb{P}^{n^2-1}$$

for Brauer-Severi varieties S.

Idea:

Take elt $\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix}$; ask if we can pull it back to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$

The obstruction map, applied to $\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix}$, gives:

$$Ob\left(\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix}\right) = \text{class of } S \text{ in } Br(K)$$

<u>Strategy</u>

Describe $Sel^n(E/k)$ using Galois cohomology

&

Base change

$\Downarrow$

Study Galois action

Use "black box" — For any algebra known to be isomorphic to $M_n(k)$, the black box exhibits an isomorphism.