GTP Seminar
3/11/16

- Cremona, Fisher, O'Neil, Simon, Stoll
  "Explicit n-descent on elliptic curves I, II, III"

E/k (k = fraction of coord ring of B)

$$\begin{tikzcd} E \arrow[d] \\ \operatorname{Spec} k \arrow[u, bend right] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d] \\ B \arrow[u, bend right, dashed] \end{tikzcd}$$

Mordell-Weil group of $E \cong E[k]$

Tate-Shafarevich group of $E \sim E$-torsors

n-selmer group of E.

$$0 \longrightarrow E[k]/nE[k]$$

$$\shortparallel$$

$$E[n] = n\text{-torsion points}$$

$$\longrightarrow \mathrm{Sel}^{(n)}(E/k) \longrightarrow Ш\, E/k[n] \longrightarrow 0$$

F-theory:

$Ш(E/k)$ is closely related to group of CY E-torsors

$$\shortparallel$$

$\pi_0(G)$

$\subseteq$ Gauge group of F-theory model.

G = compact Lie group

$\mathfrak{g}$ = easily computable from geometric data.

$G/[G,G] \cong A$, max abelian quotient

$T \subseteq G$

$\shortparallel$

max abelian subgroup.

$T \longrightarrow A$ has finite kernel.

$G \xrightarrow{\alpha} A$ $\ker\alpha$ is semi-simple

$\ker\alpha = \prod G_j$ $\quad |\pi_1(G_j)| < \infty$

$\pi_1(G)$ = finitely gen. group

free part $\longleftrightarrow A$

torsion $\longleftrightarrow \prod \pi_1(G_j)$

$\mathfrak{g}$ and $\pi_1(G)$

$\rightsquigarrow$ connected compact Lie group.

$\mathfrak{g}_{ss} \longleftrightarrow$ Kodaira type of any fibers

$\pi_1(G) \cong MW(E/k)$

$\shortparallel$

$E[k]$

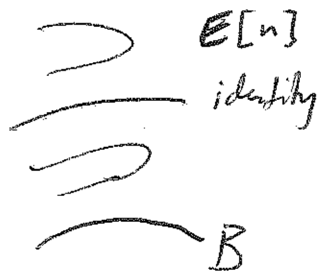
fn. field of $E[n]$

$|$

K = ground field.

$$\begin{tikzcd} E \supseteq E[n] \arrow[d] \arrow[d] \\ \operatorname{Spec} k \end{tikzcd}$$

n^2 pts if $k = \bar{k}$



n-torsion points on elliptic curves

$\underline{k = \mathbb{C}}$

$$H^1(E,\mathbb{Z}) \times H^1(E,\mathbb{Z}) \to H^2(E,\mathbb{Z}) \cong \mathbb{Z}$$

$\langle \alpha, \beta \rangle \in \mathbb{Z}$

skew-symmetric.

$$H^1(E, \mathbb{Z}/n\mathbb{Z}) \cong E[n] \times E[n] \to \mu_n$$

K arbitrary:

Weil pairing:

$$E[n] \times E[n] \longrightarrow \mu_n = \operatorname{Spec} k[x]/(x^n - 1)$$

R = etale algebra of $E[n]$

= product of fixed fields of $E[n]_j$ where

$E[n] = \cup E[n]_j$ ($E[n]_j$ irred.)

$W_1 : H^1(K, E[n]) \longrightarrow R^\times/(R^\times)^n$

injective, image is known.

$W_2 : H^1(K, E[n]) \to (R \otimes R)^\times / \partial R^\times$

injective, image is known.

Ob: $H^1(K, E[n]) \longrightarrow Br(K)$

Warning: Ob is not a homomorphism, $\ker(\text{Ob})$ need not be a group.

($Br(K)$ is Brauer group of k)

$W : E[n] \to R^\times = \operatorname{Map}(E[n], \bar{k}^\times)$

$W(S)(T) = \langle S, T \rangle_n$

(Weil Pairing)

$$w(\xi_\sigma) = \frac{\sigma(\gamma)}{\gamma}, \quad \alpha := \gamma^n \in R^\times, \quad \rho = \partial\gamma \in R^\times \otimes R^\times$$

$$E[n] \xrightarrow{w} R^\times \xrightarrow{\partial} (R \otimes R)^\times$$

$$\partial_\alpha(T_1, T_2) := \frac{\alpha(T_1)\alpha(T_2)}{\alpha(T_1 + T_2)}$$

$$w_1(\xi) = \alpha \cdot (R^\times)^n$$

$$w_2(\xi) = \rho \cdot \partial R^\times$$

Goal: represent each element of

$Sel^{(n)}(E(K))$

by an "elliptic normal curve"

$C \subset \mathbb{P}^{n-1}$, which is an E-torsor

$$|\mathcal{O}_E(nP)| : C \hookrightarrow \mathbb{P}^{n-1}$$

(if $n=2$ $C \to \mathbb{P}^1$ is $2:1$)

Elements in $H^1(K, E[n])$ can be represented by

$$\begin{array}{c} C \\ \downarrow \\ S \end{array} = \text{a Brauer-Severi}$$

variety over K of dim $n-1$

$$\begin{array}{ccc} \mathbb{P}^{n-1}_{\bar K} \cong \bar S & \longrightarrow & S \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar K & \longrightarrow & \operatorname{Spec} K \end{array}$$

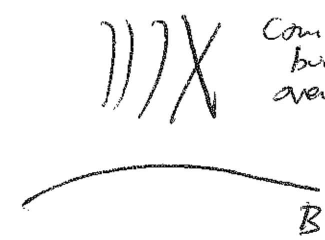
dim 1

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & S \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar K & & \operatorname{Spec} K \end{array}$$

conics with no points

$$S \xhookrightarrow{|\omega_S^{-1}|} \mathbb{P}^2$$

$K^\times / (K^\times)^2$



$$\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \searrow$$

$$S \times S \longrightarrow \mathbb{P}^{n^2-1}$$

$$\operatorname{Ob}\begin{pmatrix} C \\ \downarrow \\ S \end{pmatrix} = \text{class of } S \text{ in } Br(K)$$

"Black box": For any algebra known to be isomorphic to $M_n(K)$, exhibit an isomorphism