Twisted coned sum constructions of 62 manifolds

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Algebraic Geometry preprint

$S \equiv K3$  
$C \subseteq S$ alg. curve, nonsingular  
$C^2 = 2g - 2$ so if $g > 2$, $C^2 > 0$

$\begin{array}{rcl}
\varphi_{1|C} : S & \to & \mathbb{P}^3 \\
M & \mapsto & \{ P \}
\end{array}
$  
$K3$ with rational double points at $\varphi_{1|C}^{-1}(0)$

We have  
$H \cdot \varphi(S) = C' = C$  
and  
$K_{C'} = (K_S + 1) \cdot C' = N_{C'/S}

\text{such that } \begin{array}{rcl}
|C'| & = & |K_{C'}| \\
\varphi_{1|C'} : C' & \to & \mathbb{P}^3
\end{array}

Question: Is every algebraic curve the hyperplane section of some $K3$ surface?
\[ \dim M_{K3}^{(2g-2)} = 19 = \dim(P^g)^* \]
\[ \Rightarrow \dim(C \mid C \cap S \leq 19 + g) \]

\[ \dim M_g = 3g - 3 \quad \& \quad 19 + g > 3g - 3 \quad \Rightarrow \quad g \leq 11 \]

Therefore no hope that every curve of genus \( h \leq g \)
has no chance.

**Q:** Is
\[ 3(C \cap C)^3/100 \quad \Rightarrow \quad 3(C \cap C) \]

subjective in other cases?

If \( Y \) is a \textbf{Fano} 3-fold, i.e., \(-K_Y\) is ample and
\( S \) is a smooth divisor in \(-K_Y\), then
\[ K_S = (K_Y + S)_f = (K_Y + K_Y)_f = 0 \]
(\text{In fact, } S \text{ is } K_3)

\[ Y \subset S \subset C \]
\[ C = Y \cap H \cap H' \quad \Rightarrow \quad S = Y \cap (c_1 H \cap c_2 H) \quad \text{no } K_3 \text{ and } C \]

\[ \Rightarrow \text{ we have a 1 parameter family in the fibre of } C \]
\( M_{(4,5)} \rightarrow M_{k3}^N = \mathbb{P} \times K3 \cup N \in \pi_c(S)^3 / \) 

\( \text{Pic}(Y) \rightarrow \text{Pic}(S) \leq \mathbb{Z}^{3N} \quad N = \text{im}(\alpha) \quad \text{eg } N = (3,2) \)

Bauville's theorem \( \rightarrow \) this map is generically surjective for my Fano 3-fold

Generally \( \dim M_{k3}^N = 2d - \text{rc}(W) \)

CHNP: generalize to \( Y \) being semi-Fano.

(End AC proof)

\[ G2 \text{ manifolds} \]

Given 11-D spacetime that turn as

\[ \text{compact 7-manifold} \times \text{Minkowski 4-space} \]

\[ \text{Spinors} \]

\[ \text{covariant constant w.r.t. } g_7 \]

\[ \text{Spinors} \]

If we can do the holonomy is contained in \( G_2 \)
We can also do this with a 10-dim `2

Spacetime; the analogous result says that the
holonomy is contained in \( \text{SO}(9) \), which eventually leads
to the Calabi-Yau condition.

Holonomy:

Take loops in space

\[ \bigcirc \]

parallel transport along loop &
get a map \( T_p \to T_p \) preserving
inner product \( \to \text{ESO}(T_p) \)

\( \text{SO}(T_p) \) has a double cover called spin group, \( \text{Spin}(T_p) \)

Spinor representation

\[ \text{SO}(T_p) \xrightarrow{\text{Spin}} \text{Spin}(T_p) \]

Spinor rep
Question: Given \( SO(7) \hookrightarrow \text{Spin}(7)_0 \), what is the stabilizer of \( \alpha_0 \) under \( \text{Spin}(7) \)? It turns out that the answer is \( G_2 \) regardless of the given \( \alpha_0 \).

\[
\begin{align*}
0 & \cong & V & \simeq T_p \\
G_2 \text{ acts on } V &; \text{ indec. 7-dim. rep of } \text{ Spin}(7) \\
G_2 & \cap \Lambda^2 V = 7 \ \cong \ \text{adj}(G_2) \\
G_2 & \cap (\Lambda^2 V)_0 \ \text{ indec. 27-dim. rep: 27} \\
G_2 & \cap (\Lambda^3 V) = 27 + 7 + 1 \\
\end{align*}
\]

\( G_2 \) stabilizes a 3-form on 7-manifold

In fact, \( G_2 \cong \text{Stab} \) (This 3-form in \( \Lambda^3 \mathbb{R}^7 \))
\{3 - \text{forms} \subseteq \text{SU} \times G_2 \} \subseteq \Lambda^3 \mathbb{R}^7

\text{"Positive 3-forms"}

\textbf{Examples}

1) \(T^7 = \mathbb{R}^7/\mathbb{Z}^7\)

\(h_0 = \text{SU}(3) \leq G_2\)

2) \(T^5 \times S^1, \ S = K3\)

\(h_0 = \text{SU}(2) \leq G_2\)

\(\Phi = d\omega_1 \wedge d\omega_2 \wedge d\omega_3 + \frac{1}{3!} d\phi_1 \wedge d\phi_2 \wedge d\phi_3\)

Where \(\omega_1, \omega_2, \omega_3\) are orthonormal basis of positive harmonic forms.

3) \(S^1 \times T^2\) where \(T^2 = \text{CY 3-fold}\)

\(\Phi = d\Theta \wedge \omega + \text{Re}(\Omega), \text{ where } \Theta \text{ is Kahler form + 2}\)

\(A = \text{holo 3-forms}\)

\(h_0 \text{holo 3-form} = \text{SU}(3) \leq G_2\)
1) \( T^7 / G \) where \( G \) is a finite group

whose fixed points in real codimension 4

\[ \text{stab}(p) \leq SU(2) \leq SO(4) \]

Joyce orbifolds

\( T^7 / G \)  

(like a res. of \( \text{H}^3 \) \( \text{w} \))

This gave the first class of \( G_2 \) manifolds

\[ \text{Holonomy} = G_2 \]

2') \( (S^1 \times \mathbb{Z}) / \mathbb{Z} \): complex \( \mathbb{Z} \) acts by complex on \( \mathbb{C} \) and

\[ 0 \to 0 \to \mathbb{R} \]  

(it \( G_2 \) is defined)

This is smooth if \( \mathbb{Z} / (\mathbb{Z} \times \mathbb{R}) \to \mathbb{Z} \) is simply connected and

\[ \text{holonomy} \approx \text{SU}(3) \times \mathbb{Z} \subset G_2 \]

\[ \mathbb{Z}_2 \] locus.
$2/G_2$ cylinder

$\phi \times K3 \mapsto \text{unwarp}$

$\phi \times K3$

$\int d\theta_1 d\theta_2 \wedge \theta_1 \wedge \theta_2 + d\theta_1 \wedge \omega_1 + d\theta_2 \wedge \omega_2 + d\theta_3 \wedge \omega_3$

Donaldson matching:

$X \to X$, $\theta_1 \to -\theta_2$, $\omega_1, \omega_2, \omega_3 \to \omega_2, \omega_1 - \omega_3$

Preserves $\phi$

Asymptotically cylindrical $G_2$ manifold
\((Y_+, S_+) \times S^1 \rightarrow \text{Cyl}_+ \times S^1\)

\((Y_-, S_-) \times S^1 \rightarrow \text{Cyl}_- \times S^1\)

\(\mathcal{B}_+ Y = \mathcal{S} \quad \mathcal{S} \backslash \mathcal{S} = \text{noncompact CY 3-fold}\)

Variant of Yau's theorem: there is a Ricci flat metric, asymptotically canonical

\(\text{A weak Fano 3-fold is non-singular } Y \text{ such that } \langle -K_Y \rangle \geq 0 \quad (K_Y)^3 > 0 \quad \text{(!big!)}\)

Choose \(S, S'\), generate an anticanonical pencil:

\(Z = \mathcal{B}_{S,S'} Y \quad \text{Z fibered over } \mathbb{P}^1, \text{ fibers are K3 surfaces.}\)
to $y$, $P_2 \neq 1$, so, as above $\theta = \theta_0$.

Partial closure gives $y_t$ of $N^{\mathbb{Z}_2}$.

Small closure gives $y_t$ of $N^{\mathbb{Z}_2}$.

Since $F_1 + F_2 - 3 \theta_0 = X$, $x_2 F_2 = 2 \theta_0 F_1 = 2 \theta_0 F_2 = 3 \theta_0$.

Case with a quadratic yield $\mathbb{Z}_n$.

Example:

Start with a quadratic field $F_4$. Which can be done as $P_2 = 3 F_2$.

Assume line yields $\mathbb{Z}_n$.