5d SCFTs:
Symmetries and Moduli Spaces

Sakura Schäfer-Nameki

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Based on Papers BC (Before Corona)

1906.11820, 1907.05404,1909.09128 with
Fabio Apruzzi, Craig Lawrie, Ling Lin, Yi-Nan Wang
1912.04264 with Fabio Apruzzi, Yi-Nan Wang

...and DC (During Corona)

2004.15007 with Julius Eckhard, Yi-Nan Wang
2005.12296 with Dave Morrison, Brian Willett
→ SSN: Italian String Webinar 5/2020
2007.15600 with Cyril Closset, Yi-Nan Wang
→ Cyril: String-Math 2020
2008.NNNNN with Marieke van Beest, Antoine Bourget, Julius Eckhard

5d Gauge Theories

1. $d > 4$: Gauge coupling in

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}F^2 + \cdots$$

   mass dimension $[g^2] = d - 4 < 0$

2. The interactions are irrelevant at long distances ('IR free'). Naive expectation: these are boring theories.

3. No interacting CFTs?

Evidence to the contrary.

5d CFTs

1. UV: $g \to \infty$
2. 5d gauge theories: effective theories on extended Coulomb branch
3. Evidence:
   Find description that extrapolates to strong coupling
   $\Rightarrow$ string/M-theory

5d $\mathcal{N} = 1$ Gauge Theories and SCFTs

5d $\mathcal{N} = 1$ are IR descriptions of 5d $\mathcal{N} = 1$ SCFTs in the UV:

- ⋆ Gauged: G_{gauge}
- ⋆ Global: $G_{\text{F}}^{\text{IR}} \times U(1)_T \subset G_{\text{F}}^{\text{UV}}$.
- ⋆ Reps:

$$\text{Vector multiplet in the adjoint of } G_{\text{gauge}}: \quad \mathcal{A} = (A_\mu, \phi, \lambda)$$

$$\text{Hyper-multiplet in } (\mathbf{R}, \mathbf{R}_{\text{F}}) \text{ of } G_{\text{gauge}} \times G_{\text{F}}^{\text{IR}}: \quad \mathbf{h} = (h \oplus h^c, \psi)\,.$$

- ⋆ Vacuum moduli spaces:
  1. Coulomb branch (CB): vevs of ϕ and masses of m_F of $\mathbf{h}$.
  2. Higgs branch (HB): vevs of the hyper-multiplets.

Example: Rank 1 Seiberg Theories

- $G_{\text{gauge}} = SU(2)$ with N_F fundamental hyper-multiplets, $N_F = 0, \cdots, 7$
- $G_{\text{F}}^{\text{IR}} = SO(2N_F)$
- UV: enhanced 'super-conformal flavor symmetry'

$$G_{\text{F}}^{\text{IR}} \times U(1)_T \hookrightarrow G_{\text{F}}^{\text{UV}} = E_{N_F+1}$$

- N_F to $N_F - 1$ by giving mass m_F for a matter multiplet and decoupling ($m_F \to \infty$).
  $\Rightarrow (\phi, m_F)$ parametrize the extended Coulomb branch

5d SCFTs and Canonical Singularities

A 5d superconformal field theory is defined as [Seiberg][Morrison, Seiberg]

$$\mathcal{T}^{5\mathrm{d}}(\mathbf{X}) = \text{M-theory on } \mathbf{X} \times \mathbb{R}^{1,4}\,,$$

where **X** = canonical singularity (isolated or not).

$$\begin{aligned} \text{Canonical singularity} &\longleftrightarrow \text{SCFT} \\ \text{Kähler cone} &\longleftrightarrow \text{(Extended) Coulomb Branch} \\ \text{Complex deformations} &\longleftrightarrow \text{Higgs Branch} \end{aligned}$$

5d SCFTs and SQFTs

5d QFT, geometry and webs:

[Seiberg][Morrison, Seiberg], [Intrilligator, Morrison, Seiberg][Klemm, Mayr, Vafa][Aharony, Hanany, Kol][Bergman, Rodriguez-Gomez][Bergman, Zafrir]

And recent works by [Kim, Lee, Hayashi, Zafrir, Bergman, Yagi, Hwang, Park, Yonekura, Tachikawa, Rodriguez-Gomez, Hanany, Bourget, Cabrera, Yagi]...

Recently, approach using 6d SCFT on S^1:

[Xie, Yau][Del Zotto, Heckman, Morrison][Jefferson, Kim, Vafa, Zafrir][Jefferson, Katz, Kim, Vafa][Bhardwaj, Jefferson][Apruzzi, Lin, Mayrhofer][Closset, Del Zotto, Saxena] [Apruzzi, Lawrie, Lin, SSN, Wang]^3[Apruzzi, SSN, Wang] [Bhardwaj][Eckhard, SSN, Wang]....

Today's Goals:

Geometric description of the 5d:

1. Coulomb branch
2. Higgs branch
3. Symmetries

Coulomb Branch

$\mathbf{X}$ admits resolutions (crepant or with residual terminal singularities)

$$\widetilde{\mathbf{X}} \longrightarrow \mathbf{X}$$

- Gauge Symmetry:
  (compact) exceptional divisors

$$\mathcal{S}_a \,, \qquad a = 1, \cdots, r = b_4(\widetilde{\mathbf{X}}) = \text{rank of the SCFT}$$

- Global (flavor) symmetry:
  non-compact divisors D_α, $\alpha = 1 \cdots, f$ = flavor rank,

$$b_2(\widetilde{\mathbf{X}}) = r + f \,.$$

- Free hypermultiplets: for $\mathbb{P}^1 \hookrightarrow \mathcal{S}_a \to \Sigma_{g_a}$,

$$b_3(\widetilde{\mathbf{X}}) = 2 \sum_a g_a \,,$$

  contribute $b_3/2$ free hypers.

- Dynamics on the Coulomb Branch:

  Pre-potential: ϕ^i, $i = 1, \cdots, r$ CB vevs

$$\mathcal{F} = \left(\frac{1}{2g^2} C_{ij}\phi^i\phi^j + \frac{k}{6} d_{ij\ell}\phi^i\phi^j\phi^\ell \right) + \frac{1}{12}\left(\sum_{\alpha \text{ roots}} |\phi\cdot\alpha|^3 - \sum_{\lambda_{\text{F}}\in\mathbf{R}_{\text{F}}} |\lambda_{\text{F}}\cdot\phi + m_{\text{F}}|^3 \right) ,$$

  $C_{ij} = \mathrm{Tr}_{\mathrm{F}} T_i T_j$, $d_{ijk} = \frac{1}{2}\mathrm{Tr}_{\mathrm{F}}((T_i(T_jT_k + T_kT_j))$, T_i = Cartans of G_{gauge}.

  The prepotential determines the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = G_{ij}\, d\phi^i \wedge \star d\phi^j + G_{ij}\, F^i \wedge \star F^j + \frac{c_{ij\ell}}{24\pi^2} A^i \wedge F^j \wedge F^\ell$$

  where $G_{ij} = \partial_i\partial_j\mathcal{F}$ and CS-levels $c_{ij\ell} = \partial_i\partial_j\partial_\ell\mathcal{F}$.

  Relation to Geometry: [Intriligator, Morrison, Seiberg]

$$\partial_i\partial_j\partial_\ell\mathcal{F} = \mathcal{S}_i \cdot_{\widetilde{\mathbf{X}}} \mathcal{S}_j \cdot_{\widetilde{\mathbf{X}}} \mathcal{S}_k \, .$$

- Wrapped M2-branes on rational curves:
  1. normal bundle degree $(-2, 0)$: W-bosons
  2. normal bundle degree $(-1, -1)$: matter hypermultiplets
- SCFT:

$$\frac{1}{g_a^2} \sim \text{Volume}(\mathcal{S}_a) \to 0$$

Many geometric tools:

- Toric CY
- Elliptic fibrations
- Characterize collapsable complex surfaces
- Isolated Hypersurface Singularities (IHS)

Geometric Setup	CB	HB	Symmetries	Scope
Toric CY	✓	✓	✓	Limited Class of models
Elliptic CY	✓	-	✓	All known examples (from 6d)
Collapsable Surfaces	✓	-	Some	Bottom-up, not CY geometry
IHS	✓	✓	'✓'	Special class, new effects
Brane-Webs Not always Geo.	✓	✓	✓	After Elliptic CY: largest class

Toric CY3

Toric fan Σ= set of cones in the 3d lattice $N \cong \mathbb{Z}^3$.

1d cones (rays) $\mathbf{v} = (v_x, v_y, v_z = 1)$: divisors

Project to a convex polygon in the xy-plane:

Internal: compact divisors $\widehat{\mathbf{v}}_i$, $i = 1, \cdots, r$

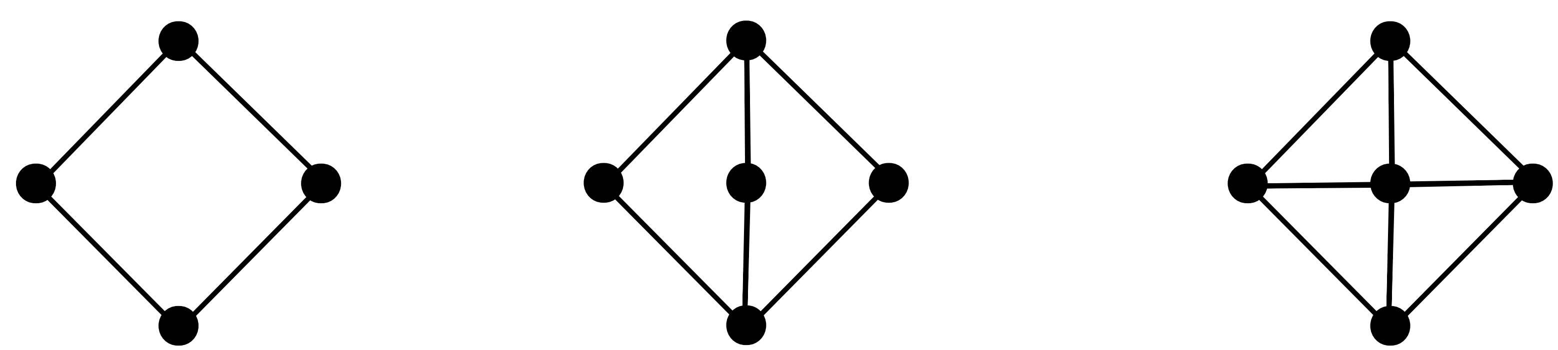
External: non-compact divisors $\mathbf{v}_\alpha$, $i = 1, \cdots f + 3$, f flavor rank

2d cones: complete intersection curves $C_{ij} = D_i \cdot D_j$,

3d cones: intersections $D_i \cdot D_j \cdot D_k$

$$E_1:\ SU(2)_0$$

Toric geometry of E_1 Seiberg theory: $\mathbf{v}_\alpha : \{(\pm 1, 0, 1), (0, \pm 1, 1)\}$. In the xy-plane:



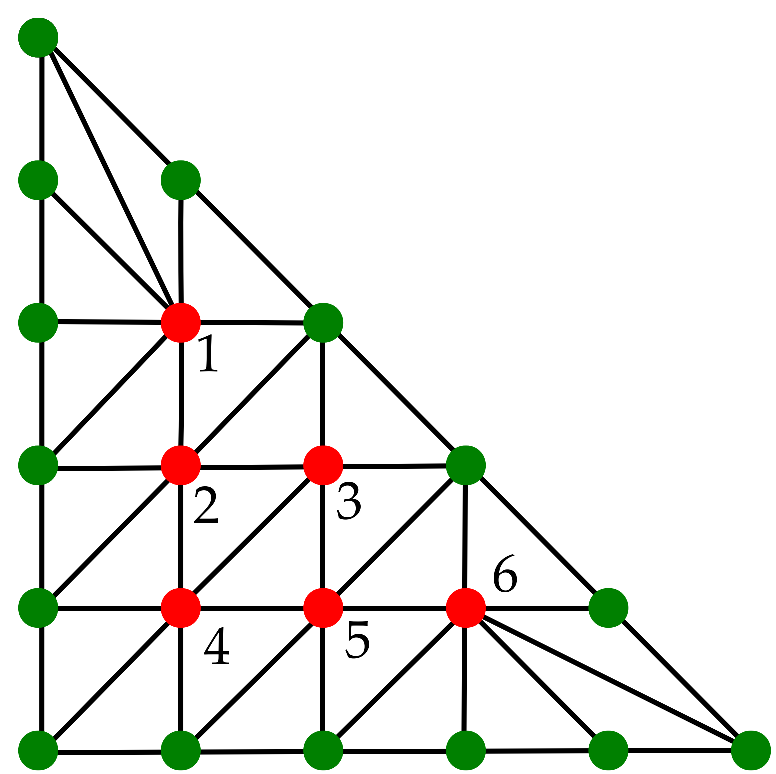
Singular (SCFT), partially resolved ($SU(2)$ gauge theory description), fully resolved ($U(1)$ Coulomb branch)

T_N Theories

M-theory on $\mathbf{X} = \mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ [Benini, Benvenuti, Tachikawa].

Toric realization: external rays

$\mathbf{v}_\alpha$: $((N-1,0,1),(1,N-1,1),(0,1,1))$. E.g. $N=5$:



Gauge, rank $r=6$. Non-compact $\supset$ Flavor.

Symmetries from CB

0-form symmetries:

Gauge Theory has global symmetry (IR flavor symmetry) G_F^{IR} and topological $U(1)_T$

$$j = \frac{1}{8\pi^2} \star \mathrm{Tr} F \wedge F$$

Examples:
$SU(N_c) + N_F \mathbf{F}$ has $G_F^{\mathrm{IR}} = U(N_F)$, $Sp(N) + N_F \mathbf{F}$ has $G_F^{\mathrm{IR}} = SO(2N_F)$.

UV fixed points:

$$G_F^{\mathrm{UV}} \supset G_F^{\mathrm{IR}} \times U(1)_T$$

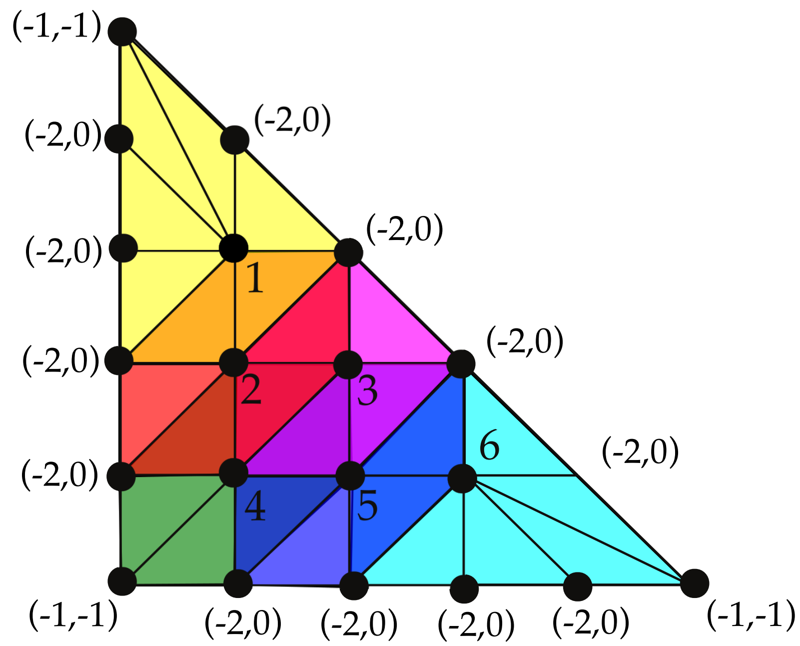
G_F^{UV}: Encoded in the Combined Fiber Diagram (CFD):

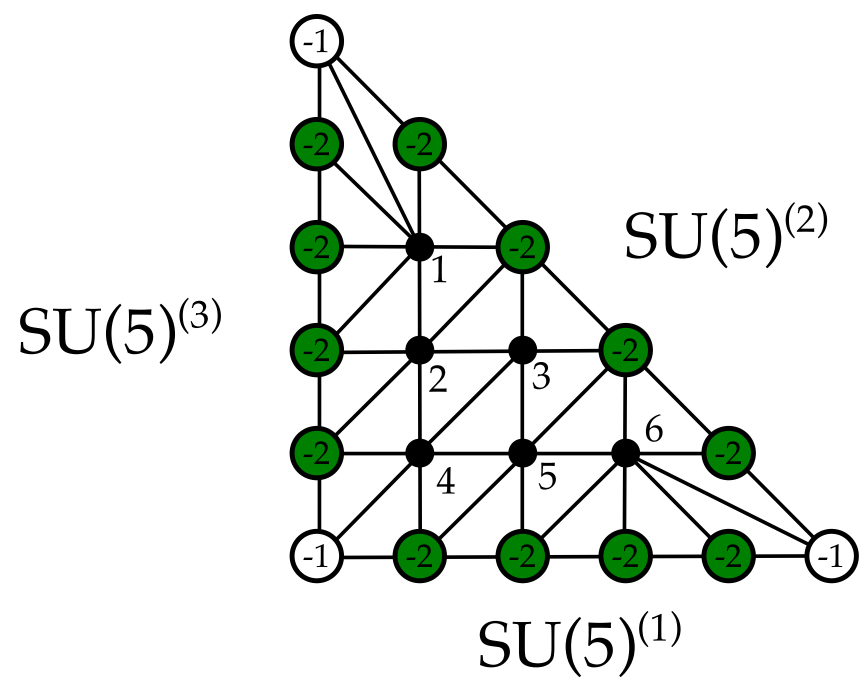
Graph made of rational curves $C_i = D_i \cdot (\sum_\alpha \mathcal{S}_\alpha)$, where $(-2, 0)$ curves are marked vertices, and intersections give rise to G_F, and $(-1, -1)$ curves are hypermultiplets.

[Series of papers with: Apruzzi, Lawrie, Lin, Yi-Nan Wang, Eckhard, SSN]

T_N

$(-2,0)$ are C = compact $\cap$ non-compact divisor $\Rightarrow$ Dynkin diagram of G_F
$(-1,-1)$ Hypermultiplet matter:





Here: $G_F^{UV} = SU(5)^3$, for T_N, $G_F^{UV} = SU(N)^3$.

From the CFD: read off the UV flavor, but also the entire decoupling tree, by flops on the $(-1,-1)$ curves.

Higher-Form Symmetries

Gauge theories can have generalized global symmetries
[Gaiotto, Kapustin, Seiberg, Willett].

In d dimensions: 0-form symmetry (ordinary symmetry), charged operator that is point-like with

$$q = \int_{S^{d-1}} \rho$$

A q-form symmetry: charged operators are dimension q and with topological surface operators of dimension $d-1-q$.

Higher form symmetries for 5d SCFTs: [Morrison, SSN, Willett]
[Albertini, Garcia-Extebaria, Hosseini, Del Zotto] [Closset, SSN, Y-N Wang]

Higher Form Symmetries in Gauge Theories

5d Gauge Theories:

- Gauge theories (no matter) with simply-connected gauge group G and center Z have an (electric) 1-form symmetry $\Gamma = Z$.

  Charged operators:
  Wilson loops in rep **R**, transform under Γ as **R** does under Z.

- If $\pi_1(G) = \Gamma_m \neq 1$ then the theory as a 2-form (magnetic) symmetry.

Can pass from one to the other by gauging (sum over background values of gauge field $H^2(M_5, \Gamma)$).

Example:
$SU(N)$ has a $\Gamma = \mathbb{Z}_N$, $SU(N)/\mathbb{Z}_N$ has $\Gamma_m = \mathbb{Z}_N$ 2-form symmetry.

q-Form Symmetry from Geometry

M-theory on $\mathbf{X}$, boundary five-manifold $\partial\mathbf{X}$. 1-form symmetry:

- # M2-branes on compact 2-cycles: $H_2(\mathbf{X})$
  mass $m < \infty$ particles in 5d
- # M2-brane on non-compact 2-cycle: $H_2(\mathbf{X}, \partial\mathbf{X})$
  infinite mass particle, worldline defines line operator.
  Some line operators could be screened by dynamical particles:

$$\Gamma^{(1)} = H_2(\mathbf{X}, \partial\mathbf{X})/H_2(\mathbf{X})$$

For q-form symmetry: e = M2, m = M5-branes wrapped

$$\Gamma_e^{(q)} = \mathfrak{h}_{(k=3-q)} \qquad \mathfrak{h}_{(k)} = \text{Torsion}\left(H_k(\mathbf{X}, \partial\mathbf{X})/H_k(\mathbf{X})\right)$$

$$\Gamma_m^{(q)} = \mathfrak{h}_{(k=6-q)}$$

$$
\begin{aligned}
q = -1: &\qquad \Gamma_e^{(-1)} = \mathfrak{h}_{(4)} \\
q = 0: &\qquad \Gamma_e^{(0)} = \mathfrak{h}_{(3)}\,, \qquad \Gamma_m^{(0)} = \mathfrak{h}_{(6)} \\
q = 1: &\qquad \Gamma_e^{(1)} = \mathfrak{h}_{(2)}\,, \qquad \Gamma_m^{(1)} = \mathfrak{h}_{(5)} \\
q = 2: &\qquad \Gamma_e^{(2)} = \mathfrak{h}_{(1)}\,, \qquad \Gamma_m^{(2)} = \mathfrak{h}_{(4)} \\
q = 3: &\qquad \Gamma_e^{(3)} = \mathfrak{h}_{(0)}\,, \qquad \Gamma_m^{(3)} = \mathfrak{h}_{(3)} \\
q = 4: &\qquad \qquad\qquad\qquad\quad\;\; \Gamma_m^{(4)} = \mathfrak{h}_{(2)}\,.
\end{aligned}
$$

$\Gamma^{(1)}$ from intersection theory

$$\Gamma^{(1)} = H_2(\mathbf{X}, \partial\mathbf{X})/H_2(\mathbf{X})$$

This can be computed on the Coulomb branch. Poincaré-Lefschetz duality maps this to

$$\Gamma^{(1)} = \mathbb{Z}^{b_4}/\mathcal{M}_4\mathbb{Z}^{b_2}$$

where $\mathcal{M}_4$ is the intersection matrix between compact curves C and compact divisors $\mathcal{S}$ in **X**:

$$\mathcal{M}_4 = (\mathcal{S} \cdot C)_{r\times(r+f)}$$

Toric CY3

Recall: toric fan defined by external vertices $\mathbf{v}_\alpha$, $\alpha = 1, \cdots, f+3$ and internal vertices $\hat{\mathbf{v}}_i, i = 1, \cdots, r$. Can further simply Γ to

$$\Gamma = \mathbb{Z}^{f+3}/\mathrm{Im}A, \qquad A = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{f+3} \end{pmatrix}$$

Compute Smith normal form of A to find

$$\Gamma = \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \mathbb{Z}_{\alpha_3} \,.$$

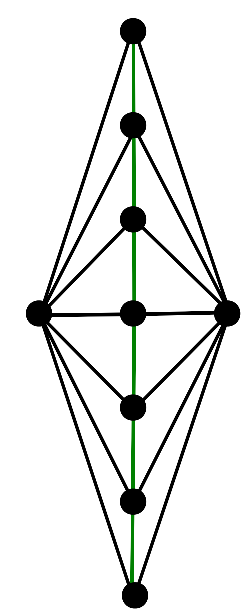
Examples

- $SU(2)_0$: $A = ((1,0,1),(-1,0,1),(0,1,1),(0,-1,1))$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \Gamma = \mathbb{Z}_2$.

- Likewise $SU(N)_k$: $\Gamma = \mathbb{Z}_{\gcd(N,k)}$ ✓



1-Form symmetries beyond Gauge Theories

Advantages of the geometric formulation:

1. Γ depends only on data of external vertices, but can be equally computed on the CB.
   $\Rightarrow$ seems to be applicable to UV fixed point
2. Non-Lagrangian theories: Rank 1: $\mathbb{P}^2$-Seiberg theory.
   Toric fan: $A = ((-1,0,1),(0,-1,1),(1,1,1)$ results in

$$\mathbb{P}^2 : \qquad \Gamma = \mathbb{Z}_3 \qquad (G_F = 1)\,.$$

We will see examples of 3-form symmetries in 5d SCFTs on the Higgs branch.

Coulomb branch

- Gauge theory descriptions:
  Not necessarily unique, from rulings of compact divisors.
- SCFT flavor symmetry and decoupling:
  manifest in terms of CFDs. Generalization to non-toric models, such as elliptic fibrations [Apruzzi, Lawrie, Lin, SSN, Y-N Wang]
- Higher form symmetries:
  Computable in terms of the $\partial \mathbf{X}$ or resolution.

Higgs Branch

Higgs Branch (HB) of the SCFT $\mathcal{T}^{\text{5d}}(\mathbf{X})$ is a hyper-Kähler cone

$$\text{HB}[\mathcal{T}^{\text{5d}}(\mathbf{X})]\,.$$

$\text{Dim}_{\mathbb{H}} = d_H$. Unlike the CB, metric on the HB receives quantum corrections from M2-instantons.

Geometric characterization in terms of deformation of **X**

$$\widehat{\mathbf{X}}$$

Geometric Framework:

- Isolated toric CY3
- Isolated Hypersurface Singularities (IHS)

Complementary approach:
recent progress using branewebs, determining the *magnetic quiver* and Hasse diagram for 5d SCFTs with gauge theory IR descriptions [Bourget, Cabrera, Hanany, Grimminger, Yagi, Zhong...]

Interlude: Deformations from Tropical Geometry

$\mathbf{X}$ =(Generalized) toric.

$$\mathbf{X} \quad \xrightarrow{\text{dual graph}} \quad \mathcal{W}_{\mathbf{X}} = \text{5-brane web (tropical geometry)}$$

Conjecture [Cabrera, Hanany, Yagi]:
$\exists$ 3d $\mathcal{N} = 4$ quiver gauge theory $\text{MQ}^{(5)}$ associated $\mathcal{W}_{\mathbf{X}}$, determined by irreducible subwebs, such that

$$\text{CB}\left[\text{MQ}^{(5)}\right] = \text{HB}\left[\mathcal{T}^{(5\text{d})}(\mathbf{X})\right] .$$

Quantum corrections on LHS are understood [Cremonesi, Hanany, Zaffaroni], and in mathematics [Nakajima][Braverman, Finkelberg, Nakajima].

Natural question: what is the $\text{MQ}^{(5)}$ in terms of $\mathbf{X}$?

- Strictly convex: deformations in terms of Minkowski sums [Altmann].
- Formulate the rules on $\mathcal{W}_{\mathbf{X}}$ in terms of the generalized toric polygon $P_{\mathbf{X}}$ for $\mathbf{X}$: algorithm to determine the MQ and Hasse diagram [van Beest, Bourget, Eckhard, SSN, to appear].

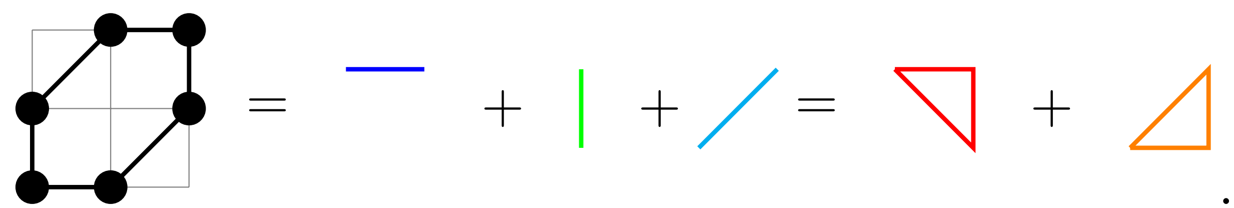
Higgs branch from Tropical Geometry

An Example from [van Beest, Bourget, Eckhard, SSN, to appear]:
Isolated toric CY **X**, with strictly convex polygon P_{X}

1. Determine all Minkowski sum decompositions of P_{X}: $P_{\mathsf{X}} = +_i P_{c_i}$
2. Each summand is associated with a color c_i, and each Minkowski sum decomposition induces an edge coloring

   E_3-theory (i.e. $SU(2) + 2\mathbf{F}$ IR description):

   E_3 :



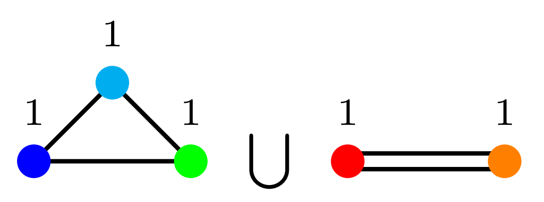
3. An edge coloring is consistent if it extends to a tasselation of P into triangles of one color and bi-colored parallelograms:

4. Each color $\to \overset{1}{\bullet}$ in the MQ.
5. The number of edges k_{c_1,c_2} between nodes associated to c_1 and c_2 are determined by the mixed volume, i.e.

$$k_{c_1,c_2} = \text{Area}(G_{c_1,c_2}) = \text{area of the } c_1, c_2 \text{ bicolored paralellogram}\,.$$

   In the dual tropical geometry: stable intersection of tropical curves.

$\text{MQ}(E_3) =$



Higgs branch: $\mathfrak{a}_2 \oplus \mathfrak{a}_1$ (minimal nilpotent orbits).

This algorithm generalizes to any not strictly convex toric, and generalized toric ('dot diagram') polygon.

Open question: derive these rules from the deformation theory of **X**.

Deformations of Isolated Hypersurface Singularities (IHS)

We propose [Closset, SSN, Y-N Wang] a geometric approach for IHS to determine the MQ from $\widehat{\mathbf{X}}$.

Let $\mathbf{X}$ be a canonical IHS. Classified by [Yau, Yu][Xie, Yau]:

$$\mathbf{X}: \qquad \{F(x_1, x_2, x_3, x_4) \equiv F(\mathbf{x}) = 0\} \quad \subset \quad \mathbb{C}^4 \,.$$

1. F is quasi-homogeneous, i.e. $x_i \to \lambda^{q_i} x_i$ then

$$F(\lambda^{q_i} x_i) = \lambda F(\mathbf{x}) \,, \qquad q_i \in \mathbb{Q}_{>}0 \,.$$

2. Singular at an isolated point.
3. Canonical $\Rightarrow \sum_{i=1}^4 q_i > 1$ or $\widehat{c} = \sum_{i=1}^4 (1 - 2q_i) < 2$ [Shapere, Vafa]

19 different families of IHS, with many redundancies.

Deformation

Deformation $\widehat{\mathbf{X}}$ is characterized by the Milnor ring,

$$\mathcal{M}(F) = \mathbb{C}[x_1, x_2, x_3, x_4]/dF\,,$$

which is finitely generated for IHS of dimension $\mu = \prod_{i=1}^{4}(q_i^{-1} - 1)$, with

$$\widehat{\mathbf{X}}: \qquad F(\mathbf{x}) + \sum_{l=1}^{\mu} t_l \mathbf{x}^{\mathbf{m}_l} = 0\,, \qquad \mathbf{x}^{\mathbf{m}_l} \in \mathcal{M}(F)\,.$$

Deformed space has additional three-cycles

$$H_3(\widehat{\mathbf{X}}, \mathbb{Z}) = \mathbb{Z}^{\mu}\,, \qquad \mu = \text{Milnor number}\,.$$

Define the spectral numbers ℓ_l:

$$\ell_l = Q_l + \sum_i q_i - 1\,, \qquad Q_l = \sum_{i=1}^{4} q_i \mathbf{m}_{l,i}\,.$$

Mixed Hodge structure from monodromy acting on H_3:

$$\begin{aligned} \ell_l < 1: &\qquad \dim H^{1,2}(\mathbf{X}, \mathbb{Z}) = \widehat{r} \\ \ell_l = 1: &\qquad \dim H^{2,2}(\mathbf{X}, \mathbb{Z}) = f \\ \ell_l > 1: &\qquad \dim H^{2,1}(\mathbf{X}, \mathbb{Z}) = \widehat{r}\,. \end{aligned}$$

Higgs branch is given by the number of dynamical hypermultiplets, which arise from $\ell_l \leq 1$ [Gukov, Vafa, Witten]

$$\dim_{\mathbb{H}}(\mathcal{M}_H) = d_H = \widehat{r} + f\,.$$

f = flavor rank, as on the CB, via conifold transitions.

Higgs Branch from Duality

To determine the hyper-Kähler structure: use dualities.

Proposal in [Closset, SSN, Y-N Wang]:
Consider Type IIB on **X**. This is a 4d $\mathcal{N}=2$ SCFT $\mathscr{T}^{\rm 4d}(\mathbf{X})$.

Compactify both theories to 3d $\mathcal{N}=4$, the 'electric quiver(ine)s'

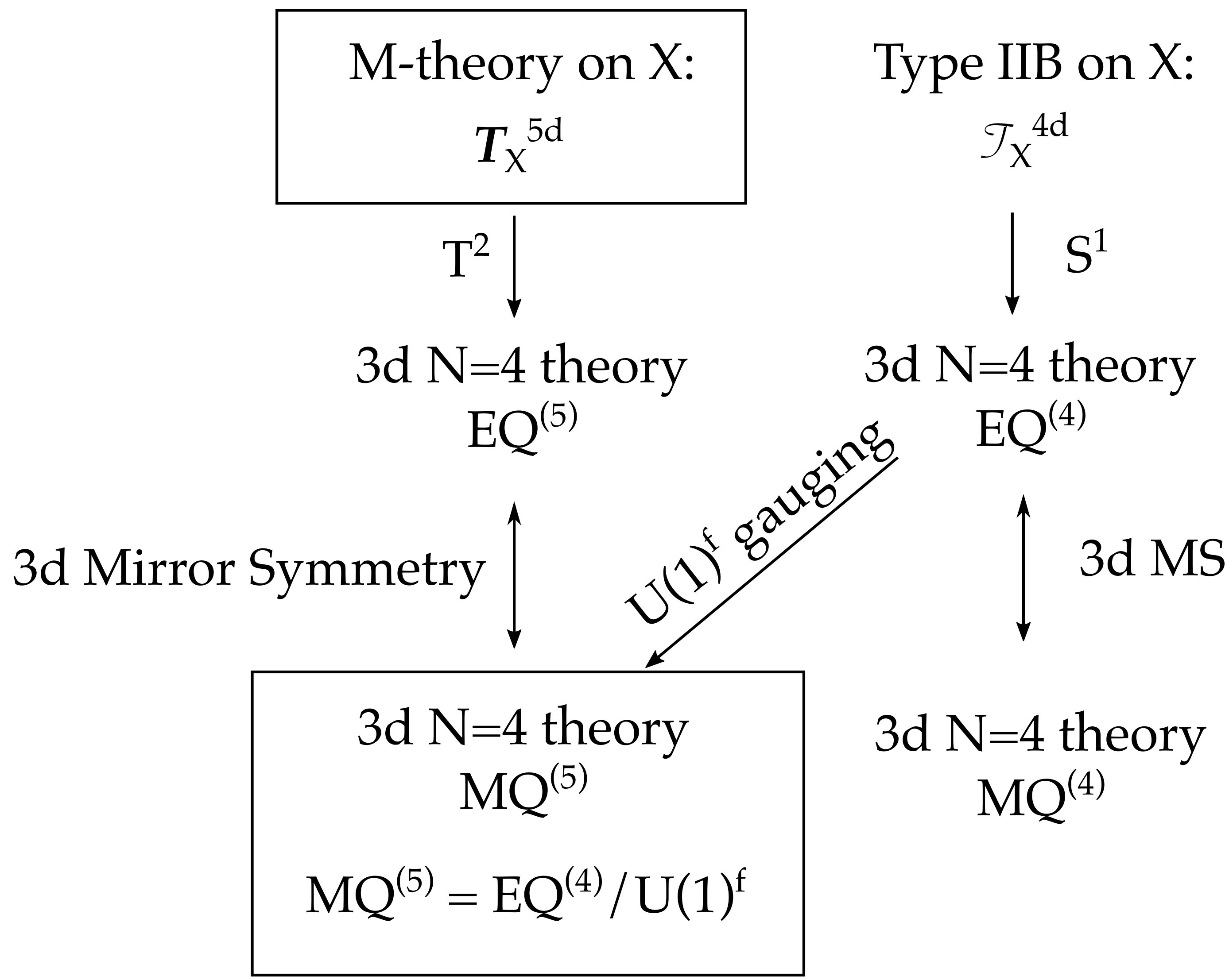
- $\mathrm{EQ}^{(5\mathrm{d})} \equiv \mathcal{T}^{5\mathrm{d}}(\mathbf{X})$ on T^2
- $\mathrm{EQ}^{(4\mathrm{d})} \equiv \mathscr{T}^{4\mathrm{d}}(\mathbf{X})$ on S^1

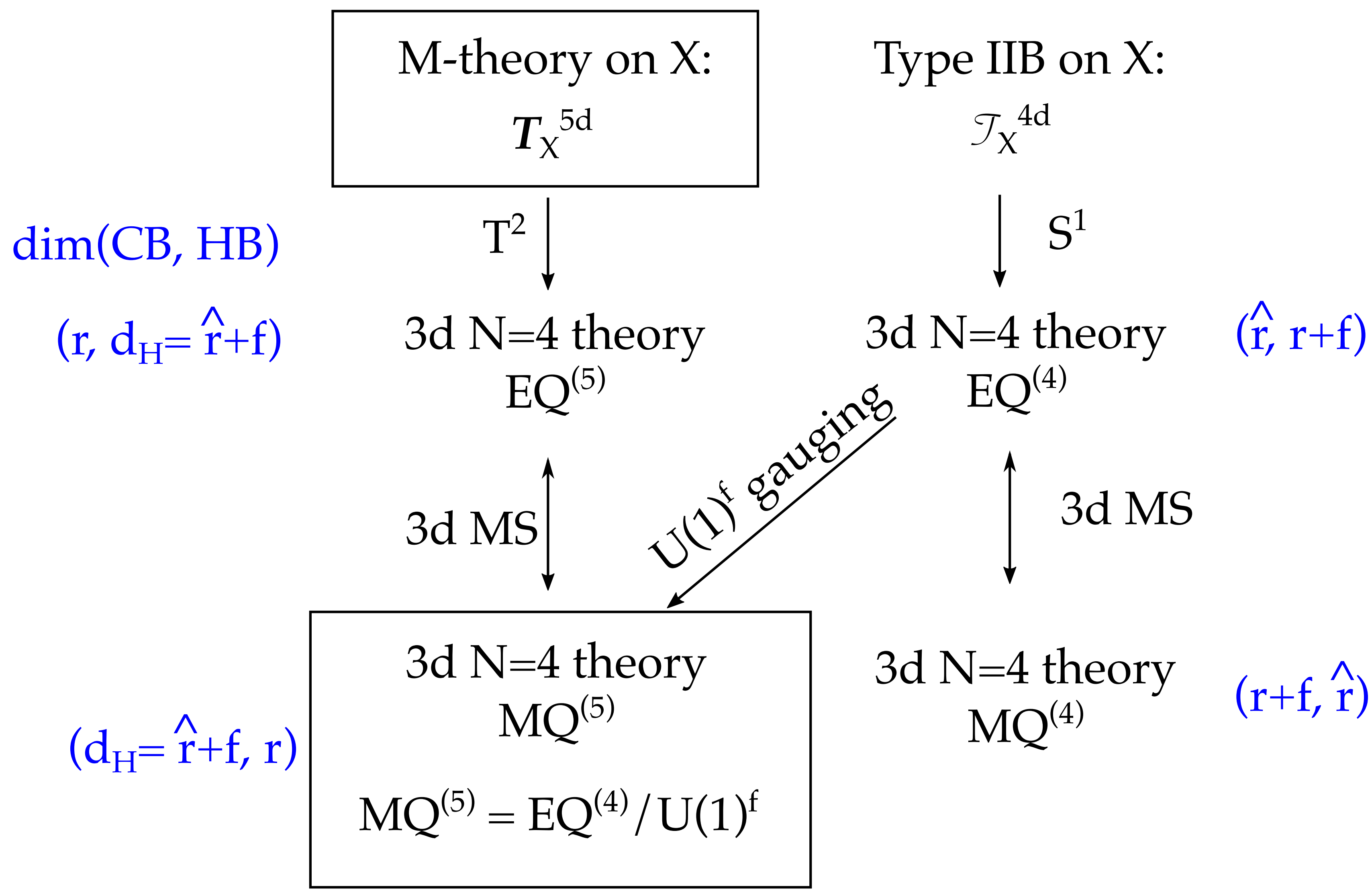
These theories are related by T-duality, which realizes 3d mirror symmetry [Hori, Ooguri, Vafa].

Let: $\mathrm{MQ}^{(5\mathrm{d})} \equiv$ 3d mirror of $\mathrm{EQ}^{(5\mathrm{d})}$, $\mathrm{MQ}^{(4\mathrm{d})} \equiv$ 3d mirror of $\mathrm{EQ}^{(4\mathrm{d})}$.

Conjecture:

$$\mathrm{MQ}^{(5\mathrm{d})} = \mathrm{EQ}^{(4\mathrm{d})}/U(1)^f\,, \qquad \mathrm{MQ}^{(4\mathrm{d})} = \mathrm{EQ}^{(5\mathrm{d})}/U(1)^f\,,$$


M-theory on X:
$\boldsymbol{T}_X{}^{5d}$
Type IIB on X:
$\mathcal{T}_X{}^{4d}$
T^2
S^1
3d N=4 theory
$EQ^{(5)}$
3d N=4 theory
$EQ^{(4)}$
3d Mirror Symmetry
$U(1)^f$ gauging
3d MS
3d N=4 theory
$MQ^{(5)}$
$MQ^{(5)} = EQ^{(4)}/U(1)^f$
3d N=4 theory
$MQ^{(4)}$



M-theory on X:
T_X^{5d}
Type IIB on X:
$\mathcal{T}_X^{4d}$
T^2
S^1
dim(CB, HB)
$(r, d_H = \hat{r}+f)$
3d N=4 theory
$EQ^{(5)}$
3d N=4 theory
$EQ^{(4)}$
$(\hat{r}, r+f)$
$U(1)^f$ gauging
3d MS
3d MS
3d N=4 theory
$MQ^{(5)}$
$MQ^{(5)} = EQ^{(4)}/U(1)^f$
$(d_H = \hat{r}+f, r)$
3d N=4 theory
$MQ^{(4)}$
$(r+f, \hat{r})$

Magnetic Quiver(ine)s and 5d Higgs Branch

From this conjecture we identify the MQ= as the magnetic quiver(ine) of the 5d SCFT $\mathcal{T}^{5\mathrm{d}}$, which whenever $\mathrm{MQ}^{(5\mathrm{d})}$ is a Lagrangian quiver should agree with [Bourget, Cabrera, Hanany, Grimminger, Yagi, Zhong...]

We derive this from a geometric point of view:

$$\mathrm{HB}\left[\mathcal{T}^{5\mathrm{d}}(\mathbf{X})\right] = \mathrm{CB}\left[\mathrm{MQ}^{(5\mathrm{d})}\right] = \mathrm{CB}\left[\mathrm{EQ}^{(4\mathrm{d})}/U(1)^f\right].$$

Bonus 4d result: $\mathrm{HB}(\mathscr{T}^{4\mathrm{d}}) = \mathrm{CB}(\mathrm{MQ}^{(4\mathrm{d})})$.

The M2-instantons, which quantum correct the metric on the classical Higgs branch are encoded in the monopole operators studied in [Cremonesi, Hanany, Zaffaroni] in 3d $\mathcal{N} = 4$.

5d Higgs branch from EQ for 4d SCFT

The strategy to compute the Higgs branch of $\mathcal{T}^{5\text{d}}(\mathbf{X})$:

- Consider 4d SCFT $\mathscr{T}^{4\text{d}}$ = IIB on $\mathbf{X}$.
- Compute $\text{EQ}^{(4\text{d})}$:
  4d SCFT Lagrangian SCFT, then EQ simply dimensional reduction.
  Using geometric engineering in 4d: [Shapere, Vafa][Shapere, Tachikawa]
  1. CB of $\mathscr{T}^{4\text{d}}$: Deformations $\widehat{\mathbf{X}}$
  2. CB spectrum of operators from spectrum of the singularity

$$\Delta_l = \frac{Q_l}{\sum_{i=1}^{4} q_i - 1}$$

- Gauge $U(1)^f$ to obtain $\text{MQ}^{(5)}$, whose CB is the HB of $\mathcal{T}^{5\text{d}}$

Example 1: E-strings

Rank N E_8 Seiberg theories:

$$\mathbf{X}_{E_6}: \qquad x_1^3 + x_2^3 + x_3^3 + x_4^{3N} = 0$$
$$\mathbf{X}_{E_7}: \qquad x_1^2 + x_2^4 + x_3^4 + x_4^{4N} = 0$$
$$\mathbf{X}_{E_8}: \qquad x_1^2 + x_2^3 + x_3^6 + x_4^{6N} = 0$$

IR-description: $Sp(N) + (n-1)\mathbf{F} + \mathbf{AS}$.
Resolution by N exceptional divisors.

	f	r	d_H	$\widehat{r}$	$\widehat{d}_H$
E_6	6	N	$12N-1$	$12N-7$	$N+6$
E_7	7	N	$18N-1$	$18N-8$	$N+7$
E_8	8	N	$30N-1$	$30N-9$	$N+8$

Example: Rank N E_6-theory

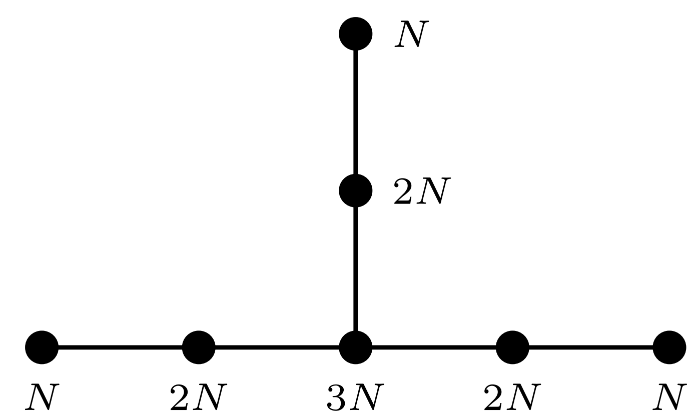
The 4d SCFT on $\mathbf{X}_{E_6}$ was shown in [Katz, Mayr, Vafa] to have a gauge theory description

$$G = \prod_{d_k} SU(d_k N)\,.$$

The spectrum e.g. for $N = 2$:

Δ	6	5	4	3	2
#	1	1	4	4	7

$SU(L)$ contributes:
$\Delta = \{L, L-1, \cdots, 2\}$.



This quiver with $SU(d_k N)$ gauge nodes is the electric quiver $\text{EQ}^{(4)}$. The same quiver with $U(d_k N)$ nodes is the magnetic quiver $\text{MQ}^{(5)}$.

Example 2: Rank 2 with $G_F = E_8$

$$F(x) = x_1^2 + x_2^5 + x_3^{10} + x_3 x_4^3 = 0\,, \qquad (q_1, q_2, q_3, q_4) = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{3}{10}\right)$$

$$\mu = 84\,, \quad r = 2\,, \quad f = 8\,, \quad d_H = 46\,, \quad \widehat{r} = 38\,.$$

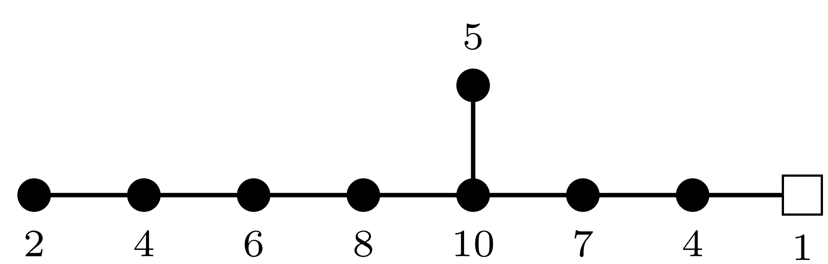
Computing the scaling dimensions of $\mathscr{T}^{\text{4d}}$:

Δ	1	2	3	4	5	6	7	8	9	10
#	8	8	7	7	5	4	3	2	1	1

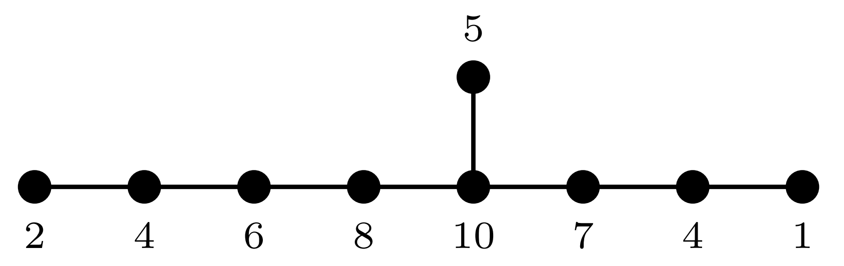
consistent with a 4d Lagrangian SCFT with

$$G = SU(10) \times SU(8) \times SU(7) \times SU(6) \times SU(5) \times SU(4)^2 \times SU(2)\,.$$

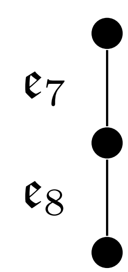
4d Quiver, which is the same as EQ$^{\text{(4d)}}$ in 3d, with $SU(L)$ nodes:



$\mathrm{MQ}^{(5)}$ given by gauging $U(1)^8$, i.e. the quiver with $U(L)$ nodes:



The Higgs branch of $\mathcal{T}^{(5d)}$ has a Hasse diagram – the partially ordered set of symplectic leaves of the HB – following from this



Implies $G_F = E_8$.

Coulomb branch and IR-description

By resolving the singularity

$$(x_1^{(2)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}; \delta_1)$$
$$(x_1^{(3)}, x_4^{(2)}, x_3^{(1)}, \delta_1^{(1)}; \delta_2)\,,$$

We find that the geometry is $\mathbb{P}^2 \cup \mathrm{Bl}_8\mathbb{F}_3$, with a ruling yieling a 5d IR description

$$SU(2)_0 - SU(2) - [5]$$

$G_F = E_8$ agrees with [Apruzzi, Lawrie, Lin, SSN, Wang].

Using 5-brane webs we can confirm the MQ
[van Beest, Bourget, Eckhard, SSN, wip].

Note this is a descendant of the rank $N = 2$ E_8-theory, which also has a description as $[1] - SU(2) - SU(2) - [5]$.

Example 3:

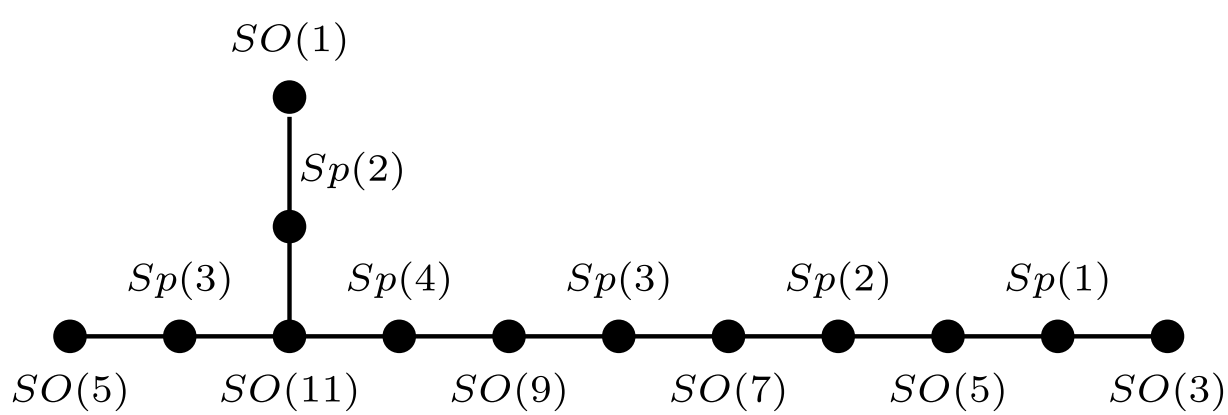
Applications to theories that so far have no MQ using brane-webs:

$$F(x) = x_1^2 + x_2^5 + x_3^5 + x_4^5 = 0 \,, \qquad (q_1, q_2, q_3, q_4) = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \,.$$

Crepant resolution yields that the IR description is one of the three 'dual' gauge theories:

$$SU(3)_{\frac{9}{2}} + 5\mathbf{F} \,, \quad Sp(2) + 3\mathbf{F} + 2\mathbf{AS} \,, \qquad G_2 + 5\mathbf{F} \,.$$

Using the same logic as above we find the $\mathrm{MQ}^{(5)}$ to be



Curiosities: Rank 0 theories

Many IHS have remanent terminal singularities:

IHS with no crepant blowups are rank 0 theories.
For higher rank there can be remnant terminal singularities – coupling of rank 0 to higher rank.

Example: Close cousin to the E_6 rank 1 theory:

$$F(x) = x_1^3 + x_2^3 + x_3^3 + x_4^5 = 0\,.$$

One blowup

$$(x_1, x_2, x_3, x_4; \delta_1)\,.$$

The resolved singularity is has a terminal singularity at $\delta_1 = x_1 = x_2 = x_3 = 0$:

$$x_1^3 + x_2^3 + x_3^3 + \delta_1^2 = 0\,.$$

This is in fact in IIB the theory of type Argyres-Douglas (AD) $[A_2, D_4]$.

This theory has

$$r = 1, \qquad f = 0, \qquad d_H = 16\,, \qquad \Gamma_m^{(3)} = \mathbb{Z}_5\,.$$

Interpretation in IIB is simply that the AD theory $[D_4, E_8]$ has along a sublocus on the HB a residual SCFT of type AD $[A_2, D_4]$.

In 5d further analysis of this model needs to determine, whether this is a new rank 1 5d SCFT. Possibly this is a discrete $\mathbb{Z}_5$-gauging of a Seiberg theory.

This effect is rather prominent even within the IHS class of canonical singularities.

Summary and Outlook

5d SCFTs provide a perfect setup, where geometric methods inform our understanding of QFTs.

To fully explore their moduli spaces, and properties, we need a variety of geometric and string theoretic tools: resolution and deformation of singularities, brane-webs/dualities.

Key open questions:

- Role of the rank 0 theories: e.g. as discrete gauging, and relation to 3-form symmetries.
- Mixed Coulomb/Higgs branches
- Generalization of deformations beyond IHS, e.g. to generalized toric models, and identify the associated geometry.