Electric-Magnetic Duality for Periods and L-functions

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Overview

Describe a perspective on number theory
(periods of automorphic forms)
inspired by physics
(boundaries in supersymmetric gauge theory).

Based on joint work with
• Yiannis Sakellaridis (Johns Hopkins U.) and
• Akshay Venkatesh (IAS)
Outline

Arithmetic and Quantum Field Theory

Periods and L-functions

Relative Langlands Duality
Automorphic forms: QM on arithmetic locally symmetric spaces

e.g., $\Gamma = SL_2(\mathbb{Z}) \circlearrowright \mathbb{H} = SL_2(\mathbb{R})/SO(2)$, study

$\Delta \circlearrowleft L^2(\Gamma \backslash \mathbb{H})$

(+ twisted variants)
Arithmetic Quantum Mechanics

General story:

$G$ reductive (e.g., $GL_n$, $Sp_n$, $E_8$,..)

$\mapsto$

study QM (e.g. spectral decomposition of $L^2$) on arithmetic locally symmetric space

\[
[G] = G(\mathbb{Z}) \backslash G(\mathbb{R}) / K
\]

(and variants)
Structure in Arithmetic Quantum Mechanics

Very special QM problem:

- Hecke operators

Huge commutative algebra of symmetries ("quantum integrable system"): Hecke operators $T_p$, $p$ prime

General $G$:
Hecke operators at $p \leftrightarrow$ reps of Langlands dual group $G_C^\vee \circlearrowleft V$
• Can vary ramification:
\[ \Gamma = SL_2(\mathbb{Z}) \cong \text{subgroups defined by congruences mod } N, \text{ study} \]
\[ \Gamma_N \backslash \mathbb{H} \]

• Can vary number field:
\[ F/\mathbb{Q} \text{ finite} \cong \]
\[ [SL_2]_F = SL_2(\mathcal{O}_F) \backslash SL_2(F \otimes \mathbb{R})/K \]
(e.g., \( F = \mathbb{Q}(\sqrt{-d}) \cong \text{arithmetic quotients of } \mathbb{H}^3 \))
Structure in Arithmetic Quantum Mechanics

- **Langlands correspondence:**
  
  Automorphic forms $\leftrightarrow$ Galois representations

  representation varieties of Galois groups
  \[
  \{ \text{Gal}(\overline{F}/F) \to G^\vee \}
  \]

  **Spectral side** “solves” quantum integrable system on $[G]_F$:

  Hecke operators at $p$
  $\leftrightarrow$
  trace functions of Frobenius at $p$

  [both labeled by reps of $G^\vee$]
Automorphic forms as gauge theory

Much richer paradigm for automorphic forms:

arithmetic 4d quantum field theory

Specifically: 4d $\mathcal{N} = 4$ super-Yang-Mills, gauge group $G_c$ in “A-type” topological twist$^1$

Arithmetic extension of Kapustin-Witten interpretation of Geometric Langlands Program

$^1$GL twist at $\Psi = 0$
Arithmetic gauge theory

- **Arithmetic Topology:**

  $F$ number field (or $\text{Spec}(\mathcal{O}_F)$)

  $\leftrightarrow$

  3-manifold $M$

  Primes in $\mathcal{O}_F$

  $\leftrightarrow$

  Knots in $M$

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$^2$Mazur, Morishita, Kapranov, Reznikov. M. Kim: arithmetic Chern-Simons
Arithmetic gauge theory

Vague idea: QM on \([G]_F\)

\(\leftrightarrow\)

Hilbert space of QFT on \(M \times \mathbb{R}\) –
QM on space of connections on \(M\)

Better ansatz – **Topological twist**:
Pass from QM \(L^2([G]_F)\) \(\Longrightarrow\) Topological QM \(H^*([G]_F)\), \(^3\)

\(\leftrightarrow\)

Hilbert space of TQFT on \(M \times \mathbb{R}\) –
topology of space of connections on \(M\), \(^4\)

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\(^3\)i.e., study *cohomological* automorphic forms, e.g., classical modular forms

\(^4\)Or, symplectic topology of space of Higgs bundles
Arithmetic gauge theory: Hecke and Loop Operators

- Hecke operators at prime $p$

$\leftrightarrow$

1d defects: 't Hooft loop operators along a knot $K$:

insert magnetic monopole along $K \times \frac{1}{2} \subset M \times [0, 1]$

singularity measured by rep of dual group $G^\vee_C$. 
Arithmetic gauge theory: Ramification and Surface Operators

- **Ramification mod** $N$

  $\leftrightarrow$

  2d defects$^5$: solenoids along a link $L \times \mathbb{R} \subset M \times \mathbb{R}$

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$^5$Gukov-Witten
Arithmetic gauge theory

- Langlands correspondence $\leftrightarrow$ Electric-Magnetic Duality

for twisted $\mathcal{N} = 4$ SUSY Yang-Mills theory

4d A-model $\mathcal{A}_G \sim$ topology of spaces of connections
$\leftrightarrow$
4d B-model $\mathcal{B}_{G^\vee} \sim$ algebraic geometry of spaces of flat connections / monodromy representations

$\{\pi_1(M) \to G^\vee\}$

\[6\text{Montonen-Olive S-duality}\]
## Langlands / Electric-Magnetic Duality

<table>
<thead>
<tr>
<th>automorphic</th>
<th>spectral</th>
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</thead>
<tbody>
<tr>
<td>magnetic</td>
<td>electric</td>
</tr>
<tr>
<td>$\mathcal{A}_G$</td>
<td>$\mathcal{B}_G^\vee$</td>
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<tr>
<td>topology:</td>
<td>algebraic geometry:</td>
</tr>
<tr>
<td>- of spaces of connections</td>
<td>- of flat connections</td>
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<tr>
<td>- of arith. loc. sym. spaces</td>
<td>- of Galois representations</td>
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| 1d defects (loops):  | Wilson / trace     |
| Hecke / ’t Hooft    |                   |

| 2d defects (solenoids): | singularity of flat connections |
| congruence subgroups   |                                 |

| BZSV: 3d defects (bdry conditions) | L-functions |
| periods                        |              |
Outline

Arithmetic and Quantum Field Theory

Periods and L-functions

Relative Langlands Duality
Our work (BZ-Sakellaridis-Venkatesh):

Apply paradigm to theory of integral representations of L-functions as periods:

understand using E-M duality for 3d defects (boundary conditions) in gauge theory\(^7\).

\(^7\)Gaiotto-Witten
Integral representations

**L-functions:** fundamental invariants of Galois representations

\[ L(\rho, s) := \prod_p \frac{1}{\det(1 - p^{-s}\rho(F_p))} \]

e.g. \( \rho \) trivial \( \leadsto \) Riemann \( \zeta \)-function.

**L-values** (e.g. \( L(s, 0) \)) capture deep arithmetic information.

**Integral representation:** most important tool to access \( L \)-functions (analytic continuation, functional equation, ...):

e.g., Riemann:

\[
\pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_0^{\infty} y^{s/2} \sum_{n=0}^{\infty} e^{-n^2 \pi y} dy
\]
Hecke period: \( \varphi \) cusp form on \( \mathbb{H} \)

\[
P_T(\varphi) := \int_0^\infty \varphi(iy)y^s \frac{dy}{y}
\]

\( T \subset SL_2\mathbb{R} \) torus\(^8 \) \( \mapsto \) T-period – integral over \([T] \subset [G]\)

\(^8 N \subset SL_2\mathbb{R} \mapsto \) integral over \([N] \subset [G]\), constant term/Eisenstein period
Hecke Period as L-function

\[ P_T(\varphi) = \frac{\Gamma(s)}{(2\pi)^s} L(\varphi, s) \]

produces L-function of modular form:

If \( \varphi \leftrightarrow \rho \) 2d Galois representation

\[ L(\varphi, s) = L(\rho, s) := \prod \frac{1}{\det(1 - p^{-s} \rho(F_p))} \]
More general periods

$H \subset G$ subgroup $\Rightarrow$
define $\mathcal{P}_H$ as integral over $[H]_F \subset [G]_F$

More generally, $\mathcal{P}_X$ function of $G$-space $X$ (e.g., $G/H \leadsto \mathcal{P}_H$)

Iwasawa, Tate express abelian $L$-functions$^9$ as periods for
$G = GL_1 \circlearrowleft X = \mathbb{A}^1$:

Riemann’s integral of $\Theta$-series
$\leadsto$
Integrate over $[GL_1]_F$ against push-forward from (adèlic) $\mathbb{A}^1$

$^9$Riemann and Dedekind $\zeta$-, Dirichlet and Hecke $L$-functions
Spherical varieties

Finiteness condition: $X$ needs to be spherical $G$-variety

Spherical variety: nonabelian version of toric variety

$G \circ X$ is a spherical variety if Borel $B \subset G$ has finitely many orbits.

- “Tate”: Toric varieties
- “Eisenstein”: Flag varieties
- Symmetric spaces
- “Group” $G = H \times H \circ X = H$
- “Whittaker” $G \circ (G/N, \psi)$
- “Branching laws”\(^{10}\) $GL_{n+1} \times GL_n \circ GL_{n+1}$,
  $SO_{2n+1} \times SO_{2n} \circ SO_{2n+1}$, \cdots

\(^{10}\)Gan-Gross-Prasad
Periods vs. L-functions

Problem: Basic mismatch of data!

$L$-functions of $\rho: Gal \rightarrow G^\vee$ naturally labeled by representations $V$ of $G^\vee$

(product of inverted char. polys of Frobenius $\rho_V(F_p)$)

Completely unrelated to data of spherical varieties $G \bowtie X$

$\mathcal{P}_{X=??}(\varphi) \leftrightarrow L(s, \rho, V)$?

Huge collection of examples of integral formulas, lack coherent theory
Outline

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Periods and L-functions

Relative Langlands Duality
Relative Langlands Program

Sakellaridis-Venkatesh: tie global theory of periods with local theory: harmonic analysis of $G \otimes L^2(X)$ over local fields

Extract from $X$ algebraic data (subgroup\(^{11}\) $G^\vee_X \subset G^\vee$ and representation\(^{12}\) $G^\vee_X \otimes V_X$) controlling $X$-relative Langlands program:

- when $\mathcal{P}_X(\varphi) \neq 0$,
- which $G$-reps appear on $X$,
- $X$-Plancherel measure, etc.

\(^{11}\) cf. also Knop, Gaitsgory-Nadler
\(^{12}\) cf. Sakellaridis
Boundary conditions

BZSV: Periods, L-functions \( \subset \) richer structure of boundary theories for \( \mathcal{A}_G, \mathcal{B}_G \): 

QFT on \( M \times [0, 1] \) with local boundary condition produces state on any \( M \)

\( \mathcal{A}_G \) boundary theory uniformly encodes relative Langlands:

- global (period \( P_X \), \( \Theta \)-series, Relative Trace Formula)
- local (rep \( L^2(X) \), Plancherel measure)
E-M duality of boundary conditions

Duality $\mathcal{A}_G \simeq \mathcal{B}_G^\vee$ identifies boundary theories on two sides.

Gaiotto-Witten: SUSY boundary in $\mathcal{N} = 4$ SYM for $G$

$\leftrightarrow$

holomorphic Hamiltonian $G$-spaces$^{13}$

($\sim$ couple to SUSY 3d $\sigma$-models with $G$-symmetry)

$\mapsto$ identify Hamiltonian actions of Langlands dual groups!

Discover symmetry between automorphic and spectral periods ($L$-functions)

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$^{13}$Roughly. Also have e.g. Nahm pole $\leftrightarrow$ Arthur $SL_2$
Boundary conditions from periods

Subgroups $H$ (spherical)
$\leadsto X = G/H$
$\subset$

$G$-varieties $X$ (spherical)
$\leadsto M = T^*X$

QM on $X$ really microlocal.
e.g. Tate: fun. eqn. for $\zeta(s) \leftrightarrow$ Fourier transform on $\mathbb{A}^1$.

$\subset$

Hamiltonian $G$-varieties $M$ (hyperspherical)
Many more examples: Whittaker periods$^{14}$, $\Theta$-correspondence...

$\leadsto \sigma$-model into $M$
$\subset$

Boundary theories for $\mathcal{A}_G$

$^{14} \leftrightarrow$ principal Nahm pole
Boundary conditions from L-functions

\[ G^\vee \text{ representations } V \]

\[ \frac{1}{\text{det}(1 - t \rho(F))} = \text{Tr}_{gr}(F, \text{Sym}^n V = \mathcal{O}(V^\vee)) \]

\[ \leadsto X^\vee = V^\vee \]

\[ \subset \]

\[ G^\vee \text{-varieties } X^\vee \]

\[ \leadsto M^\vee = T^* X^\vee \]

\[ \subset \]

Hamiltonian \( G^\vee \)-varieties \( M^\vee \)

\[ \leadsto \sigma\text{-model into } M^\vee \]

\[ \subset \]

Boundary theories for \( B_{G^\vee} \)
Duality for boundary conditions

How to see duality?

Build moment map

$$M^\vee / G^\vee \xrightarrow{\mu} g^{\vee^*} / G^\vee$$

directly out of $G \circlearrowleft M$

as “moduli space of vacua\(^{15}\) for boundary theory”

[Geometrization of Plancherel measure for $L^2(X)$]

• create algebra (bulk) and module (boundary)
  out of line operators in extended TFT,

• describe spectrally

\(^{15}\)Coulomb branch, cf. Braverman-Finkelberg-Nakajima
Reconstructing the dual

OPE of line defects on boundary ($\sim \text{Shv}(LX/LG_+)$)
forms tensor category,
linear over 't Hooft line defects ($\sim \text{Shv}(LG_+\backslash LG/LG_+)$)

$G^\vee$-equivariant sheaves $\text{QC}(M^\vee/G^\vee)$
form tensor category,
linear over Wilson line defects\(^{16}\) $\text{QC}(g^\vee*/G^\vee)$

\(^{16}\text{Mirkovic-Vilonen, \ldots, Bezrukavnikov-Finkelberg}\)
Comparison with Sakellaridis-Venkatesh data\textsuperscript{17} $X \mapsto (G_X^\vee \circ V_X)$

$\Rightarrow$ precise conjectures for duality:

$$M^\vee = G^\vee \times_{G_X^\vee} V_X$$

$\Rightarrow$ local and global predictions identifying

- $M$-relative automorphic theory (periods, harmonic analysis,..)

and

- $M^\vee$-relative spectral theory (L-functions, alg.geom. of $M^\vee$)

\textsuperscript{17}More general form includes Arthur $SL_2$ / Nahm pole
Thank You!

Image by Rok Gregoric: Scylla (Number Theory) and Charybdis (Physics)