Electric-Magnetic Duality for Periods and L-functions

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Overview

Describe a perspective on number theory
(periods of automorphic forms)
inspired by physics
(boundaries in supersymmetric gauge theory).

Based on joint work with
- Yiannis Sakellaridis (Johns Hopkins U.) and
- Akshay Venkatesh (IAS)

Outline

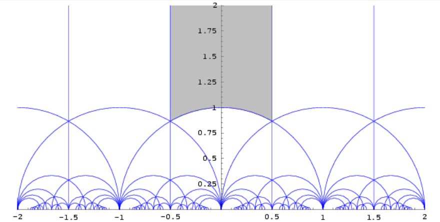
Arithmetic Quantum Mechanics

$\boxed{\text{Automorphic forms: QM on arithmetic locally symmetric spaces}}$

e.g., $\Gamma = SL_2(\mathbb{Z}) \circlearrowright \mathbb{H} = SL_2(\mathbb{R})/SO(2)$, study

$$\Delta \circlearrowright L^2(\Gamma\backslash\mathbb{H})$$

(+ twisted variants)



Arithmetic Quantum Mechanics

General story:

G reductive (e.g., GL_n, Sp_n, E_8,..)

$\rightsquigarrow$

study QM (e.g. spectral decomposition of L^2) on arithmetic locally symmetric space

$$[G] = G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

(and variants)

Structure in Arithmetic Quantum Mechanics

Very special QM problem:

- Hecke operators

Huge commutative algebra of symmetries
("quantum integrable system"):
Hecke operators T_p, p prime

General G:
Hecke operators at $p \leftrightarrow$ reps of Langlands dual group $G^\vee_{\mathbb{C}} \circlearrowright V$

- Can vary ramification:

$\Gamma = SL_2(\mathbb{Z}) \rightsquigarrow$ subgroups defined by congruences mod N, study

$$\Gamma_N \backslash \mathbb{H}$$

- Can vary number field:

$F/\mathbb{Q}$ finite $\rightsquigarrow$

$$[SL_2]_F = SL_2(\mathcal{O}_F) \backslash SL_2(F \otimes \mathbb{R})/K$$

(e.g., $F = \mathbb{Q}(\sqrt{-d}) \rightsquigarrow$ arithmetic quotients of $\mathbb{H}^3$)

Structure in Arithmetic Quantum Mechanics

- Langlands correspondence:

$\boxed{\text{Automorphic forms} \leftrightarrow \text{Galois representations}}$

representation varieties of Galois groups
$\{Gal(\overline{F}/F) \to G^\vee\}$

Spectral side "solves" quantum integrable system on $[G]_F$:

Hecke operators at p
$\leftrightarrow$
trace functions of Frobenius at p

[both labeled by reps of $G^\vee$]

Automorphic forms as gauge theory

Much richer paradigm for automorphic forms:

$\boxed{\text{arithmetic 4d quantum field theory}}$

Specifically: 4d $\mathcal{N} = 4$ super-Yang-Mills, gauge group G_c in "A-type" topological twist[1]

Arithmetic extension of Kapustin-Witten interpretation of Geometric Langlands Program

[1] GL twist at $\Psi = 0$

Arithmetic gauge theory

- Arithmetic Topology: [2]

F number field (or $Spec(\mathcal{O}_F)$)

$\leftrightarrow$

3-manifold M

Primes in $\mathcal{O}_F$

$\leftrightarrow$

Knots in M

[2] Mazur, Morishita, Kapranov, Reznikov. M. Kim: arithmetic Chern-Simons

Arithmetic gauge theory

Vague idea: QM on $[G]_F$

$\leftrightarrow$

Hilbert space of QFT on $M \times \mathbb{R}$ –
QM on space of connections on M

Better ansatz – Topological twist:
Pass from QM $L^2([G]_F) \Longrightarrow$ Topological QM $H^*([G]_F)$, [3]

$\leftrightarrow$

Hilbert space of TQFT on $M \times \mathbb{R}$ –
topology of space of connections on M [4]

[3] i.e., study *cohomological* automorphic forms, e.g., classical modular forms
[4] Or, symplectic topology of space of Higgs bundles

Arithmetic gauge theory: Hecke and Loop Operators

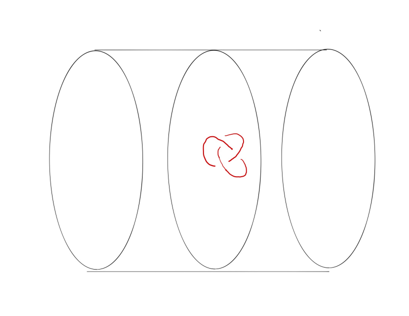
- Hecke operators at prime p

$\leftrightarrow$

1d defects: 't Hooft loop operators along a knot K:

insert magnetic monopole along $K \times \frac{1}{2} \subset M \times [0, 1]$

singularity measured by rep of dual group $G_{\mathbb{C}}^{\vee}$.

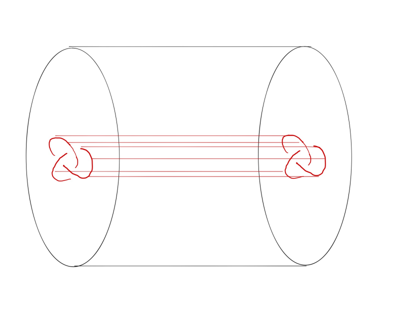


Arithmetic gauge theory: Ramification and Surface Operators

- Ramification mod N

$\leftrightarrow$

2d defects[5]: solenoids along a link $L \times \mathbb{R} \subset M \times \mathbb{R}$



[5] Gukov-Witten

Arithmetic gauge theory

- Langlands correspondence $\leftrightarrow$ Electric-Magnetic Duality [6]
for twisted $\mathcal{N} = 4$ SUSY Yang-Mills theory

4d A-model $\mathcal{A}_G \sim$ topology of spaces of connections
$\leftrightarrow$
4d B-model $\mathcal{B}_{G^\vee} \sim$ algebraic geometry of spaces of flat connections / monodromy representations

$$\{\pi_1(M) \to G^\vee\}$$

[6] Montonen-Olive S-duality

Langlands / Electric-Magnetic Duality

automorphic	spectral
magnetic	electric
$\mathcal{A}_G$	$\mathcal{B}_{G^\vee}$
topology: - of spaces of connections - of arith. loc. sym. spaces	algebraic geometry: - of flat connections - of Galois representations
1d defects (loops): Hecke / 't Hooft	Wilson / trace
2d defects (solenoids): congruence subgroups	singularities of flat connections
BZSV: 3d defects (bdry conditions) periods	L-functions

Outline

BZSV

- Our work (BZ-Sakellaridis-Venkatesh):

Apply paradigm to theory of integral representations of L-functions as periods :

understand using E-M duality for 3d defects (boundary conditions) in gauge theory[7].

[7] Gaiotto-Witten

Integral representations

L-functions: fundamental invariants of Galois representations

$$L(\rho, s) := \prod_p \frac{1}{det(1 - p^{-s}\rho(F_p))}$$

e.g. ρ trivial $\rightsquigarrow$ Riemann ζ-function.

L-values (e.g. $L(s,0)$) capture deep arithmetic information.

Integral representation: most important tool to access L-functions (analytic continuation, functional equation,...):
e.g., Riemann:

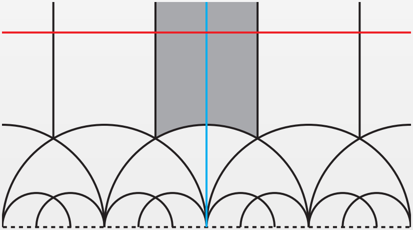
$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty y^{s/2} \sum_{n=0}^\infty e^{-n^2\pi y} dy$$

Hecke Period

Hecke period: φ cusp form on $\mathbb{H}$

$$\mathcal{P}_T(\varphi) := \int_0^\infty \varphi(iy) y^s \frac{dy}{y}$$

$T \subset SL_2\mathbb{R}$ torus[8] $\rightsquigarrow$ T-period – integral over $[T] \subset [G]$



[8] $N \subset SL_2\mathbb{R} \rightsquigarrow$ integral over $[N] \subset [G]$, constant term/Eisenstein period

Hecke Period as L-function

$$\mathcal{P}_T(\varphi) = \frac{\Gamma(s)}{(2\pi)^s} L(\varphi, s)$$

produces L-function of modular form:

If $\varphi \leftrightarrow \rho$ 2d Galois representation
$\Rightarrow$

$$L(\varphi, s) = L(\rho, s) := \prod \frac{1}{det(1 - p^{-s}\rho(F_p))}$$

More general periods

$H \subset G$ subgroup $\Rightarrow$
define $\mathcal{P}_H$ as integral over $[H]_F \subset [G]_F$

More generally, $\mathcal{P}_X$ function of G-space X (e.g., $G/H \rightsquigarrow \mathcal{P}_H$)

Iwasawa, Tate express abelian L-functions[9] as periods for
$G = GL_1 \circlearrowright X = \mathbb{A}^1$:

Riemann's integral of Θ-series
$\rightsquigarrow$
Integrate over $[GL_1]_F$ against push-forward from (adèlic) $\mathbb{A}^1$

[9] Riemann and Dedekind ζ-, Dirichlet and Hecke L-functions

Spherical varieties

Finiteness condition: X needs to be spherical G-variety

$\boxed{\text{Spherical variety: nonabelian version of toric variety}}$

$G \circlearrowright X$ is a spherical variety if Borel $B \subset G$ has finitely many orbits.

- "Tate": Toric varieties
- "Eisenstein": Flag varieties
- Symmetric spaces
- "Group" $G = H \times H \circlearrowright X = H$
- "Whittaker" $G \circlearrowright (G/N, \psi)$
- "Branching laws"[10] $GL_{n+1} \times GL_n \circlearrowright GL_{n+1}$, $SO_{2n+1} \times SO_{2n} \circlearrowright SO_{2n+1}, \cdots$

[10] Gan-Gross-Prasad

Periods vs. L-functions

Problem: Basic mismatch of data!

L-functions of $\rho : Gal \to G^\vee$ naturally labeled by representations V of $G^\vee$
(product of inverted char. polys of Frobenius $\rho_V(F_p)$)

Completely unrelated to data of spherical varieties $G \circlearrowright X$

$$\mathcal{P}_{X=??}(\varphi) \longleftrightarrow L(s, \rho, V)?$$

Huge collection of examples of integral formulas,
lack coherent theory

Outline

Relative Langlands Program

Sakellaridis-Venkatesh: tie global theory of periods with local theory: harmonic analysis of $G \circlearrowright L^2(X)$ over local fields

Extract from X algebraic data (subgroup[11] $G_X^\vee \subset G^\vee$ and representation[12] $G_X^\vee \circlearrowright V_X$) controlling X-relative Langlands program:

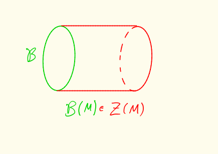
- when $\mathcal{P}_X(\varphi) \neq 0$,
- which G-reps appear on X,
- X-Plancherel measure, etc.

[11] cf. also Knop, Gaitsgory-Nadler
[12] cf. Sakellaridis

Boundary conditions

BZSV: Periods, L-functions $\subset$ richer structure of boundary theories for $\mathcal{A}_G$, $\mathcal{B}_{G^\vee}$:

QFT on $M \times [0,1]$ with local boundary condition produces state on any M



$\mathcal{A}_G$ boundary theory uniformly encodes relative Langlands:
- global (period $\mathcal{P}_X$, Θ-series, Relative Trace Formula)
- local (rep $L^2(X)$, Plancherel measure)

E-M duality of boundary conditions

Duality $\mathcal{A}_G \simeq \mathcal{B}_{G^\vee}$ identifies boundary theories on two sides.

Gaiotto-Witten: SUSY boundary in $\mathcal{N} = 4$ SYM for G

$\longleftrightarrow$

holomorphic Hamiltonian G-spaces[13]

($\sim$ couple to SUSY 3d σ-models with G-symmetry)

$\rightsquigarrow$ identify Hamiltonian actions of Langlands dual groups!

Discover symmetry between automorphic and spectral periods (L-functions)

[13]Roughly.. Also have e.g. Nahm pole $\leftrightarrow$ Arthur SL_2

Boundary conditions from periods

Subgroups H (spherical)
$\rightsquigarrow X = G/H$
$\subset$
G-varieties X (spherical)
$\rightsquigarrow M = T^*X$

QM on X really microlocal.
e.g. Tate: fun. eqn. for $\zeta(s) \Leftarrow$ Fourier transform on $\mathbb{A}^1$.

$\subset$
Hamiltonian G-varieties M (hyperspherical)
Many more examples: Whittaker periods[14], Θ-correspondence...

$\rightsquigarrow \sigma$-model into M
$\subset$
Boundary theories for $\mathcal{A}_G$

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[14] $\leftrightarrow$ principal Nahm pole

Boundary conditions from L-functions

$G^\vee$ representations V

$$\frac{1}{det(1 - t\rho(F))} = Tr_{gr}(F, Sym^\bullet V = \mathcal{O}(V^\vee))$$

$\rightsquigarrow X^\vee = V^\vee$

$\subset$

$G^\vee$-varieties $X^\vee$

$\rightsquigarrow M^\vee = T^*X^\vee$

$\subset$

Hamiltonian $G^\vee$-varieties $M^\vee$

$\rightsquigarrow \sigma$-model into $M^\vee$

$\subset$

Boundary theories for $\mathcal{B}_{G^\vee}$

Duality for boundary conditions

How to see duality?

Build moment map

$$M^\vee/G^\vee \xrightarrow{\mu} \mathfrak{g}^{\vee *}/G^\vee$$

directly out of $G \circlearrowright M$

as "moduli space of vacua[15] for boundary theory"

[Geometrization of Plancherel measure for $L^2(X)$]

• create algebra (bulk) and module (boundary)
out of line operators in extended TFT,

• describe spectrally

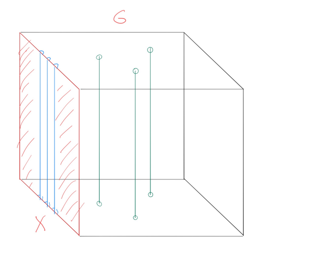
[15] Coulomb branch, cf. Braverman-Finkelberg-Nakajima

Reconstructing the dual

OPE of line defects on boundary ($\sim Shv(LX/LG_+)$)
forms tensor category,
linear over 't Hooft line defects ($\sim Shv(LG_+ \backslash LG/LG_+)$)

$$\longleftrightarrow$$

$G^\vee$-equivariant sheaves $QC(M^\vee/G^\vee)$
form tensor category,
linear over Wilson line defects[16] $QC(\mathfrak{g}^{\vee *}/G^\vee)$



[16] Mirkovic-Vilonen, ..., Bezrukavnikov-Finkelberg

Relative Langlands Duality

Comparison with Sakellaridis-Venkatesh data[17] $X \mapsto (G_X^\vee \circlearrowright V_X)$

$\Rightarrow$ precise conjectures for duality:

$$M^\vee = G^\vee \times^{G_X^\vee} V_X$$

$\Rightarrow$ local and global predictions identifying

- M-relative automorphic theory (periods, harmonic analysis,..)

and

- $M^\vee$-relative spectral theory (L-functions, alg.geom. of $M^\vee$)

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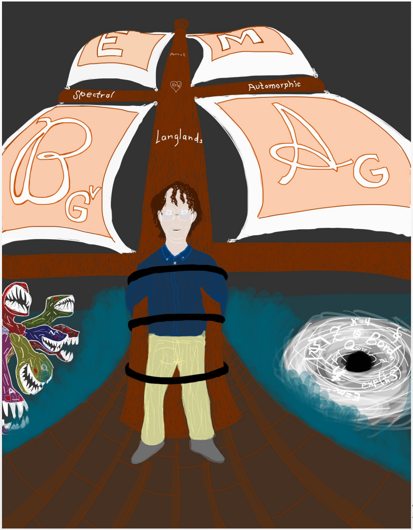
[17] More general form includes Arthur SL_2 / Nahm pole

Thank You!



[18]

[18] Image by Rok Gregoric: Scylla (Number Theory) and Charybdis (Physics)