Calabi-Yau modularity and Feynman Graphs

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Allredt Klein
Bethe & Hausdorff centers Uni Boun
International Group de Travail on differential equation (in Paris).

with Kilian Böhmisch, Fabian Fischbach
I. Feynman graphs and Calabi-Yau motives

There is growing evidence that Feynman integrals are quite generally related to (relative) period integrals, i.e., related to the moduli dependent pairing between parts of the (relative) cohomology
$H^*(W,F)$ and the (relative) homology $H_* (W,L)$ of (degenerate) algebraic varieties $\Rightarrow$ mixed variation of (mixed) Hodge structures or more generally motives

Growing evidence that Calabi-Yau motives play an important role; Preliminary dictionary

Complex moduli of $W \Leftrightarrow$ physical parameters of EG external momenta and masses
dimension of $W \iff$ loop order of Feynman graph

Monodromy groups $\iff$ analytic properties

Mumford degenerations $\iff$ high momentum regions

Periods over different $\Omega \in H^k(W)$

different master integrals for a class of FG

Gauss-Manin system $\iff$ differential relations between master integrals

$\Lambda$-classes $\iff$ (multiple) zeta values

Factorisation of Hasse-Weil z-function $\iff$ special arithmetical values of Feynman integrals
integrality of instanton expansion

\[ \leftrightarrow \]

modularity of Feynman integrals

dimensional regularization

\[ \varepsilon = \frac{1}{2} (D - D_0) \]

elliptic Fekl modularity

Aim of this talk is to show some of these exiting relations for
a simple infinite family of higher order diagrams, the Banana Feynman integrals.

\[ \begin{array}{c}
\text{P} \\
M_1 \\
M_2 \\
\vdots \\
M_k \\
M_{k+1} \\
\text{P}
\end{array} \]

\( k \)-loop diagram with 2 vertices and \( k+1 \) propagators with different masses \( M_j \) \( j = 1, \ldots, k+1 \) and external momentum \( p \).
\[ \sum P_i = 0 \]

\[ m_1 m_3 \ll m_2 m_4 \]

\[ \Rightarrow 1 \text{-loop resonance} \]

\[ I_G = \int \frac{d^4K}{[K^2-m_1][K+P_1]^2-m_2][K+P_1+P_2]^2-m_3][K-P_4]^2-m_4] \]

Needs dimensional regularization

\[ D_0 = 4 \quad \Rightarrow \quad D = D_0 - 2\varepsilon \]

General structure

\[ \frac{1}{\lambda} \int d^D K : \frac{1}{(\varepsilon+\delta)^n} \]
\[ I_0 = \sqrt{\prod_{i=1}^{n} \left( \frac{i}{\pi} \right)^{3/2}} e^{- \frac{1}{4} \sum_{e} \left[ \omega_e^2 - m_e^2 \right] X_e} \]

**Regulated integral**

**Propagators**

\( X_e \in \mathbb{N} \)

**Edge multiplicity**

\( \leftrightarrow \) different master integrals

**Feynman integral representations**

\[ I_G(P,m) \sim \int \mathcal{M}_{n-1} \frac{\omega - \frac{D}{2}}{G \omega(P,m)} \prod_{i=1}^{n} X_i^\nu_i - 1 \]  

(1)

\[ \omega = \sum_{i=1}^{n} \nu_i - \frac{D}{2} \]

\# vert = \( n - \ell + K - 1 \)
Integration domain:
\[ Q_{n-1} = \left\{ [x_1, \ldots, x_n] \in \mathbb{R}^{n-1} \mid x_i \in \mathbb{R}_{\geq 0}, \forall i \right\} \]

Measure: top form on \( \mathbb{R}^{n-1} \)
\[ m_{n-1} = \frac{1}{2^{n-1}} \sum_{k=1}^{n} (-1)^{k+1} x_k \, dx_k \wedge \cdots \wedge dx_k \wedge \cdots \wedge dx_n \]

Observation from Gelfand, Kapranov and Zelevinsky (1988) that "practically all integrals that arise in perturbative QFT" are related to A-hypergeometric functions (\( GKZ - \) systems).
of rational functions in toric varieties

II The Family of Banana graphs

For the $l$-loop Banana Feynman integral, it simplifies to

$$I_e(p,m) \propto \int \frac{U_e \frac{l+1 - l+1}{2} D}{G_e \frac{l+1 - \frac{1}{2} D}{2} \mu e}$$

with

$$U_e = \left( \prod_{i=1}^{l+1} x_i \right) \left( \prod_{i=1}^{2} i \right) \left( \prod_{i=1}^{l+1} x_i \right)$$

$$G_e = \left( \prod_{i=1}^{l+1} x_i \right) \left( \sum_{i=1}^{l+1} s_i x_i \right) \left( \sum_{i=1}^{l+1} \frac{1}{x_i} \right)$$

$$t = \frac{p^2}{\mu^2} \quad s_i = \frac{m_i}{\mu}$$
Laurent polynomial of a reflexive Polyhedron $\Delta \setminus \bar{\Delta}$.
can construct mirror pairs of CY manifolds

$$W_{e-1} = \sum P_{\Delta e} = 0 \implies \sum P_{\Delta e} = W_{e-1}$$

Mirror Symmetry

1-loop

Graph

$\Delta e$

2-loop

Graph

$W_{e-1}$ 2 pt

3-loop

Elliptic curve with 1 modulus 3 residue values

Graph
\[ u = e^{-2.5i} \]

\[ \Delta_e \]

\[ W_2 \text{ K3 surface with } 3 \text{ complex structure def} \]

\[ l=2 \text{ case } S. \text{ Bloh, M. Kerr and P. Vanhove (2016)} \]

massive case Griffith reduction method

III Inhomogeneous Gauss-Manin System

for the Banana Graphs
We solved the l=3,4 loop massive case using the GKZ system. Generally one proceeds with the following steps.

**Step 1:** Determine the Gauss-Manin connection and solve for the periods

\[ \Pi_k = \int_{\Gamma_k} \Omega(\ldots) \]

Where \( \Gamma_k \) is some basis of \( H^{l-1}_{\text{or}}(W_{k-1}, \mathbb{C}) \) and \( \Omega \in H^{l-1,0}(W_{k-1}) \). [1352] later started with the GKZ-system and...
derived the Picard–Fuchs D-module \( \Xi \mathcal{D} \), i.e., a complete set of linear differential operators characterizing \( \Pi_k \) by the homogeneous diff eqs

\[ \mathcal{D}_s \Pi_k = 0 \]

so \( \cdots \text{rank}(\mathcal{D}_s) \)

---

Step 2: Apply the \( \mathcal{D}_s \) to the chain integral (i) [Note \( \mathcal{D}_s \) is not closed] to \( a \cdot \mathcal{I}_k \{ t \in \mathcal{G} \} \) to get the inhomogeneity. The corresponding integrals can be performed at least numerically.
Actual Feynman graph fill fill a system of inhomogeneous diff eqn.

(2) \[ D_s \mathcal{I}_k(t, \xi) = \sum_i P_i(2) \cdot q_i(\log(2)) \quad \text{so 1...End}(8) \]

In the large radius coordinates \( Z_t \) of the mean point, with \( P_i, q_i \) polynomials.

**Step 3:** Find the actual basis of solutions that corresponds to the Feynman integral in terms of local Frobenius bases \( \omega_k \).
\[ I_e (k, \mathfrak{g}) = \sum_k \left( \prod_{\text{loc}} \Theta^\text{loc} (\mathfrak{g}^\text{loc}) \right) \]

For this we found the \(A\)-class of \(I_e (k, \mathfrak{g})\). Was latter confirmed in [GJ] 2011.0591 math ArXiv.

This might become generally a very usefull tool!

IV A better Calabi-Yau motive

A problem with the naive geometric description \(W_{-1} = \mathfrak{g} \mathfrak{p}_{-1} = 0 \subset \mathfrak{p}_{-1}\)?
is the great redundancy of the geometric moduli and (co)homology groups

physical parameters: \( p^2, m_1 \# = \text{hl} \)

complex moduli: \( \dim(W_{\text{c}}, TW_{\text{c}}) = h^{3,1} = \ell^2 \)

The better motive for the Bannau graphs is the primitive vertical quantum cohomology of a infinite series of Calabi-Yau spaces
In particular in the large-volume region the complexified Kähler parameter of the \( R^i \) 's can be identified with the logarithm of the physical parameter.
\begin{align*}
\log \left( \frac{Z_K}{\Delta \Gamma} \right) &= \log \left( \frac{m_K^2}{p^2} \right) = T_K = \frac{1}{4\pi i} \int \left[ i(\omega - b) + \mathcal{O}(e^{-T}) \right]_{\mathbf{p}^2}^\uparrow \mathbf{b} \cdot \mathbf{F}^\uparrow \\
Z_K &= 0 \iff p^2 \gg m_K^2
\end{align*}

Mean point

This yields the solution \( \ell_K \) without redundancy in terms of the GKZ system of the complete intersection CY We-1 analysed in [EJ Haseo, Kleen, Theisen, You 840 6055]

A Hypergeometric system with \( \ell \)-vector
\[ L = \left[ \begin{array}{c} \ell_1 \ell_2 \cdots \ell_n \end{array} \right] \quad \ell_j \in \mathbb{Z}^2 \]

\[ e_1 = (-1, -1, 1, 1, 0, 0, \cdots, 0, 0) \]

\[ e_0 = (-1, -1, 0, 0, 0, 0, 1, 1) \]

\( \Rightarrow \) **All solutions from Frobenius deconotition method** form:

\[ \omega_0 (z_1, g) = \sum_{u_1, \ldots, u_n \geq 0} C(u + g) \sum_{u_1, \ldots, u_n \geq 0} \]

\[ \text{with} \]

\[ C(u + g) = \prod_{i=1}^{2} \Gamma \left( -\frac{l_i + 1}{2} \right) \left( u_{k_i + g_{k_i}} + 1 \right) \]

\[ \frac{\alpha l + 2}{2} \quad \ell_i \begin{pmatrix} \frac{\alpha l + 2}{2} \end{pmatrix} \]
\[ \sum_{i=1}^{\infty} \xi_k^{n_k} (u_k + 8n_i + 1) \]

Structure of solutions:

1. Indom Solution

\[ \Theta_0(z) = w_0(z) \xi_1^{s_1} \xi_2^{s_2} \cdots \xi_s^{s_s} \]

\[ s = \dim (H_{\text{prim}}) \]

k-th order logarithmic solution \( k \in \{1, \ldots, L-1\} \)

\[ \Theta_k^{(s)} = \sum_{\xi_1, \ldots, \xi_s} \frac{\partial^k}{\partial \xi_1^{s_1} \cdots \partial \xi_s^{s_s}} \left[ w_0(z) \xi_1^{s_1} \xi_2^{s_2} \cdots \xi_s^{s_s} \right]_{\xi=0} \]

\[ \alpha_{i_1, \ldots, i_k} \]

are easily determined by the classical intersection numbers of the order \( k \) Chow ring elements.
generating the primitive homology of $W = c F_e$. In addition one has an inhomogeneous solution of order $l$ in $\log(2)$... that we call $\Phi^{(l)}_e$.

The $F^i$-class of the Feynman integral

To fix the actual basis of the above solution we need some geometric understanding of the Feynman integral. The relevant real part of the Feynman integral is
The imaginary part of the Feynman integral corresponds precisely to the linear combination with

corresponds to the vanishing cycle

determined by the $\hat{F}$-Class

\( I_e(T(p,m)) = \int e^{\omega \hat{T}} \frac{R(1-si)}{R(1+ai)} \cos(\pi q) + \Theta e^i \)

\[ \text{Re}(T(p,m)) = \sum_{W_{k-1}} e^{\omega T} \hat{F}(TW_{k-1}) + \Theta e^i \]
From this we can find the precise linear combination

\[ I_{e}(p,m) = \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \lambda_{k,s} \Theta_{s}^{(k)} \]

For example, one finds

<table>
<thead>
<tr>
<th>l</th>
<th>( \lambda_{0,1}^{(l)} )</th>
<th>( \lambda_{1,1}^{(l)} )</th>
<th>( \lambda_{1,2}^{(l)} )</th>
<th>( \lambda_{1,3}^{(l)} )</th>
<th>( \lambda_{1,4}^{(l)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>18 ( g(2) )</td>
<td>( 2 \pi i )</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-16 ( g(3) + 24 \pi i g(3) )</td>
<td>-18 ( g(3) )</td>
<td>-18 ( s(2) )</td>
<td>-8 ( \pi i )</td>
<td>1</td>
</tr>
</tbody>
</table>
The reason that we calculate so simply is that all Chern classes \( C_i \) have to calculate

\[
\hat{A}(F) = \exp \left( \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \text{ch}_k(F) \right)
\]
\[ c_{1k} = (-1)^{k+1} k \left[ \log \left( 1 + \sum_{i=1}^{\infty} c_i x^i \right) \right]_k \]

are by the adjunction formula very simple:

\[ c_k(W_{-1}) = (-1)^k k! \sum_{j=0}^{k} \frac{(-2)^j (k+1-j)}{j!} S_k(H) \]

\[ c_k(F_2) = (-1)^k k! \sum_{j=0}^{k-1} \frac{(-2)^j}{j!} S_k(H) \]

\[ \text{Symmetric k-order polynomial of hypersurface classes in } \mathbb{P}^1 \]

Remark: 1) The topological data
of the underlying Calabi-Yau model appear in the leading logarithms of the high-energy limit of the Feynman graph.

Remark 2) This gives the full analytic structure of this family of 1-loop integrals.

This due to a 30 year study of the Banana-graph in the low energy limit by Broadhurst, Laporta and Remiddi, Broeders, Salomonson, Wei and Bauer, Varchi.
\[
I_e(t, \mathbf{u}) = 2^e \int_0^\infty \sum_{k=1}^l I_0(\sqrt{l+1}z) \prod_{k=1}^l K_0(\sqrt{s_k}z) \, dz
\]

\[
\sum_{k=1}^l s_k < \frac{2^e}{s_{max}}
\]

Bessel function

Analytic structure: \( m_k = \mu \) \( \forall k \Rightarrow s_k = 1 \)

One modulus case:

\begin{align*}
&X-X-X-X-X \quad \text{Conifold} \\
&t = 0 \\
&\text{Lose monodromy} \\
&t = \infty
\end{align*}

New analytic \quad Exact analytic
Remark III. We had calculated in $D_o = 2$. How is this relevant to 4-dim quantum field theory?

General Fact: There is a difference relation between $D_o \to D_o + 2$ that one can use if one knows all master integrals.
\[ I_2 \left( D-2, z \right) = \frac{4}{(D-6)^2 (z+1) (z+8)} \left[ (D-3)^2 \beta^3 (D-2)^2 + 2 (D^2) z \right] \]

\[ I_3 (D, z) = 2 (D-3) \frac{d}{dz} S(D, z) + \frac{3 (z+3)}{4(D-4)^2} \]

Master integral

\[ \uparrow \]

Here the differential is determined by the \( \ell \)-polynomial and one needs the full set of master integrals [3] Böhmisch, Dauth, Fischbach, Nega, A. K.

Choco ring of \( W_1 \) not too hard
Recall IV Arithmetic properties and Calabi-Yau modularity

1. Deduce modular properties in the instanton expansion at large radius

\[ g_{kx} = 1 \]

\[ I_3(T) = \text{r-class-term} + \sum_{k=1}^{a_0} \frac{1}{2} n_k \text{Li}_3(e^{4\pi T}) \]

Corresponds to Fano F_3

\[ \nu_{kx} = \sum 48, 36, 32, 48, 48, 24, 48, 48, 32, 36, 48, 32 \]
\[ \frac{d^4}{dt^4} I_3 = \frac{1}{5} \left( 121 + 2E_4(q) - 2E_4(q^2) - 2E_4(q^3) \right) \]
\[ + 16E_4(q^4) + 18E_4(q^6) - 144E_4(q^{12}) \]
\[ \text{h=4 form of } \Pi_0(q) \]

(2) Arithmetic (equal mass case)
$t = 1$

$t_{\pm} = (33 \pm 8 \sqrt{17})$  \[ l = 4 \]

For example at the attractor point the numerator of the Hasse-Weil Zeta function becomes \[ 7 \] Canadas, de la Ossa Elmi, van Stalen 1812, 061146

\[ P_{3}(\overline{\mathbb{F}_1}, T) = (1 - a_p(pT) + p(pT^2))(1 - b_p pT + p^3T^3) \]
$a_p, b_p \in \mathbb{Z}$ and according to the Weil conjectures $|a_p| \leq 2p$

$|b_p| \leq 2p^{3/2}$ moreover one finds that the $a_p, b_p$ are weight 2 and coefficients

4 Hecke eigenforms in $S^\text{new}_2 \left( \Gamma_0(34), \chi_{68}(3,1) \right)$

In [23] we checked that the modular periods $\omega^+ \omega^-$

modular quasi periods $\gamma^+ \gamma^-$

determine all maximal cusp master integrals exactly!