Western Hemisphere Colloquium on Geometry & Physics

Thomas Dumitrescu (UCLA)

2 - Group Global Symmetry in Quantum Field Theory

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w/ C. Cordova & K. Intriligator

Familiar ex. of flavor symmetry in QFT: U(1) flavor symmetry (e.g. baryon number in QCD).

→ Noether current \( j^\mu \), \( \mathcal{L} \cdot j^\mu = 0 \)

Convention: this is an ordinary 0-form global symmetry \( U(1) = U(1)^{(0)} \)

\( j^{(1)} = j \cdot dx^\mu \) is its 1-form current.

Operator conservation eq.: \( d \ast j^{(1)} = 0 \).
Can define **topological charge** operators. Consider a 4d QFT on $\mathcal{M}_4$ (e.g. $\mathcal{M}_4 = \mathbb{R}^4$).

$$Q_3(\Sigma_3) = \int_{\Sigma_3} \star \mathbf{j}^{(1)}$$

**conservation of** $\mathbf{j}^{(1)}$ \[ \Rightarrow \text{topological invariance of } Q_3(\Sigma_3) \]

**Charged objects** = local operators $\mathcal{O}(x)$

$$\mathcal{Q}(\Sigma_3) \xrightarrow{\text{QFT}} \mathcal{O}(x)$$

**$U(1)$ charge** of $\mathcal{O}(x)$

$\mathcal{O}(x)$ can create point particles w/ $U(1)$" charge.

**One more layer needed:** (standard for global symmetries in QFT)

**Couple** $\mathbf{j}^{(1)}$ to a non-**dynamical** classical source:

$$\Delta S = \int j^m A_m = \int A^{(1)} \wedge \star \mathbf{j}^{(1)}$$
Standard fact:
\[
\begin{align*}
\{ & \text{current conservation:} \\
& d \ast j^{(1)} = 0
\end{align*}
\]
\[
\begin{align*}
\{ \text{partition function invariant} \\
& Z [A^{(1)}] \text{ under } A^{(1)} \rightarrow A^{(1)} + dx^{(0)}
\end{align*}
\]

Recently this familiar story has been generalized in important ways, e.g.

Higher form ("generalized") global symmetries [Kapustin, Seiberg; Gaiotto, Kapustin, Seiberg, Willett]

Simplest 4d example:
\[ U(1)^{(1)} = \text{continuum 1-form global symm.} \]
\[ J^{(2)} = \text{conserved 2-form current} \]
\[ d \ast J^{(2)} = 0 \quad (\partial^m J^{(2)}_{m=1}=0) \]

Topological surface operators (\(\text{dim} - 2\))
\[ Q (\Sigma_2) = \int_{\Sigma_2} J^{(2)} \]

\[ \int_{\Sigma_2} J^{(2)} \]
The charged line defects/operators can create dynamical strings charged under U(1)\(_{\text{EM}}\).

Simplest 4d example: free Maxwell theory

\[ f^{(2)} = d a^{(1)} \]

dynamical U(1) connection \[ d \ast f^{(2)} = 0 \] (e.o.m.)

\[ d \ast f^{(2)} = 0 \]

\[ J^{(2)} \sim f^{(2)} \]

\[ F_E (E_2) = \int_{E_2} \ast J^{(2)} \sim \int_{E_2} \ast f^{(2)} \]

Electric charge

\[ U(1)_{\text{EM}} : J_m \sim \ast f^{(2)} \]

\[ Q_m (E_2) = \int_{E_2} \ast J_m^{(2)} \sim \int_{E_2} \ast f^{(2)} \]

Magnetic charge
**Charged op's:** Wilson & 't Hooft lines
  (electric) (magnetic)

**Final Step:** Couple $J^{(2)}_E, J^{(2)}_m$ to background
gauge fields $\sim$ 2-form $B$-fields.

$$\Delta S = \int B^{(2)}_E \wedge \ast J^{(2)}_E + \int B^{(2)}_m \wedge \ast J^{(2)}_m$$

$\sim \int B^{(2)}_E \wedge f^{(2)}$

$\sim \int B^{(2)}_m \wedge f^{(2)}$

Standard "BF" or
Green–Schwarz (GS)
coupling.

Again: $d \ast J^{(2)}_{E,m} = 0$

$\Rightarrow$ invariance under 1-form background
gauge transformations:

$$B^{(2)}_{E,m} \rightarrow B^{(2)}_{E,m} + d A^{(1)}_{E,m}$$

This is the beginning of a
rich story, with many recent
results.

**Key point:** Symmetries are key in analyzing
$\chi$FTs (esp. at strong coupling). So more, and
new kinds of, symmetries are better.
This talk: yet a further generalization of global symmetry to higher n-group symmetries; simplest case = 2-group.

Basic intuition: in many examples, the ordinary 0-form symmetries and the higher-form symmetries do not talk to each other. Higher group symmetries arise when they do.

A prototypical example: 4d QED$_{N_f}$, i.e. U(1) Maxwell theory coupled to Nf Dirac electrons of mass $m = 0$ and charge $g = 1$, $\Psi^i_D = \left( \frac{4 i}{X^i_{D}} \right) \ i = 1, \ldots, N_f$

Classical symmetries:

<table>
<thead>
<tr>
<th></th>
<th>U(1)$_{\text{gauge}}$</th>
<th>U(1)$^{(0)}_A$</th>
<th>SU(CN)$_{L}^{(0)}$</th>
<th>SU(CN)$_{R}^{(0)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4_i$</td>
<td>1</td>
<td>1</td>
<td>$\Box$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$X_{\kappa_1}$</td>
<td>-1</td>
<td>1</td>
<td>$\emptyset$</td>
<td>$\Box$</td>
</tr>
</tbody>
</table>

Note: $X_{\kappa_i}$ is a $\Omega$ of SU(CN)$_{\text{diag}}^{(0)}$. 
QED retains the $U(1)_M$ magnetic 1-form symmetry since $d^2 f^{(2)} = 0$ still, but $U(1)_E$ is broken completely since $d^* f^{(2)} \sim \ast j^{(1)}_E \neq 0$.

To discover the fate of the remaining symmetries, let us consider their perturbative anomalies $\sim$ anomaly polynomial $I^{(6)}$:

$$I^{(6)} = I^{(6)}_{\text{gauge}} + I^{(6)}_{\text{ABJ}} + I^{(6)}_{\text{global}} + I^{(6)}_{\text{mixed}}.$$  

calculate via fermion $\Delta$'s

1. $I^{(6)}_{\text{gauge}} \sim K_{\text{gauge}} \left(C_4(f^{(2)})\right)^3$

$$\frac{q^3 - q_1^3}{q^4 - q_1^4} = 1 - 1 = 0.$$  

These pure gauge anomalies cancel $\Rightarrow$ required for consistency of gauge thry.  

$$\Rightarrow I^{(6)}_{\text{gauge}} = 0.$$
2.) \[ I_{ABJ}^{(6)} \sim K_{ABJ} \frac{C_1(F_A^{(2)})}{4} \frac{C_1(f^{(2)})}{X} \]

\[ K_{ABJ} = \frac{1}{4} (1)^2 + \frac{1}{4} (-1)^2 = +2 \neq 0. \]

**Upshot:** \[ d \neq j_A^{(1)} \sim K_{ABJ} f^{(2)} \land f^{(2)} \neq 0 \]

Thus \( j_A^{(1)} \) is **not** conserved and we no longer consider the \( U(1)_A \) symmetry.

3.) \[ I_{global}^{(6)} \sim C_3 \left( F_{SU(3)_L}^{(2)} \right) - C_3 \left( F_{SU(3)_R}^{(2)} \right) \]

\[ \sim \text{ 't Hooft anomalies for global SU(3)_L and SU(3)_R symmetries.} \]
Key fact: 't Hooft anomalies do not break global symmetries; currents conserved as operators: \( d \times j_{\gamma}^{(1)} = d \times j_{\nu}^{(1)} = 0 \) at separated points. But: subtle c-number violations at coincident points of background gauge invariance.

4.) \( I_{\text{mixed}}^{(6)} \sim (C_2(F_{\text{L}}^{(2)}) - C_2(F_{\text{R}}^{(2)})) \times C_1(f^{(2)}) \)

\[ \begin{array}{c}
\text{dyn. fields} \\
\text{ULF\ Maxwell} \\
\text{field.}
\end{array} \]

This diagram neither destroys gauge invariance (like \( \Delta \)) nor is it an ABJ anomaly that destroys conservation of \( j_L^{(1)} \) (like \( \Delta \)). But the monomeron diagram still has interesting effects on the global symmetries:

\[ d \times j_{\Omega L R}^{(1)} \sim \pm dA_{\Omega L R}^{(4)} \times C_1(f^{(2)}) \text{ C-number operator} \sim *J_{\Omega}^{(2)} \]
in the absence of b.g. fields \( A_{L,R}^{(1)} = 0 \) of the \( SU(N)_L \times SU(N)_R \) currents we
conserved operators: \( d \times j_{L,R}^{(1)} = 0 \).

But their current algebra is deformed:

\[
\alpha \times j_{L,R}^{(1)}(x) \ , j_{L,R}^{(1)}(y) \sim \pm \partial \Phi^{(2)}(x-y) \, f^{(2)}(y)
\]

integrating this gives a (singular) OPE:

\[
\hat{j}_{L}^{(1)}(x) \hat{j}_{L}^{(1)}(y) \sim \frac{J_{M}^{(2)}(y)}{(x-y)^{4}} \ , \ \hat{j}_{R}^{(1)}(x) \hat{j}_{R}^{(1)}(y) \sim -\frac{\overline{J}_{M}^{(2)}(y)}{(x-y)^{4}}
\]

This fusion of two 1-form currents into a 2-form current is the defining feature of (continuous) 2-group global symmetry.

Background fields:

\[
S = \int A_{L,R}^{(1)} \times j_{L,R}^{(1)} + \int B_{M}^{(2)} \times \overline{J}_{M}^{(2)}
\]

normally:

\[
A_{L,R}^{(1)} \rightarrow A_{L,R}^{(1)} + d \lambda_{L,R}^{(0)}
\]

\[
B_{M}^{(2)} \rightarrow B_{M}^{(2)} + d \lambda_{M}^{(1)}
\]
with 2-group symmetry, extra term: \( \overset{(11)}{\text{etc}} \)

\[ B_m \rightarrow B_m^{(2)} + dM_m + \frac{1}{4\pi} \text{tr} (\chi_L dA_L^{(1)}) \]

needed to cancel RHS of \( d \star j_{12}^{(2)} + dA_{L12}^{(1)} \cdot f^{(2)} \).

Thus 2-group global symmetry \( \Leftrightarrow \)

Green-Schwarz mechanism for background fields \([\text{Kapustin, ...}]\)

We have seen that anomalies of type I\(^{(6)}\) mixed inevitably give rise to 2-group global symmetry in 4d.

Remainder of talk: sketch some recent applications of 2-group symmetry to 6d Supersymmetric Little String Theories and SCFTs.
The $U = (1,1)$ LST:

decoupling limit of NS5 stack in IIB

$R^4 \rightarrow \text{6d MSYM with gauge group } g = U(N)$. We will decouple the C.O.M. and take $g = SU(W)$.

Has $U(1)^{(1)}_I$ instanton 1-form symmetry:

$J^{(2)} \sim * c_2(f^{(2)} \wedge f^{(1)}) \rightarrow \text{d} * J^{(2)} = 0$

(present in any 6d gauge theory)

instantons $\sim$ little strings $\sim$ fundamental

Compute (via diagram):

$I^{(g)}_{\text{mixed}} = N \left( C_2(L) - C_2(R) \right) \cdot C_2(f^{(2)})$

$\text{SU(2)}_{\text{L,R background fields}}$

$\text{dynamical SU(N) gauge field}$

$\Rightarrow 2$-group symmetry!
This is generic in 6d gauge theories (with or without SUSY). We saw it happens in \( W = (1,1) \) LSTs with SU(N) gauge group. We also checked other LST examples, and even more were checked by [del Zotto-Ohmori]

**Emerging Picture:** all LSTs seem to have a U(1)\(^{(1)}\) symmetry (little string charge)

- generically it is part of a 2-gp.
- [dZ- O] matched this structure for many T-dual pairs of LSTs (meaningful!)

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**What happens in 6d SCFTs?**

All SCFTs with gauge fields on their tensor branch have a GS coupling:

\[
S_{GS} = i \sqrt{c} \sum b^{(2)} \wedge c_2 \left( f_g^{(2)} \right) *
\]

\* Dynamical s.d. 2-form in a \((1,0)\) tensor-multiplet \( (\phi, \psi, b^{(2)}) \)
The GS coupling has many important roles:

(i) contributes \( I_{GS}^{(g)} = + C \cdot C_2(f_g^{(2)})^2 \) to \( I^{(8)} \), which cancels \( I_{1-loop}^{(8)} = - C \cdot C_2(f_g^{(2)})^2 \) from matter.

(ii) gauge \( J_{I}^{(2)} \) no longer global symmetry!

SCFT unitary rep. theory says that a conserved 2-form current is not allowed \( [\text{CDT, 1612, 00309}] \).

\[ \Rightarrow \begin{cases} \text{no } U(1)^{(1)}_I \text{ or 2-gp symmetries in } \} & \text{unitary SCFTs} \\ \end{cases} \]

(iii) Superpartner of dilaton coupling \(~\sqrt{C} \not\times tr(f_g^{(2)} \wedge f_g^{(2)})\) which makes YM term conformal \( (\not\times \sim \text{dilaton}) \)

The absence of 2-group symmetries in SCFTs has non-trivial consequences:

\[ S_{GS} = i \oint b^{(2)} \wedge (\sqrt{C} \cdot C_2(f_g^{(2)}) + K \cdot C_2(R) + \ldots) \]

\( \text{dynamical} \) \( \frac{\text{b.g. fields for}}{\text{global symmetries}} \)
\[ I_{\text{GS}}^{(8)} = c (C_2(f_\text{g}^{(8)}))^2 + k^2 (C_2(R))^2 \]
\[ + 2 \sqrt{\kappa} \frac{C_2(f_\text{g}^{(8)})}{\text{gauge}} \frac{C_2(R)}{\text{global}} \]
\[ \quad \text{\`t Hooft anomaly} \]

But: in \( I_{\text{total}}^{(8)} = I_{1\text{-loop}}^{(8)} + I_{\text{GS}}^{(8)} \) all mixed gauge-global terms must cancel: otherwise there would be a \( Z \)-gp in the SCFT!

Thus: Can calculate coefficients like \( k \) (and thus \`t Hooft anomalies of the SCFT) by insisting that mixed terms in \( I_{\text{GS}} \) cancel those in \( I_{1\text{-loop}} \). This justifies the procedure of [Ohmori, Shimizu, Tachikawa, Yonekura] for computing \`t Hooft anomalies of SCFTs from the low-energy spectrum on their tensor branch.

Application: the \( \alpha^\prime \)-theorem in 6d

\[ \langle T^M \rangle \sim a E_6 + \ldots \]

Weyl anomaly

Cardy Conjecture ("\( \alpha^\prime ") theorem")

\[ \Delta a = a_{u} v - a_{1R} > 0 \]

under RG: CFT\(_{\text{IR}} \rightarrow \text{CFT}_{\text{IR}} \)
recall: intution: $a \sim $ effective d.o.f. (16.) along RG flow

* proven in 2d [Z] and 4d [KS]
* no general proof in $d \geq 6$ (without SUSY)

Some time ago [CDI 1506.03807] we made progress for SUSY RG flows, all of which look like $SCFT_{1,\nu} \rightarrow (\text{Moduli Space})_{IR}$ (no SUSY-preserving relevant ops.)

we showed: \[ \Delta a \sim b^2 > 0 \] (a-thm)

where $b \sim GS$ coefficients $\sim k b^{(2)} \wedge c_2(Q)$ and $\sim b^{(2)} \wedge P(CT)$ on tensor branch.

If IR theory is a free SCFT (only tensors + hypers, e.g. $W = (2,0)$ SCFTs) then $a_{IR} > 0 

\Rightarrow a_{SCFT} = a_{IR} + \frac{\Delta a}{\nu > 0} = 0

It is expected that $a > 0$ in unitary CFTs (d.d.o.f.), but it is distinct from $\Delta a > 0$, and no general p.f. (e.g. in 4d $a > 0$ follows from [HMJ])
In fact, there is a problem with vector multiplets:

$$a_{1R} = \frac{1}{210} (11 \, n_H + 199 \, n_T - 251 \, n_V)$$

formally (since vectors $\neq$ SCFT). Enough vectors can lead to $a_{1R} < 0$ (happens in many examples! In fact generic).

What about $a_{\text{SCFT}} > 0$? Only way out is that $\Delta a > |a_{1R}| > 0$ (*

(stronger than a-thm).

Apply previous results: unitary SCFT $\rightarrow$

$\rightarrow$ no $Z$-gp symmetry $\rightarrow$ no mixed
gauge-global anomalies $\rightarrow$ fix GS terms

$\sim b^{(2)} \wedge C_2 (R)$ and $b^{(2)} \wedge \rho_3 (T)$ to compute

$$\Delta a \sim \frac{1}{|k R^2|} \left( k g^2 R^2 - k g^2 p^2 \right)^2$$

with $\mathcal{I}_{1\text{-loop}}^{(g)} \geq C_2 (f_g) \left( k g^2 R^2 C_2 (f_g) + k g^2 R^2 \rho_3 (T) \right)$
The formula for $\Delta a$ only depends on the massless spectrum on the tensor branch (no details of massive states needed).

Bound $\Delta a$ from below:

$$k g^2 r^2 = -h^2 g, \quad k g_2 r^2 = -2h^2 g + \text{(hypers)}$$

$$k(g^2)^2 = -Ug + \text{(hypers)}$$

$$\text{Tr}_{ad}(f g)^4 \geq u g^2 \left( \text{Tr}(f g^2) \right)^2$$

Get a bound:

$$\Delta a > \frac{23(h^2 g)^2}{u g^2} \quad \text{"universal"?}$$

$\Delta a$ always wins! (check $U g$)

$\alpha_R, \text{vector} \approx -1.2 \dim(g^2)$

Thus:

- prove a $\Delta a > 0$ for unitary 6d SCFTs!
- proof works for all ranks (here: rank 1)
- does not rely on detailed classification (general QFT result)
- Crucial: ability to compute G.S couplings by enforcing no 2-gp symm.