From gapped phases of matter to Topological Quantum Field Theory and back again

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Outline

- Gapped phases of matter
- Invertible TQFT and invertible phases of matter
- Berry-Wess-Zumino-Witten cohomology classes for families of gapped lattice models (based on work with Lev Spodyneiko, arXiv:2001.03454)

States of matter

In elementary school, we learn about qualitatively different "states of matter": solid, liquid and gas.

Solids are crystals on a microscopic scale. The distinction between solids and other states of matter is based on symmetry:

- Liquids and gases are invariant under arbitrary translations and rotations of the Euclidean space $\mathbb{R}^3$.
- Crystals are invariant under some infinite discrete subgroup of the symmetry group of $\mathbb{R}^3$.

There are also states of matter ("liquid crystals") which are invariant under arbitrary translations, but not under arbitrary rotations. Liquid crystal displays (LCD) can be found in many TVs and computer monitors.

Phases of matter

Is there some qualitative distinction between liquid and gas? No: one can "deform" one into another by varying both temperature and pressure.

Allowed deformations are those which preserve a natural property: correlations between physical quantities at points p and p' become exponentially small when $|p - p'|$ is large:

$$\left|\langle A(p)B(p')\rangle - \langle A(p)\rangle\langle B(p')\rangle\right| \leq Ce^{-|p-p'|/\xi}.$$

Loci in the parameter space where this fails are called phase transitions. After removing these loci, the parameter space may fall into several disconnected components. These are called phases of matter.

Liquid and gas belong to the same phase of matter. Crystals with different symmetry groups are examples of distinct phases of matter.

Quantum systems

Traditionally, physicists studied phases at positive temperature. More recently, they turned to phases at zero temperature.

A quantum system is described by a triple $(\mathscr{A}, \tau_t, \psi)$:

- $\mathscr{A}$ is a C^* algebra
- τ_t is a 1-parameter family of automorphisms of $\mathscr{A}$
- ψ is a state on $\mathscr{A}$ invariant under τ_t

ψ gives rise to a representation of $\mathscr{A}$ by bounded operators in a Hilbert space V, so that $\psi(A) = \langle 0|A|0\rangle$ for some cyclic vector $|0\rangle \in V$.

τ_t gives rise to a Hamiltonian $\hat{H}$ (unbounded self-adjoint operator on V) which annihilates $|0\rangle$.

The "zero-temperature" condition is: ψ is pure and $\hat{H} \geq 0$. For infinite volume systems one usually assumes $|0\rangle$ is the only vector annihilated by $\hat{H}$. Then ψ is called a ground state for τ_t.

Mind the gap!

If 0 is an isolated eigenvalue of H, the system is said to be in a gapped phase. Two systems are in the same gapped phase if they can be deformed into each other without "closing the gap".

Typically, it is assumed that $\mathscr{A}$ is an infinite tensor product $\otimes_{p\in\Lambda}\mathscr{A}_p$, where Λ is an infinite discrete subset of $\mathbb{R}^d$, and $\mathscr{A}_p$ is a matrix algebra. $A \in \mathscr{A}$ is a local observable localized on a finite subset $X \subset \Lambda$ if $A \in \mathscr{A}_X = \otimes_{p\in X}\mathscr{A}_p$.

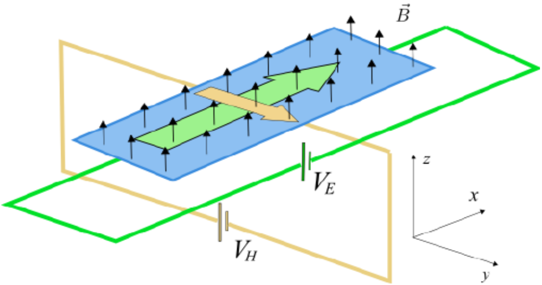
Under some assumptions about the Hamiltonian H ("exponentially decaying interactions") a gap in the spectrum of H implies that for any finite subsets X and Y and any $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ one has

$$|\langle AB\rangle - \langle A\rangle\langle B\rangle| \leq ||A|| \cdot ||B|| \cdot |X| \cdot |Y| e^{-\mathrm{dist}(X,Y)/\xi}.$$

Thus the notion of a "gapped phase" is compatible with the notion of a phase based on correlation decay.

Examples of gapped phases

- Trivial phase (deformation class of a gapped system where $H = \sum_p H_p$ with $H_p \in \mathscr{A}_p$, and the state ψ is "factorized", i.e. $\psi(AB) = \psi(A)\psi(B)$ if $A \in \mathscr{A}_p$, $B \in \mathscr{A}_q$ and $p \neq q$.)
- Fractional Quantum Hall phases of 2d insulating systems in a magnetic field
- Integer Quantum Hall phases of 2d insulating systems in a magnetic field



Fractional Quantum Hall phases

- Hall effect at zero temperature with rationally "quantized" Hall conductance
- Localized excitations with fractional charge! (For example, charge $e/3$, where e is the charge of the electron)
- Enthropic Hall effect (flow of entropy perpendicular to temperature gradient in the limit $T \to 0$).
- Localized excitations with "fractional statistics"! (local excitations cannot be classified as bosons or fermions)

Integer Quantum Hall phases

- Integrally quantized Hall conductance at zero temperature
- Enthropic Hall effect (flow of entropy perpendicular to temperature gradient in the limit $T \to 0$).

IQHE phases can be distinguished from the trivial phase only by their response to external "fields" (voltage and temperature).

Both FQHE and IQHE phases have $U(1)$ symmetry, but remain non-trivial even if one allows deformations which break this symmetry.

Topological Quantum Field Theory

It was realized in the late 1980s - early 1990s that both FQHE and IQHE can be described by Chern-Simons field theory.

Chern-Simons field theory is an example of a 3d TQFT.
Can one use TQFT in $d+1$ space-time dimension to describe gapped phases of matter in d spatial dimensions in general?

TQFTs have been axiomatized (Atiyah,..., Bayez, Dolan,...,Lurie).
Can one use TQFT to classify gapped phases of matter?

There is actually an infinity of classification problems, labeled by the dimension of space d and the symmetry group G. IQHE phases correspond to $d = 2$ and $G = U(1)$.

Extended TQFT

Extended TQFT in n space-time dimensions is defined as a symmetric monoidal functor from one symmetric monoidal (∞, n) category ("source") to another ("target").

The "source" (∞, n) category Bord_n has compact 0-manifolds as objects, compact 1-manifolds with boundaries as 1-morphisms, compact 2-manifolds with corners as 2-morphisms,..., all the way up to n-manifolds. Higher morphisms are multi-parameter families of n-morphisms. Symmetric monoidal structure arises from disjoint union.

The "target" (∞, n) category $\mathcal{C}_n$ can vary, but its morphisms of degree $n-1$ are finite-dimensional vector spaces and morphisms of degree n are linear maps between them. Symmetric monoidal structures is related to tensor product.

Invertible TQFTs

It is very difficult to classify TQFTs in general dimensions (although some dimensions might be easier than others). One problem is the absence of a natural choice of a target (∞, n)-category.

An interesting class of TQFTs is invertible TQFTs. TQFTs form a monoid, invertible TQFTs are those which have an inverse.

If all morphisms in an ∞-category are invertible, we get an ∞-groupoid which can be replaced with an ∞-loop space ("loop spectrum").

This is a sequence of pointed spaces $X_0, X_1, \ldots,$ and homotopy equivalences

$$\Omega X_n \sim X_{n+1}.$$

Classification of invertible TQFTs

Freed, Hopkins, 2016:

- For unitary invertible TQFTs, can replace Bord_n with the oriented Thom spectrum (for spin-TQFTs, spin Thom spectrum)
- In the invertible case, there is a "natural" choice of a target (a particular loop spectrum).

This leads to a classification of deformation classes of unitary invertible TQFTs in $d+1$ space-time dimensions:

$$Inv_{d+1}^{B} = \pi_0(X_{d+1}),$$

where $X_0, X_1, \ldots$ is the "Anderson dual" of the oriented Thom spectrum.

For unitary invertible spin-TQFTs:

$$Inv_{d+1}^{F} = \pi_0(Y_{d+1}),$$

where $Y_0, Y_1, \ldots$ is the dual of the spin Thom spectrum.

- $X_0 = K(\mathbb{Z}, 1), \quad Y_0 = K(\mathbb{Z}, 1)$
- $X_1 = K(\mathbb{Z}, 2), \quad Y_1 = \mathbb{Z}/2 \times K(\mathbb{Z}, 2),$
- $X_2 = K(\mathbb{Z}, 3), \quad Y_2 = \mathbb{Z}/2 \times K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3)$
- $X_3 = \mathbb{Z} \times K(\mathbb{Z}, 4), \quad Y_3 = \mathbb{Z} \times E$

Freed-Hopkins result implies that the deformation class of a unitary invertible TQFTs in n dimensions is determined by its partition functions on all possible closed n-manifolds (oriented or spin).

In the bosonic case, first non-trivial deformation classes appear for $d = 2$. In the fermionic case, they first appear for $d = 0$. IQHE phases are believed to be described by non-trivial invertible spin-TQFTs in three space-time dimensions (even if one ignores $U(1)$ symmetry).

Gapped phases vs TQFT

For some time if was believed (without much evidence) that any gapped phase can be described by a TQFT.

Recently, intricate examples of gapped systems in 3d have been constructed ("fractons") which are counter-examples to this belief.

The 1-1 relation between TQFTs and gapped phases might still be true if one restricts to invertible TQFTs and invertible gapped phases.

Invertible gapped phases

Gapped lattice systems can be "stacked" (composed):

$$(\mathscr{A}, \tau_t, \psi), (\mathscr{A}', \tau'_t, \psi') \mapsto (\mathscr{A} \otimes \mathscr{A}', \tau_t \otimes \tau'_t, \psi \otimes \psi').$$

A system $(\mathscr{A}, \tau_t, \psi)$ is said to be an invertible phase if it can be stacked with another system to obtain a system in a trivial phase.

Gapped phases form a commutative monoid, invertible phases form a commutative group.

IQHE phases are invertible. FQHE phases are not.

Invertible gapped phases and loop spectra

A. Kitaev proposed to define a Short-Range Entangled phase as an invertible phase.

Let W_{d+1} be the space of d-dimensional systems in an invertible phase (with some fixed symmetry G). Kitaev gave a heuristic argument that the spaces $W_1, W_2, \ldots$ form a loop spectrum.

Based on some low-dimensional cases, I proposed in 2015 that this spectrum is related to Thom spectra and that invertible gapped phases can be described by TQFT.

Freed and Hopkins showed that unitary invertible TQFTs can indeed be classified using the Anderson dual of Thom spectra.

What about invertible phases?

Why it is hard to relate invertible gapped phases and invertible TQFT

- TQFTs can be defined on arbitrary $d+1$-dimesional closed manifolds
- Lattice models are formulated on $\mathbb{R}^d$ (times time) and cannot be naturally defined on a general manifold.

Every bosonic system can be viewed as a fermionic one. Thus there is a map from the set of bosonic phases to the set of fermionic phases. Restricting to invertible ones, get a homomorphism

$$Inv^B_{d+1} \to Inv^F_{d+1}.$$

On the TQFT side, get a map of spectra $X \to Y$. In particular, $\pi_0(X_3) = \pi_0(Y_3) = \mathbb{Z}$, and the map is multiplication by 16.

This is related to Rokhlin's theorem: signature of a spin 4-manifold is divisible by 16. How would lattice models on $\mathbb{R}^2$ know about this?

How to "see" a loop spectrum

Let W_{d+1} be the "space of all invertible system in spatial dimension d" (bosonic, fermionic, or with some additional symmetry). Suppose the spaces $W_1, W_2, \ldots$ form a loop spectrum. Then

$$\pi_k(W_{d+1}) = \pi_0(\Omega^k W_{d+1}) = \pi_0(W_{d+1-k}), \quad k \geq d+1.$$

Thus homotopy groups of W_{d+1} are predicted in terms of the group of invertible phases in lower dimensions.

Low-dimensional cases

W_1 is homotopy equivalent to the space of projectors with a one-dimensional image, i.e. the space of lines. Thus

$$W_1 \sim \lim_{N\to\infty} \mathbb{C}P^N = K(\mathbb{Z}, 2).$$

Therefore we predict:

$$\pi_1(W_2) = \pi_0(W_1) = 0, \quad \pi_2(W_2) = \pi_1(W_1) = 0, \quad \pi_3(W_2) = \pi_2(W_1) = \mathbb{Z}.$$

Thus the space of 1d lattice systems in a trivial phase must be a $K(\mathbb{Z}, 3)$.

In particular, since $H^3(W_2) = \mathbb{Z}$, to any family of 1d systems in an invertible phase one should be able to assign a degree-3 cohomology class on the parameter space.

Berry curvature and its higher-categorical relatives

Note that $H^2(W_1) = \mathbb{Z}$. What is the corresponding degree-2 class on the parameter space of a family of 0d invertible lattice systems?

M V. Berry (1984) noticed that to any family of gapped Hamiltonian one can associate a connection on the line bundle of its ground states (by projecting the trivial connection on the trivial Hilbert bundle).

The curvature of the Berry connection is a closed 2-form representing the 1st Chern class of this line bundle.

For 1d systems, we expect a degree-3 class which we can represent by a closed 3-form on the parameter space. This would be the Dixmier-Douady class of a gerbe.

For 2d systems, expect a 2-gerbe and a closed 4-form, etc.

One-dimensional lattice systems

Hamiltonian: $H = \sum_{p \in \mathbb{Z}} H_p$, where $H_p \in \mathscr{A}$ is uniformly bounded and approximately localized on p:

$$||[H_p, A]|| = O(|p-q|^{-\infty}), \quad \forall A \in \mathscr{A}_q.$$

Pure state $\psi : \mathscr{A} \to \mathbb{C}$ gives rise to an irreducible representation of $\mathscr{A}$ on a Hilbert space V (via the Gelfand-Naimark-Segal construction) with a cyclic vector $|0\rangle$. H gives rise to an unbounded self-adjoint operator $\hat{H}$ on V such that $\hat{H}|0\rangle = 0$. ψ is called a ground state for H if $\hat{H} \geq 0$.

A system $(\mathscr{A}, H, \psi)$ is called gapped if $|0\rangle$ is the unique vector with eigenvalue 0 and the spectrum of H is contained in $\{0\} \cup [\Delta, \infty)$ for some $\Delta > 0$.

We are interested in a differentiable family of gapped systems $(\mathscr{A}, H(s), \psi(s))$, $s \in M$ (the parameter space).

Looking for closed forms I

For a family of gapped Hamiltonian on a finite-dimensional V, we have a formula for the Berry curvature:

$$F^{(2)} = \frac{i}{2} \oint \frac{dz}{2\pi i} \mathrm{Tr}\, G(z)\, dH\, G(z)^2\, dH$$

Here $G(z) = 1/(z - H)$. $dF^{(2)} = 0$ is checked by a direct computation.

Does not work for systems for $d > 0$. We can consider

$$F^{(2)}_{pq} = \frac{i}{2} \oint \frac{dz}{2\pi i} \mathrm{Tr}\, G(z)\, dH_p\, G(z)^2\, dH_q.$$

Although $\left| F^{(2)}_{pq} \right| = O(|p-q|^{-\infty})$, the sum $F^{(2)} = \sum_{pq} F^{(2)}_{pq}$ is still divergent, thanks to the contribution of points near the diagonal.

Looking for closed forms II

Consider instead a well-defined 2-form

$$F_p^{(2)} = \sum_{q\in\mathbb{Z}} F_{pq}^{(2)}.$$

It is not closed. But is satisfies

$$dF_p^{(2)} = \sum_q F_{pq}^{(3)},$$

where $F_{pq}^{(3)}$ is a 3-form, $F_{pq}^{(3)} = -F_{qp}^{(3)}$, and $F_{pq}^{(3)} = O(|p-q|^{-\infty})$.

In turn $F_{pq}^{(3)}$ satisfies

$$dF_{pq}^{(3)} = \sum_r F_{pqr}^{(4)},$$

where $F_{pqr}^{(4)}$ is a 4-form which is skew-symmetric in p, q, r, etc.

An explicit formula for $F_{pq}^{(3)}$

$$F_{pq}^{(3)} = \frac{i}{6} \oint \frac{dz}{2\pi i} \mathrm{Tr} \left(G^2(z)\, dH\, G(z)\, dH_p\, G(z)\, dH_q \right.$$
$$\left. - G(z)\, dH\, G(z)^2\, dH_p\, G(z)\, dH_q \right) - (p \leftrightarrow q) \quad (1)$$

Vietoris-Rips chain complex

We can formalize this by introducing the Vietoris-Rips (or coarse) chain complex.

Let Λ be an infinite subset of $\mathbb{R}^d$ which is uniformly discrete and uniformly filling.

An n-chain is a skew-symmetric function of $n+1$ points on Λ which is bounded and decays rapidly away from the diagonal. Let C_n be the space of n-chains, $n \geq 1$. Define $\partial : C_n \to C_{n-1}$ by

$$(\partial A)_{p_1 \ldots p_n} = \sum_{p_{n+1}} A_{p_1 \ldots p_n p_{n+1}}.$$

Then $F_p^{(2)}$ is a 2-form with values in 0-chains, $F_{pq}^{(3)}$ is a 3-form with values in 1-chains, etc. Total degree is 2.

The descent equation

All the equations we had can be written as

$$dF^{(n)} = \partial F^{(n+1)}.$$

We will call this the descent equation.

The descent equation was proposed by A. Kitaev in the context of "classical" spin systems on a lattice. We apply it to quantum models.

To construct a closed p-form, we need to contract the p-form-valued $(p-2)$-chain $F^{(p)}$ with Vietoris-Rips cochain.

Vietoris-Rips cochain complex

An n-cochain is a bounded skew-symmetric function of $n+1$ points on Λ whose support is "co-controlled" (intersection of support with any thickened diagonal is finite). See books by John Roe on coarse geometry.

Let C^n be the space of n-cochains. The differential $\delta : C^n \to C^{n+1}$ is dual to ∂:

$$\langle \partial A, \alpha \rangle = \langle A, \delta\alpha \rangle, \quad \forall A \in C_{n+1}, \forall \alpha \in C^n.$$

If $\alpha \in C_{p-2}$ and $\delta\alpha = 0$, then

$$d\langle F^{(p)}, \alpha \rangle = \langle \partial F^{(p+1)}, \alpha \rangle = \langle F^{(p+1)}, \delta\alpha \rangle = 0.$$

If $\alpha = \delta\beta$, then $\langle F^{(p)}, \alpha \rangle$ is exact.

To get a non-trivial degree-p cohomology class on the parameter space, want α to be a closed, but not exact $(p-2)$-cochain.

Vietoris-Rips cohomology

For $\Lambda = \mathbb{Z} \subset \mathbb{R}$, the cohomology is $\mathbb{R}$ in degree 1 and zero for all other degrees. Degree-1 cohomology is spanned by $\alpha(p,q) = f(p) - f(q)$, where $f : \mathbb{Z} \to \mathbb{R}$ is any function which approaches ± 1 at $\pm\infty$.

Hence we need to let $p = 3$. Then we get a closed 3-form on the parameter space:

$$\Omega^{(3)} = \frac{1}{4} \sum_{p,q} F^{(3)}_{pq} (f(p) - f(q)).$$

Its cohomology class is independent of the choice of f. It is a higher-categorical version of the cohomology class of the Berry curvature. (Kapustin, Spodyneiko, arXiv:2001.03454)

Similarly, for a family in d dimensions get a closed $(d+2)$-form on the parameter space (Wess-Zumino-Witten form).

Quantization of periods

Do these forms have quantized periods?

In general, probably not. But invertible systems are special.

Relation with TQFT predicts that the WZW class is integral for families of invertible systems parameterized by S^{d+2}.

We gave a non-rigorous argument that this is indeed the case. Can be made rigorous (N. Sopenko, last week).

For invertible bosonic systems in low dimensions ($d < 4$) expect quantization to hold for arbitrary cycles.

Need a more powerful approach to check this.I know how to assign a gerbe to a family of 1d gapped systems. Higher gerbes must be lurking around for $d > 1$.

Some lessons

- The space of gapped lattice systems has a complicated topology
- In the invertible case, it is probably an infinite loop space
- Coarse geometry in the style of John Roe plays an important role in the study of gapped phases
- Quantum statistical mechanics (study of dynamical systems based on lattice Hamiltonians with good locality properties) gives rise to intricate geometric, topoogical, and algebraic structures