Non-Invertible Duality Defects

Symmetry in QFT $\leftrightarrow$ topological operators $G^{(q)} \mapsto U_g(M^{d-q-1})$

$\begin{align*}
q = 0 & \quad \text{ordinary} \\
q = 1 & \quad \text{1-form} \quad g \in G
\end{align*}$

Fusion algebra $U_{g_1} \times U_{g_2} = U_{g_1 g_2}$

group like
(invertible)

Fusion of top operators can be more general $U_i \times U_j = \sum_K U_K$

Prominent Examples

- Verlinde lines in $\mathcal{B}$-rational CFTs, e.g. Ising fusion category

$\begin{align*}
1, & \quad \eta, \quad \sigma \\
\text{id} & \quad \mathbb{Z}_2, \quad K \text{W duality} \quad \sigma \cdot \eta = \eta \cdot \sigma = \sigma
\end{align*}$
anyons in 3d TQFT ~ non-invertible 1-form symmetry

This talk: construct non-invertible defect ("duality defect" D)
in 3+1 d (works in general d) generalize KW

Defect Construction

\[ \text{QFT} \; \mathbb{C} \; \xrightarrow{\text{(input duality)}} \; \mathbb{C} / G^{(q)} \]

\[ (G^{(q)} = \mathbb{Z}_N^{(q)}) \]

for singlet

thy with \( G^{(q)} \) sym

sym gauged

Note: gauging \( G^{(q)} \) \( \Rightarrow \) new dual sym \( G^{(d-q-2)} \) symmetry

only possible if \( q = \frac{d-2}{2} \in \mathbb{Z} \)

\( (q=0; d=2; \quad q=1; d=4; \quad \ldots) \)
Dirichlet Interface

\[ S = \int_L \mathcal{L}_\alpha + \int_R \mathcal{L}_\alpha \left[ a^{(q+1)} \right] + S_P ^{qs} (a^{(q+1)}) \]

along interface locus \( M \) place dirichlet BC for \( a^{(q+1)} \)

\[ a^{(q+1)} \mid_M = 0 \]

topological interface \( (da^{(q+1)} = 0) \)

duality assumption \( \mathcal{Z} \cong \mathbb{Z}/G^{(q)} \) \( \Rightarrow \) topological defect in fixed theory

in \( R \) dual sym comes from ops \( \gamma (S) = \exp(i \int_S a^{(q+1)}) \)

since \( a^{(q+1)} \mid_{M=0} \Rightarrow \gamma \times 0 = 0 \times \gamma = 0 \)
\[ D \times \overline{D} \quad \text{Fusion Algebra} \]

\[
\begin{aligned}
1 + \text{id} \quad (M = S') \\
\downarrow \\
\text{say} \quad D \times \overline{D} \geq x \quad | \\
\uparrow \\
\text{unknown coeff}
\end{aligned}
\]

\[ D \times \overline{D} = x \sum_{i=0}^{N-1} \eta^i \]

\[ \eta = \exp(i \Phi \alpha_s) \]

\[ \text{expected sum over} \]

\[ \text{Wilson lines in gauge theory} \]

\[ \text{coefficient } x \text{ comes from normalization of } Z_N^{(0)} \text{ as partition function in slab} \]

\[ x = \frac{1}{|H^0(S \times \mathbb{I}, d(S \times \mathbb{I}), Z_N)|} = 1 \]

\[ \text{relative cohomology } k \text{ of Dirichlet bc.} \]

\[ g^{(0)} = Z_N^{(0)} : \eta^i \quad i = 0, \ldots, N-1 \quad ; \quad D \times \overline{D} = \sum_{i=0}^{N-1} \eta^i \]

\[ \text{(Tambora-Yamaguchi)} \]

\[ \text{(fusion category)} \]

\[ \text{Key property of } D : \]
\[
\begin{align*}
3+1 \text{ D} & \quad \mathbb{Z}_N^{(\oplus)} \\
\text{again D x D by gauging in a slab } M \times I & \\
\left[ \text{gauge fields to sum} \right] & = H^2(M \times I, O(M \times I), \mathbb{Z}_N) \cong H_2(M, \mathbb{Z}_N) \\
\text{again a sum over sym defects} & \quad D(M) \times D(M) = \times \sum_{S \in H_2(M, \mathbb{Z}_N)} \mathbb{Z}_N^{(\oplus)} \text{ sym defects} \\
\text{x = normalization} & = \frac{|H^0(M \times I, O(M \times I), \mathbb{Z}_N)|}{|H^1(\cdots)|} = \frac{1}{N}
\end{align*}
\]
\[ D(M) \times \eta = \eta \times D(M) = D(M) \quad ; \quad D(M) \times \overline{D(M)} = \frac{1}{N} \sum_{S \in H_2(M, \mathbb{Z}_N)} \eta(S) \]

\[ \left( \mathbb{Z}_N^{\ast} = G^{\ast} \right) \]

Key property

\[ \mathbb{Z}_N \]

charge loop

Example: \( U(1) \) Maxwell

\[ \text{defined by } \tau = \frac{4\pi i}{g^2} + \frac{\Theta}{2\pi} \quad ; \quad U(1)^{\ast}_c \times U(1)_m^{\ast} \]

\[ \text{gauge } \mathbb{Z}_N^{\ast} \leq U(1)^{\ast}_c \Rightarrow A \rightarrow A/\mathbb{Z}_N \Rightarrow \tau \rightarrow \tau/\mathbb{Z}_N \]

connection

\[ U(1) \quad U(1)/\mathbb{Z}_N \cong U(1) \]
Use EM duality: \( \mathfrak{z} \sim \frac{a \mathbb{Z} + b}{c \mathbb{Z} + d} \) \( a, b, c, d \in \mathbb{Z} \)
\( ad - bc = 1 \)

g.s. S-dual \(-\frac{1}{\mathfrak{z}} = \frac{\mathfrak{z}}{N^2} \implies \mathfrak{z} = iN \) \( (\theta = 0 \) \( g^2 = \frac{4\pi}{N} \)

so at special \( \mathfrak{z} \) we have duality defect D

\[
S = \frac{N}{4\pi} \int_{x < 0} dA_L \wedge x \times dA_L + \frac{N}{4\pi} \int_{x > 0} dA_R \wedge x \times dA_R + \frac{iN}{2\pi} \int_{x = 0} dA_L \wedge dA_R \] (Gaiotto-Witten)

\[
EOM \quad dA_L = -i x dA_R = \frac{1}{N} d\tilde{A}_R
\]

\[
E \mathbb{R} \mathbb{R} \quad \mathfrak{z} = iN
\]

Further Examples
- $\mathfrak{so}(\infty)$ YM has $\mathbb{Z}_2 \times \mathbb{Z}_2$ duality defects

- $\mathfrak{su}(N)$ $X=\mathbb{H}$ SYM at $\tau=i$ $\mathfrak{su}(N) \cong \mathfrak{su}(N)/\mathbb{Z}_N$

- $\mathfrak{psu}(N)$ YM at $\Theta=\pi$

- $0$ spacetime $Y \times S^1 \times S^1$