Canonical Bases for Coulomb Branches

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WHCCoP
12/14/20
Aim of the talk: Describe a tensor category $K_{c,w}(R_{\alpha,N})$ of "Koszul-perverse" coherent sheaves associated to a complex reductive group $G$ and a complex representation $N$.

Motivating features:

1) Provides a mathematical definition of the category of $\frac{1}{2}$-BPS line defects in a 4d $\mathcal{N}=2$ gauge theory with polarizable matter (following Kapustin-Saulina, Gaiotto-Moore-Neitzke).

2) Induces a canonical basis in the associated $\mathbb{R}^3 \times S^1$ Coulomb branch algebra and its quantization (following Braverman-Finkelberg-Nakajima).

3) Exhibits interesting internal structures which connect it to other topics in representation theory, such as affine quantum groups and quiver Hecke algebras.
$\text{KP}_{\text{con}}^{\text{GL}_2} (R_{\text{GL}_2, \alpha^2}) : \{ \text{irreducible objects} \} \leftrightarrow \{ (\lambda', \lambda) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \} / S_2$

Prime irreducibles, up to units:

Edges $\Rightarrow$ commutativity

Ex: $P_{-1,0} \longrightarrow P_{1,0}$

$\Rightarrow P_{1,0} \star P_{-1,0} \cong P_{-1,0} \star P_{1,0}$

Squares $\Rightarrow$ exact sequences

Ex: $P_{2,0} \longrightarrow P_{1,1}$

$\begin{array}{c}
0 \rightarrow P_{1,0} \rightarrow P_{2,0} \star P_{-1,0} \rightarrow P_{1,0} \rightarrow 0 \\
\begin{array}{c}
\downarrow \\
P_{1,0} \star P_{-1,0}
\end{array}
\end{array}$

$\begin{array}{c}
0 \leftrightarrow P_{1,0} \leftrightarrow P_{-1,0} \star P_{2,0} \leftrightarrow P_{1,1} \leftrightarrow 0 \\
\begin{array}{c}
\uparrow \\
P_{1,0}
\end{array}
\end{array}$
Coulomb branches (after Braverman–Finkelberg–Nakajima)

- $K := C((t))$, $G := C[[t]]$ as affine Grassmannian $\text{Gr}_G := G/K/G_0$
- Representation $N$ by infinite-rank bundles $V_1, V_2$ over $\text{Gr}_G$:
  $V_1 := \text{Gr}_G \times N_0$, $V_2 := \text{Gr}_G \times N_0$, $W := \text{Gr}_G \times N_K$

Def (BFN) Let $R_{\alpha,\nu}$ denote the intersection $V_1 \cap V_2$. The $K$-theoretic Coulomb branch algebra is $K^{G_0}(R_{\alpha,\nu})$ with ring structure given by convolution.

$\text{Ex}: \text{RC}^G_c = \bigoplus_{n \in \mathbb{Z}} K^G \, e^n$

$\text{Remark:}$ $V_1$ and $V_2$ not transverse in $W$ \Rightarrow derived structure on $R_{\alpha,\nu}$.

$\text{Ex}: O_K[t^n] = t^{-1}O \times (K/t^{-1}O)[[t]]$

$\text{Sym} (t^nO[[t]] \oplus (K/t^{-1}O))$

$\text{Decoh}(O_K[t^n]) = \{ \text{differential graded } C[O_K[t^n]] \text{-modules such that } H^0(K) \text{ is finitely presented over } H^0(C[O_K[t^n]]) \}$ quasi-isomorphism
Question: What kind of category $C_{\alpha,N}$ would have 
$K_0(C_{\alpha,N}) \cong K^{\alpha_0}(R\alpha,N)$ and define a basis of irreducibles? 
($D_{\text{coh}}^{\alpha_0}(R\alpha,N), Coh^{\alpha_0}(R\alpha,N)$ satisfy the former but not the latter.)

Ansatz: $C_{\alpha,N}$ is the heart of a nonstandard finite-length $t$-structure on $D_{\text{coh}}^{\alpha_0}(R\alpha,N)$ ($\Rightarrow$ abelian subcategory with finite composition series and $K_0(C_{\alpha,N}) \cong K_0(D_{\text{coh}}^{\alpha_0}(R\alpha,N))$).

Guiding example: pure gauge theory ($R\alpha_0 \cong \text{Gr}_\alpha$) 
Bezrukavnikov-Finkelberg-Mirkovic: category $P_{\text{coh}}^{\alpha_0}(\text{Gr}_\alpha) \cap D_{\text{coh}}^{\alpha_0}(\text{Gr}_\alpha)$ 
of perverse coherent sheaves on $\text{Gr}_\alpha$.

Ex: $i_\alpha': \text{Gr}_\alpha \to \text{Gr}_\alpha$ closed $\alpha_0$-orbit $\Rightarrow$ for any $\mathbb{N} \neq \mathbb{Z}$
\[ i_{\alpha'}: O(n)[\frac{1}{2} \dim \text{Gr}_\alpha] \text{ is perverse and irreducible.} \]

Features:

1. (BFM) finite length, closed under convolution and Serre duality, and $\exists$ irreducible objects $\leftrightarrow \exists \{P^\vee, P\} \in P^\vee \times P$.

2. (Curtis-W.) has duals and renormalized $r$-matrices, categories physically expected cluster algebra in type A.
Generalization to $N \neq 0$ (implicit description):

- Auxiliary choice: central subgroup $C^* \subset G$ acting on $N$ with weight one.
- Defines a weight decomposition $D_{coh}(Gr_G) \cong \bigoplus_{n \in \mathbb{Z}} D_{coh}(Gr_G)^{<n>}$.
- Can twist the perverse t-structure on $D_{coh}(Gr_G)$ by shifting its weight $n$ component by cohomological degree $n$.

Thm (Cautis-W.) There is a unique t-structure on $D_{coh}(R_{a,n})$ such that

$$D_{coh}(R_{a,n}) \xrightarrow{i_*} D_{coh}(Gr_G \times \mathbb{N}_0) \xrightarrow{\sigma^*} D_{coh}(Gr_G)$$

is t-exact with respect to the twisted perverse t-structure.

Def: $KP_{coh}(R_{a,n})$ is the heart of this t-structure.

Thm (Cautis-W.) $KP_{coh}(R_{a,n})$ is finite length, closed under convolution and duals, has renormalized S-matrices, and its irreducible objects are in bijection with those of $P_{coh}(Gr_G)$.

Ex: $i_!: R^2_{a,n} \hookrightarrow R_{a,n}$ restriction of $R_{a,n}$ to closed orbit $Gr_G \times Gr_G$

$w \mapsto i_! \tau^* \omega_{Gr_G}(w) \left[ \frac{1}{2} \dim Gr_G^2 \right]$ is Koszul-pervasive and irreducible.
Generalization to $\mathbb{N} \neq 0$ (explicit description):

- Recall the bundles $V_1 := \mathcal{G}_\alpha \times N_\mathbb{N}$, $V_2 := \mathcal{G}_\kappa \times N_\mathbb{N}$, $W := \mathcal{G}_\alpha \times N_\kappa$.
- We use the same notation for their sheaves of sections, and consider $V_2^\perp = \{ \phi \in W^* \text{ such that } \phi(V) = 0 \text{ for } v \in V_2 \}$
- Define a sheaf of algebras
  \[ \text{Cliff}_{\alpha, N} := \mathcal{T}_{\mathcal{O}_{\mathcal{G}_\alpha}}(V_2^\perp[1] \oplus V_1[-1])/(\phi, v)^2 - \phi(v) \]
  
- Fiber at $[v] \in \mathcal{G}_\alpha$ is the Clifford algebra associated to $N_\mathbb{N} \oplus gN_\mathbb{N} \subset N_\kappa \oplus N_\kappa$ with its residue pairing.

Fact: tensoring with vector bundles preserves $P_{\text{coh}}^{\mathcal{G}_\alpha}(\mathcal{G}_\alpha)$

$\Rightarrow$ tensoring with $\text{Cliff}_{\alpha, N}$ preserves $P_{\text{coh}}^{\mathcal{G}_\alpha}(\mathcal{G}_\alpha)^{\text{tw}}$

$\Rightarrow$ well-defined category of $\text{Cliff}_{\alpha, N}$-module objects in $P_{\text{coh}}^{\mathcal{G}_\alpha}(\mathcal{G}_\alpha)^{\text{tw}}$.

Thm. (Cautis-W.) $P_{\text{coh}}^{\mathcal{G}_\alpha}(\mathcal{R}_{\alpha N})$ is equivalent to the category of twisted perverse coherent sheaves of $\text{Cliff}_{\alpha, N}$-modules on $\mathcal{G}_\alpha$. 
Relation to Koszul duality

- Mirkovic-Riche: vector bundles $V_1, V_2 \in \mathcal{W}$ over $X \mapsto$ equivalence $D_{\text{coh}}(V_1 \boxdot V_2) \cong D_{\text{coh}}(\mathcal{O}_{\mathcal{W}}(V_1^+ \boxdot V_2^+)^{(2)})$

\[
\mathcal{C}[\frac{V_1 \boxdot V_2}{12}] \leftrightarrow \text{linear Koszul duality} \quad \mathcal{C}[\frac{(V_1^+ \boxdot V_2^+)^{(2)}}{12}]
\]
\[
\text{Sym}_{\mathcal{O}_X}^* (V_2^{\perp} \to V_1^*) \leftrightarrow \text{Cliff}_{\mathcal{O}_X} (V_2^{\perp} \oplus V_1[-1]) \leftrightarrow \text{Sym}_{\mathcal{O}_X}^* (V_1[-1] \to \mathcal{W}/V_2[-2])
\]

- $\text{Cliff}_{\mathcal{O}_X}(V_2^{\perp} \oplus V_1[-1])$ appears as an intermediate form of this duality, so Koszul-parverse coherent sheaves are "sheaves which become parverse after a form of Koszul duality."
\( R_{c, c} \), revisited

- Let \( R_n := O_K^* t^n O, \quad P_n := O_{R_n}^{\text{red}} \)

- Irreducibles in \( K \mathcal{P} \text{con}; \exists P_n[l] < l > ? \)

- Easy computation:
  \[
P_i \ast P_{-1} = i_\ast (O_{Z_i - i-1}) \quad P_i \ast P_i = i_\ast (O_{Z_{i-1} - i})
\]
  where we consider closed subschemes
  \[
  Z_{i-1} := tO \quad Z_{i-1} := tO
  \]

- Koszul resolution of \( Z_{i-1} \xrightarrow{\text{codim 1}} R_0 \):\[
  0 \rightarrow P_0[l] \rightarrow P_0 \rightarrow P_i \ast P_{-1} \rightarrow 0 \quad (\text{std})
  \]
  \[
  0 \rightarrow P_0 \rightarrow P_i \ast P_{-1} \rightarrow P_0[l] < l > \rightarrow 0 \quad (\text{KP})
  \]

- Isomorphism \( O_{Z_{i-1}} \cong O_{R_0} \otimes \text{Sym}(O(tO[l])) \):
  \[
  0 \rightarrow P_0[l] < l > \rightarrow P_{-1} \ast P_i \rightarrow P_0 \rightarrow 0 \quad (\text{KP})
  \]

- Koszul perspective: symmetry of extensions reflects symmetry between factors of

\[
\text{Cliff}(O^+[1] \oplus O^{-1}[1]) \cong \text{Sym}^*(O^+[1]) \otimes \text{Sym}^*(O^{-1}[1])
\]