Canonical Bases for Coulomb Branches

Harold Williams
University of Southern California

joint with Sabin Cautis

WHCGP
12/14/20

Aim of the talk: Describe a tensor category $KP^{G_O}_{coh}(R_{G,N})$ of "Koszul-perverse" coherent sheaves associated to a complex reductive group G and a complex representation N.

Motivating features:

1) Provides a mathematical definition of the category of $\frac{1}{2}$-BPS line defects in a 4d $\mathcal{N}=2$ gauge theory with polarizable matter (following Kapustin-Saulina, Gaiotto-Moore-Neitzke).

2) Induces a canonical basis in the associated $\mathbb{R}^3 \times S^1$ Coulomb branch algebra and its quantization (following Braverman-Finkelberg-Nakajima).

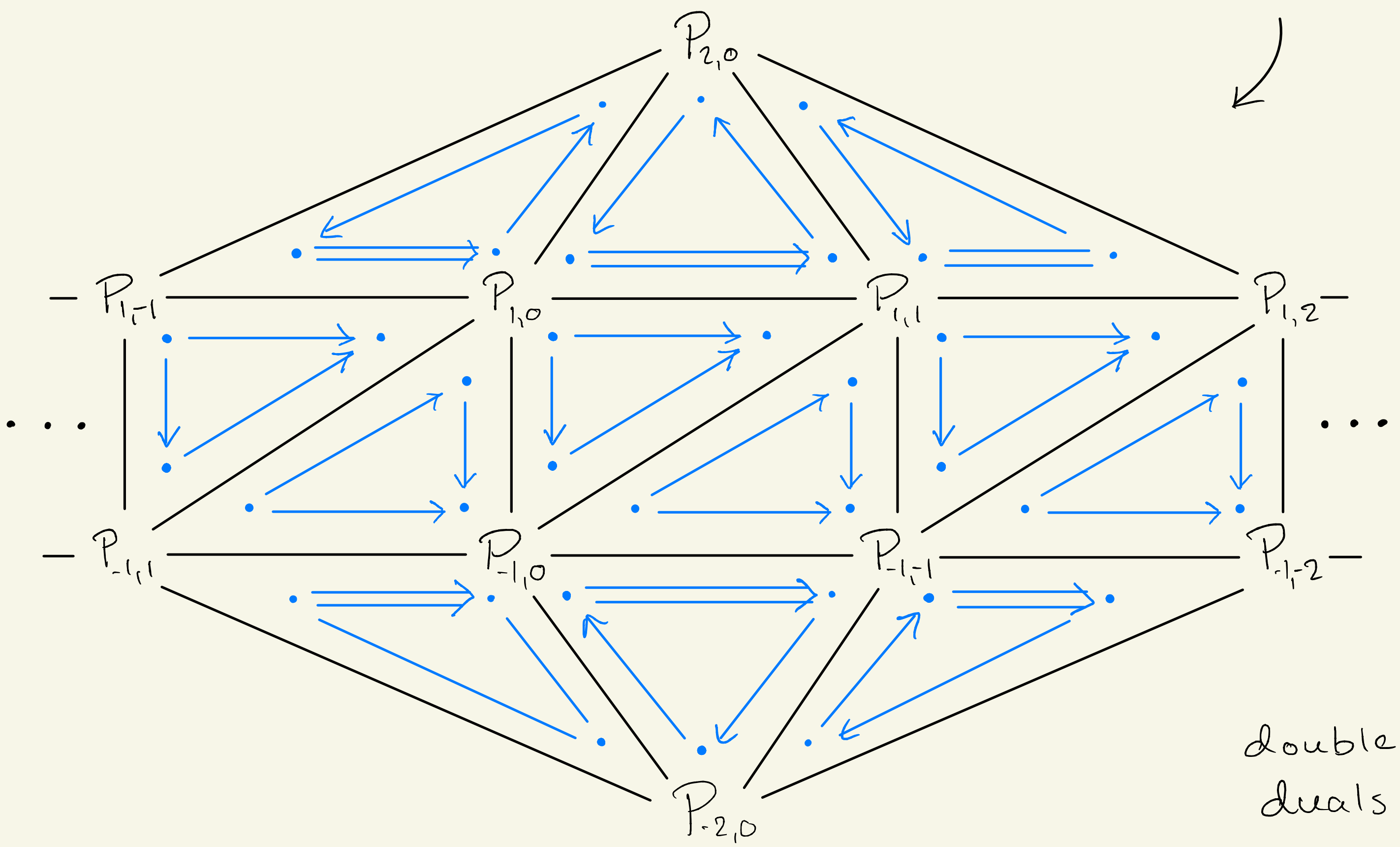
3) Exhibits interesting internal structures which connect it to other topics in representation theory, such as affine quantum groups and quiver Hecke algebras

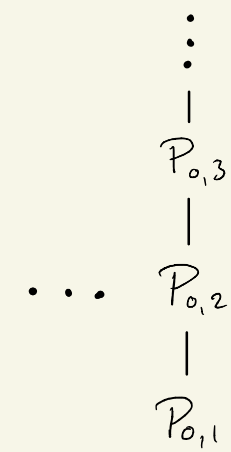
$KP^{GL_{2,0}}_{coh}(R_{GL_2,\mathbb{C}^2})$: $\{\text{irreducible objects}\} \leftrightarrow \{(\lambda^\vee, \lambda) \in \mathbb{Z}^2 \times \mathbb{Z}^2\}/S_2$

Prime irreducibles, up to units:

P^{*2} is irreducible

P^{*2} not irreducible





Rep GL_2

double duals $\rightsquigarrow$ translation symmetry

Edges $\Rightarrow$ commutativity

Ex: $P_{1,0}$ ——— $P_{-1,0}$

$\Rightarrow P_{1,0} * P_{-1,0} \simeq P_{-1,0} * P_{1,0}$

Squares $\Longrightarrow$ exact sequences

Ex:
$$\begin{array}{ccc} P_{2,0} & \text{———} & P_{1,1} \\ | & \diagup & | \\ P_{1,0} & \text{———} & P_{-1,0} \end{array} \Longrightarrow \left\{ \begin{array}{l} 0 \to P_{1,0} \to P_{2,0} * P_{-1,0} \to P_{1,1} \to 0 \\ \qquad\qquad\qquad \not\parallel \\ 0 \leftarrow P_{1,0} \leftarrow P_{-1,0} * P_{2,0} \leftarrow P_{1,1} \leftarrow 0 \end{array} \right.$$

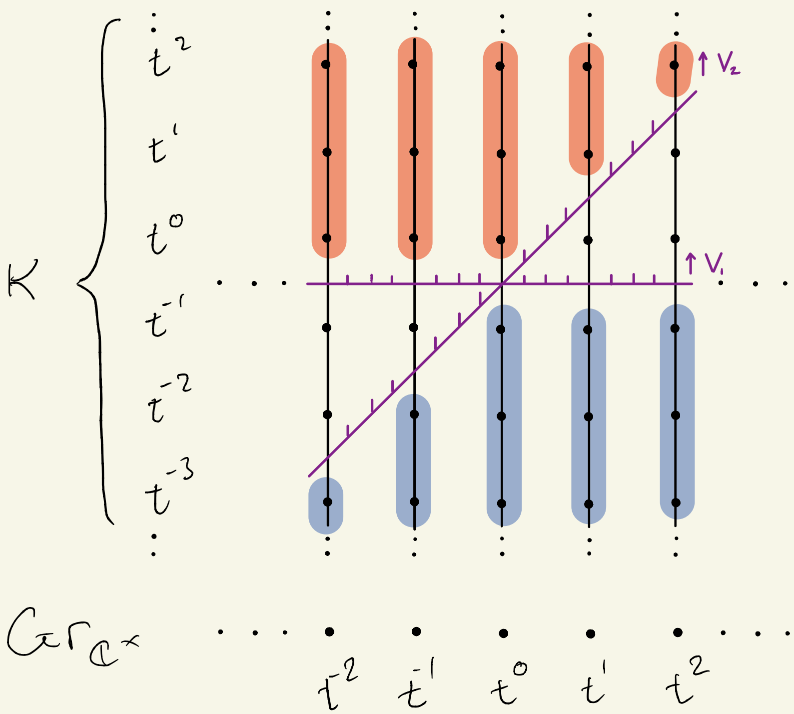
Coulomb branches (after Braverman–Finkelberg–Nakajima)

- $K := \mathbb{C}((t))$, $\mathcal{O} := \mathbb{C}[[t]]$ $\rightsquigarrow$ affine Grassmannian $Gr_G := G_K / G_{\mathcal{O}}$
- Representation N $\rightsquigarrow$ infinite-rank bundles $V_1, V_2 \subset W$ over Gr_G:

$$V_1 := Gr_G \times N_{\mathcal{O}} \qquad V_2 := G_K \times_{G_{\mathcal{O}}} N_{\mathcal{O}} \qquad W := Gr_G \times N_K$$

Def (BFN) Let $R_{G,N}$ denote the intersection $V_1 \cap_W V_2$. The K-theoretic Coulomb branch algebra is $K^{G_{\mathcal{O}}}(R_{G,N})$ with ring structure given by convolution.

Ex $R_{\mathbb{C}^\times, \mathbb{C}} = \coprod_{n \in \mathbb{Z}} \mathcal{O} \cap_K t^n \mathcal{O}$



Remark: V_1 and V_2 not transverse in W $\Longrightarrow$ derived structure on $R_{G,N}$.

Ex: $\mathcal{O} \cap_K t^n \mathcal{O} \simeq t^{n_+}\mathcal{O} \times (K/t^{n_-}\mathcal{O})[-1]$

$$\rightsquigarrow \mathbb{C}[\mathcal{O} \cap_K t^n \mathcal{O}] \simeq \mathrm{Sym}^\bullet\big(\underbrace{t^{-n_-}\mathcal{O}[1] \oplus (K/t^{-n_+}\mathcal{O})}\big)$$

$\left(\begin{array}{l} n_+ = \max\{0,n\} \\ n_- = \min\{0,n\} \end{array}\right)$ $\qquad \simeq t^{-n}\mathcal{O}[1] \xrightarrow{d} K/\mathcal{O}$

$D\mathrm{coh}(\mathcal{O} \cap_K t^n \mathcal{O}) = \{$ differential graded $\mathbb{C}[\mathcal{O} \cap_K t^n \mathcal{O}]$-modules such that $H^\bullet(M)$ is finitely presented over $H^0(\mathbb{C}[\mathcal{O} \cap_K t^n \mathcal{O}])\} /$ quasi-iso

Question: What kind of category $\mathcal{C}_{G,N}$ would have $K_0(\mathcal{C}_{G,N}) \simeq K^{G_O}(R_{G,N})$ and define a basis of irreducibles?
($D^{G_O}_{coh}(R_{G,N})$, $Coh^{G_O}(R_{G,N})$ satisfy the former but not the latter.)

Ansatz: $\mathcal{C}_{G,N}$ is the heart of a nonstandard finite-length t-structure on $D^{G_O}_{coh}(R_{G,N})$ ($\Rightarrow$ abelian subcategory with finite composition series and $K_0(\mathcal{C}_{G,N}) \simeq K_0(D^{G_O}_{coh}(R_{G,N}))$).

Guiding example: pure gauge theory ($R_{G,0} \simeq Gr_G$)

Bezrukavnikov-Finkelberg-Mirkovic: category $P^{G_O}_{coh}(Gr_G) \subset D^{G_O}_{coh}(Gr_G)$ of perverse coherent sheaves on Gr_G.

Ex: $i_\lambda : Gr^\lambda \hookrightarrow Gr_G$ closed G_O-orbit $\Rightarrow$ for any $n \in \mathbb{Z}$ $i_{\lambda *}\mathcal{O}(n)[\frac{1}{2}\dim Gr^\lambda]$ is perverse and irreducible.

Features:

1. (BFM) finite length, closed under convolution and Serre duality, and $\{\text{irreducible objects}\} \leftrightarrow \{(\lambda^\vee, \lambda) \in P^\vee \times P\}/W$.
2. (Cautis-W.) has duals and renormalized r-matrices, categorifies physically expected cluster algebra in type A.

Generalization to $N \neq 0$ (implicit description):

- Auxiliary choice: central subgroup $\mathbb{C}^\times \subset G$ acting on N with weight one.
- Defines a weight decomposition $D^{G_O}_{coh}(Gr_G) \simeq \bigoplus_{n \in \mathbb{Z}} D^{G_O}_{coh}(Gr_G)^{\langle n \rangle}$.
- Can twist the perverse t-structure on $D^{G_O}_{coh}(Gr_G)$ by shifting its weight n component by cohomological degree n.

Thm (Cautis-W.) There is a unique t-structure on $D^{G_O}_{coh}(R_{G,N})$ such that

$$D^{G_O}_{coh}(R_{G,N}) \xrightarrow{i_*} D^{G_O}_{coh}(Gr_G \times N_O) \xrightarrow{\sigma^*} D^{G_O}_{coh}(Gr_G)$$

is t-exact with respect to the twisted perverse t-structure.

$$\begin{tikzcd} R_{G,N} \arrow[d, "i"', "\text{defining map}"] \\ Gr_G \times N_O \\ Gr_G \arrow[u, "\sigma", "\text{zero section}"'] \end{tikzcd}$$

Def: $KP^{G_O}_{coh}(R_{G,N})$ is the heart of this t-structure

Thm (Cautis-W.) $KP^{G_O}_{coh}(R_{G,N})$ is finite length, closed under convolution and duals, has renormalized r-matrices, and its irreducible objects are in bijection with those of $P^{G_O}_{coh}(Gr_G)$.

Ex: $i_\lambda : R^\lambda \hookrightarrow R_{G,N}$ restriction of $R_{G,N}$ to closed orbit $Gr^\lambda \subset Gr_G$

$\rightsquigarrow i_{\lambda *} \mathcal{O}_{R^{red}_\lambda}(n)[\frac{1}{2} \dim Gr^\lambda]$ is Koszul-perverse and irreducible.

Generalization to $N \neq 0$ (explicit description):

- Recall the bundles $V_1 := Gr_G \times N_O$, $V_2 := G_K \times_{G_O} N_O$, $W := Gr_G \times N_K$.
- We use the same notation for their sheaves of sections, and consider $V_2^\perp = \{\phi \in W^* \text{ such that } \phi(v) = 0 \text{ for } v \in V_2\}$
- Define a sheaf of algebras

$$Cliff_{G,N} = T^\bullet_{\mathcal{O}_{Gr_G}}(V_2^\perp[1] \oplus V_1[-1]) / ((\phi, v)^2 - \phi(v))$$

- Fiber at $[g] \in Gr_G$ is the Clifford algebra associated to $N_O^* \oplus gN_O \subset N_K^* \oplus N_K$ with its residue pairing.

Fact: tensoring with vector bundles preserves $P^{G_O}_{coh}(Gr_G)$
$\Rightarrow$ tensoring with $Cliff_{G,N}$ preserves $P^{G_O}_{coh}(Gr_G)^{tw}$
$\Rightarrow$ well-defined category of $Cliff_{G,N}$-module objects in $P^{G_O}_{coh}(Gr_G)^{tw}$.

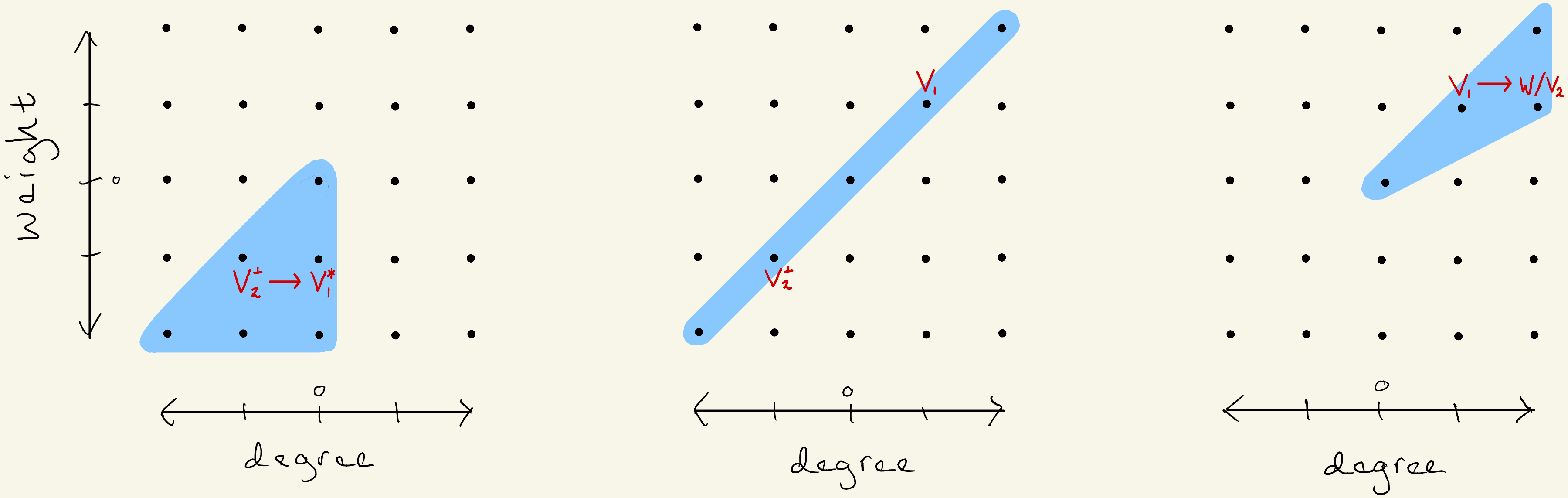
Thm: (Cautis-W.) $KP^{G_O}_{coh}(R_{G,N})$ is equivalent to the category of twisted perverse coherent sheaves of $Cliff_{G,N}$-modules on Gr_G.

Relation to Koszul duality

• Mirković-Riche: vector bundles $V_1, V_2 \subset W$ over X $\rightsquigarrow$

$\rightsquigarrow$ equivalence $D^{\mathbb{C}^\times}_{coh}(V_1 \cap_W V_2) \simeq D^{\mathbb{C}^\times}_{coh}((V_1^\perp \cap_{W^*} V_2^\perp)[2])$

$$\begin{array}{ccccc} \mathbb{C}[V_1 \cap_W V_2] & & \xleftrightarrow{\text{linear Koszul duality}} & & \mathbb{C}[(V_1^\perp \cap_{W^*} V_2^\perp)[2]] \\ \wr\!\| & & & & \wr\!\| \\ \mathrm{Sym}^\bullet_{\mathcal{O}_X}(V_2^\perp[1] \to V_1^*) & \leftrightarrow & \mathrm{Cliff}_{\mathcal{O}_X}(V_2^\perp[1] \oplus V_1[-1]) & \leftrightarrow & \mathrm{Sym}^\bullet_{\mathcal{O}_X}(V_1[-1] \to W/V_2[-2]) \end{array}$$



• $\mathrm{Cliff}_{\mathcal{O}_X}(V_2^\perp[1] \oplus V_1[-1])$ appears as an intermediate form of this duality, so Koszul-perverse coherent sheaves are "sheaves which become perverse after a form of Koszul duality".

$R_{\mathbb{C}^\times,\mathbb{C}}$ revisited

- Let $R_n := \mathcal{O} \cap_K t^n \mathcal{O}$, $P_n := \mathcal{O}_{R_n^{cl}}$
- Irreducibles in KP_{coh}: $\{P_n[\ell]\langle -\ell\rangle\}$
- Easy computation:

$$P_1 * P_{-1} = i_* \mathcal{O}_{Z_{1,-1}}, \quad P_{-1} * P_1 = i_* \mathcal{O}_{Z_{-1,1}}$$

where we consider closed subschemes

$$Z_{1,-1} := t\mathcal{O} \subset R_0 := \mathcal{O} \cap_K \mathcal{O}$$
$$Z_{-1,1} := \mathcal{O} \cap_{t^{-1}\mathcal{O}} \mathcal{O} \subset R_0$$

- Koszul resolution of $Z_{1,-1} \xrightarrow{\text{codim } 1} R_0^{cl}$:

$$0 \to P_0\langle 1\rangle \to P_0 \to P_1 * P_{-1} \to 0 \quad (\text{std})$$
$$0 \to P_0 \to P_1 * P_{-1} \to P_0[1]\langle 1\rangle \to 0 \quad (KP)$$

- Isomorphism $\mathcal{O}_{Z_{-1,1}} \simeq \mathcal{O}_{R_0^{cl}} \otimes \mathrm{Sym}^\bullet(\mathcal{O}/t\mathcal{O}[1])$:

$$0 \to P_0[1]\langle -1\rangle \to P_{-1} * P_1 \to P_0 \to 0 \quad (KP)$$

- Koszul perspective: symmetry of extensions reflects symmetry between factors of

$$\mathrm{Cliff}(\mathcal{O}^+[1] \oplus \mathcal{O}[-1]) \simeq \mathrm{Sym}^\bullet(\mathcal{O}^+[1]) \otimes \mathrm{Sym}^\bullet(\mathcal{O}[-1])$$

Ex $R_{\mathbb{C}^\times,\mathbb{C}} = \coprod_{n\in\mathbb{Z}} \mathcal{O} \cap_K t^n \mathcal{O}$