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MODULI SPACES OF PRINCIPAL 2-GROUP BUNDLES AND A CATEGORIFICATION OF THE FREED-QUINN LINE BUNDLE

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Plan: 1) Context / Motivation

2) Principal 2-group bundles: definitions and key properties

3) Applications

§1. CONTEXT AND MOTIVATION

Let G be a compact Lie group and fix $\alpha \in H^3(G, U(1))$

⇝ there are lots of fun things we can do with this data.

→ • Chern-Simons theory

• G-equivariant elliptic cohomology

→ • String structures

• Equivariant gerbes

• Higher representation theory

⋮

☆ Smooth two-group $\mathcal{G} = \mathcal{G}(G, U(1), \alpha)$ and its moduli space of principal 2-group bundles

Expectation: All of the above topics are meaningfully related to the space $\mathrm{Bun}_{\mathcal{G}}(X)$.

- Today we focus mainly on the case that G is finite

Fix input data: G - finite group

A - an abelian Lie group with trivial G-action (e.g. $A = U(1)$)

$\alpha : G^{\times 3} \longrightarrow A$ a 3-cocycle.

Chern-Simons story: $(A = U(1))$

↳ Dijkgraaf-Witten theory.

Freed-Quinn: construct a line bundle $\mathcal{L}$ on the moduli space of principal G-bundles on Riemann surfaces.

- For X a Riemann surface, we get a line bundle $\mathcal{L}_X$ on $Bun_G(X)$.
- $\Gamma(Bun_G(X), \mathcal{L}_X)$ is exactly the vector space that Chern-Simons assigns to X.

Elliptic aside Restricting $\mathcal{L}$ to the moduli space of principal G-bundles over elliptic curves, we obtain $\tilde{\mathcal{L}}$.

Ganter: $\tilde{\mathcal{L}}$ is the natural home of twisted G-equivariant elliptic cohomology.

2-groups story $[\alpha] \in H^3(G; A)$ classifies finite 2-groups that are central extensions of G by $*/\!/A$.

- To the representative α, we associate $\mathcal{G}$ with
  - objects $g \in G$
  - morphisms $Hom_{\mathcal{G}}(g,h) = \begin{cases} \emptyset & g \neq h \\ A & g = h \end{cases}$
  - $\otimes$-structure: $g \otimes h = gh$.

associativity: $(g \otimes h) \otimes k \xrightarrow{\sim} g \otimes (h \otimes k)$
$\uparrow \alpha(g,h,k) \in A$

- $\mathcal{G}$ can be viewed as a smooth 2-group, depending on the smooth structure of A.

  (i.e. a group object in Bibun)

<u>Today</u>
- we'll define a moduli space (bicategory) $\mathrm{Bun}_{\mathcal{G}}(X)$
- we'll see there is a nice map $\mathrm{Bun}_{\mathcal{G}}(X) \to \mathrm{Bun}_G(X)$
- when X is a Riemann surface, this categorifies $\mathcal{L}_X$.
  - Sections of $\mathcal{L}_X$ are then isomorphism classes of lifts from $\mathrm{Bun}_G(X)$ to $\mathrm{Bun}_{\mathcal{G}}(X)$.
- also applications to string structures/string geometry.

§2. Principal 2-group bundles

Fix a smooth (finite) 2-group $\mathcal{G}$ and smooth manifold.

Goal Define a bicategory of smooth 2-group bundles over X

<u>Our favourite definition</u>: A principal $\mathcal{G}$-bundle on X is a smooth stack $\mathcal{P} \to X$ equipped with an action of $\mathcal{G}$ which is locally trivial:

$\exists$ surjective submersion $u: Y \longrightarrow X$ and an isomorphism of $\mathcal{G}$-stacks over Y:

$$d: u^* \mathcal{P} \xrightarrow{\sim} Y \times \mathcal{G}.$$

<u>Čech data for $\mathcal{P}$</u> - tells us how to glue $\mathcal{G}$-bundles from the trivial bundle on an open cover Y.

<u>note</u>: the first level of gluing data is a bibundle Φ

$$Y \times_X Y \times \mathcal{G} \xrightarrow{\Phi} Y \times_X Y \times \mathcal{G}$$

which we can assume is trivial as an A-bundle

- $u: Y \to X$
- $g: Y \times_X Y \to G$ satisfying "cocycle" conditions.
- $\gamma: Y \times_X Y \times_X Y \to A$

Example: An A-gerbe over X is a principal $*/\!/A$-bundle over X.

Čech data: $u: Y \to X$

$\gamma: Y \times_X Y \times_X Y \to A$ a 2-cocycle.

Definition An A-2-gerbe is determined by

$u: Y \to X$

$\lambda: Y \times_X Y \times_X Y \times_X Y \to A$ a 3-cocycle.

Observe We have a forgetful functor $\pi: \mathrm{Bun}_{\mathcal{G}}(X) \to \mathrm{Bun}_G(X)$

in terms of Čech data: $(u, g, \gamma) \mapsto (u, g)$

Theorem [Berwick-Evans, C, Murray, Nakade, Phillips]

$\pi: \mathrm{Bun}_{\mathcal{G}}(X) \to \mathrm{Bun}_G(X)$ is a torsor over the symmetric monoidal bicategory $\mathrm{Gerbe}_A(X)$.

Sketch of proof The fibre over $\overset{P_{\shortparallel}}{(u, g)}$ is given by the γ's that complete the triple (u, g, γ).

- P determines a 3-cocycle $\lambda_{P,\alpha} = g^* \alpha: Y^4_X \to A$:

$(y_1, y_2, y_3, y_4) \mapsto \alpha(g(y_1, y_2), g(y_2, y_3), g(y_3, y_4))$

i.e. a 2-gerbe. ("Chern-Simons 2-gerbe")

Claims 1) the data of γ is equivalent to a trivialisation of this 2-gerbe $\lambda_{P,\alpha}$

2) the bicategory of trivialisations of a fixed 2-gerbe is a torsor over the symmetric monoidal bicategory of gerbes. $\square$

Principal 2-group bundles in terms of classifying stacks:

Expectation: $\mathcal{G}$-bundles are classified by maps $X \to B\mathcal{G}$.

$$\begin{tikzcd}
X \arrow[dr, dashed, "(g,\gamma)"] \arrow[drr, bend left, "\text{no data}"] \arrow[ddr, bend right, "g"'] & & \\
& B\mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
& BG \arrow[r, "\alpha"'] & B^3A
\end{tikzcd}$$

the diagram should commute:

$\gamma: \; g^*\alpha = \lambda_{P,\alpha} \cong \mathrm{Triv}$

Definition: A flat $\mathcal{G}$-bundle is a principal $\mathcal{G}_\delta$-bundle

discrete topology.

Recall: Flat principal G-bundles are classified by homomorphisms

$$\pi_1(X) \to G$$

Theorem [BCMNP]

For X with contractible universal cover:

$$\begin{tikzcd}
\mathrm{Bun}^b_{\mathcal{G}}(X) \arrow[r, "\sim"] \arrow[d, "\pi"] & \mathrm{Hom}_{\mathrm{Bicat}}(*/\!/\pi_1(X),\, */\!/\mathcal{G}) \arrow[d, "\pi"] \\
\mathrm{Bun}^b_G(X) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{Cat}}(*/\!/\pi_1(X),\, */\!/G) \cong \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(X), G)/\!/G
\end{tikzcd}$$

A homomorphism $\pi_1(X) \to \mathcal{G}$:

- $g: \pi_1(X) \to G$ a homomorphism

- For all $a, b \in \pi_1(X)$. $\quad \varrho(a)\varrho(b) \xrightarrow{\sim} \varrho(ab) \in \mathcal{G}$

  $\gamma(a,b) \in A$ + cocycle condition

A natural transformation $(\varrho_1,\gamma_1) \Rightarrow (\varrho_2,\gamma_2)$:

- $t \in G$ s.t. $\forall\, a \in G$, $\quad t\varrho_1(a)t^{-1} = \varrho_2(a)$
- $\forall\, a \in G$, $\quad t\varrho_1(a) \xrightarrow{\sim} \varrho_2(a)t \in \mathcal{G}$

  $\eta(a) \in A$ + cocycle condition

- 2-morphisms are given by $\omega \in A$.

Theorem [BCMNP] The action of G on $\mathrm{Hom}_{grp}(\pi_1(X), G)$ lifts to an action of G on the bicategory $\mathrm{Hom}_{bicat}(*/\!/\pi_1(X), */\!/\mathcal{G})$. This gives π the structure of a cloven 2-fibration.

§3. APPLICATIONS

3.A. Freed-Quinn line bundle $\quad (A = U(1),\ X$ a Riemann surface$)$

- We've seen that $\pi : \mathrm{Bun}^b_{\mathcal{G}}(X) \to \mathrm{Bun}^b_G(X)$ is a cloven 2-fibration with fibres equivalent to $\mathrm{Gerbe}_{U(1)_\delta}(X)$

- We can now take isomorphism classes along fibres.

  - Since isomorphism classes of gerbes are given by

    $H^2(X, U(1)) \cong U(1)$,

    we obtain a principal $U(1)$-bundle on $\mathrm{Bun}^b_G(X)$.

Theorem [BCMNP] The associated line bundle is the Freed-Quinn line bundle $\mathcal{L}_X$.

(cf. Willerton for X a torus)

Recent work in progress [Berwick-Evans, C.]

- we study categorical tori $\mathbb{T}$

$$1 \to *//U(1) \to \mathbb{T} \to \underset{\text{compact torus}}{T} \to 1 \qquad \text{(cf. Ganter)}$$

- we have

$$\begin{array}{c} \mathrm{Fun}_{\text{bicat}}\left(*//\pi_1(\mathbb{T}^2),\ *//\mathbb{T} \right) \\ \big\downarrow \\ \mathrm{Bun}^{\flat}_{T}(\mathbb{T}^2) = \mathrm{Fun}_{\text{cat}}\left(*//\mathbb{Z}^2,\ *//T \right) \cong T \times T \end{array}$$

<u>Theorem</u> (?) This fibration categorifies the line bundle from Chern-Simons, with curvature equal to the Atiyah-Bott symplectic form.

§3.B <u>string structures</u>

- We work with the string 2-group (not finite!) (cf Schommer-Pries)

$$1 \to *//U(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1.$$

- Starting from a finite group representation $\rho_0 : G_0 \to SO(n)$, we produce a 2-group:

$$\begin{tikzcd}
*//U(1) \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{String}(n) \arrow[d] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r] & G \arrow[r, "\rho"'] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spin}(n) \arrow[d] \\
& G_0 \arrow[r, "\rho_0"'] & SO(n)
\end{tikzcd}$$

<u>Recall</u> Let $P_0 \to X$ be an oriented vector bundle with structure group G.

$\Rightarrow$ a spin structure on P_0 is a lift to a G-bundle P.

<u>Definition</u> [Schommer-Pries] a string structure on P is a lift to a $\mathcal{G}$-bundle $\mathcal{P}$

Alternative definition [Waldorf] a string structure on P is a trivialization of the 2-gerbe $\lambda_{P,\alpha}$

Theorem [BCMNP] The two definitions coincide.

- Let $P \to X$ be a principal flat G-bundle and consider Chern-Simons theory for P, CS_P.

Definition [Stolz-Teichner] a geometric string structure on P is a trivialization of CS_P.

- in particular, for suitable $f: M^2 \to X$, $CS_P(f)$ is a line, and we require a non-zero point in this line.

  i.e. an isomorphism class of flat $\mathcal{G}$-bundle over f^*P

  - this could be given by $f^*\mathcal{P}$ for $\mathcal{P}$ a flat lift of P.

So (part of the data of a trivialization of CS_P)

$$\updownarrow$$

(part of the data of $\mathcal{P} \in \pi^{-1}(P) \subset \mathrm{Bun}^{\flat}_{\mathcal{G}}(X)$)

Work in progress: complete this story.

Thank you!