joint work with Dan Berwick-Evans
Laura Murray
Apuva Narade
Emma Phillips

Thanks to the AMS NRC on geometric representation theory
and equivariant elliptic cohomology.

Plan:
1) Context / Motivation
2) Principal 2-group bundles: definitions and key properties
3) Applications

§1. CONTEXT AND MOTIVATION

Let \( G \) be a compact Lie group and fix \( \alpha \in H^3(\mathcal{G}, U(1)) \)

\( \Rightarrow \) there are lots of fun things we can do with this data.

\( \Rightarrow \) Chern-Simons theory

\( \Rightarrow \) G-equivariant elliptic cohomology

\( \Rightarrow \) String structures

\( \Rightarrow \) Equivariant gerbes

\( \Rightarrow \) Higher representation theory

\( \Rightarrow \) Smooth two-group \( \mathcal{G} = \mathcal{G}(G, U(1)), \alpha \)

and its moduli space of principal 2-group bundles

Expectation

All of the above topics are meaningfully related to the space \( \text{Bun}_{\mathcal{G}}(X) \).
Today we focus mainly on the case that $G$ is finite.

**Fix input data:**
- $G$: finite group
- $A$: an abelian Lie group with trivial $G$-action (e.g., $A = U(1)$)
- $\alpha: G^3 \to A$ a $3$-cocycle.

**Chern–Simons story:**
- $(A = U(1))$
- Dijkgraaf–Witten theory.

**Freed–Quinn:** construct a line bundle $L$ on the moduli space of principal $G$-bundles on Riemann surfaces.

- For $X$ a Riemann surface, we get a line bundle $L_X$ on $\text{Bun}_G(X)$.

- $\Gamma(\text{Bun}_G(X), L_X)$ is exactly the vector space that Chern–Simons assigns to $X$.

**Elliptic aside:** Restricting $L$ to the moduli space of principal $G$-bundles over elliptic curves, we obtain $\tilde{L}$.

**Ganter:** $\tilde{L}$ is the natural home of twisted $G$-equivariant elliptic cohomology.

**2-groups story:** $\left[ \alpha \right] \in H^3(G; A)$ classifies finite 2-groups that are central extensions of $G$ by $\frac{1}{2}A$.
- To the representative $\alpha$, we associate $\hat{g}$ with
  - objects $g \in G$
  - morphisms $\text{Hom}(g, h) = \{ \emptyset, g \neq h \}$
  - $\otimes$-structure: $g \otimes h = gh$.
associativity: \[(g_0 \circ h_0) \circ k \overset{\sim}{\rightarrow} g_0 \circ (h_0 \circ k)\]

\[\Leftrightarrow \alpha(g_0, h_0, k) \in A\]

- \(G\) can be viewed as a smooth 2-group, depending on the smooth structure of \(A\).

(i.e. a group object in \(\text{Bibun}\))

Today

- we’ll define a moduli space (bicategory) \(\text{Bun}_G(X)\)
- we’ll see there is a nice map \(\text{Bun}_G(X) \rightarrow \text{Bun}_G(X)\)
- when \(X\) is a Riemann surface, this categories \(\mathcal{L}_X\).
- Sections of \(\mathcal{L}_X\) are men isomorphism classes of lifts from \(\text{Bun}_G(X)\) to \(\text{Bun}_G(X)\).
- also applications to string structures/string geometry.

§2. Principal 2-group bundles

Fix a smooth (finite) 2-group \(G\) and smooth manifold.

Goal: Define a bicategory of smooth 2-group bundles over \(X\)

**Our favourite definition:** A principal \(G\)-bundle on \(X\) is a smooth stack \(\mathcal{P} \rightarrow X\) equipped with an action of \(G\) which is locally trivial:

- a surjective submersion \(u: Y \rightarrow X\) and an isomorphism of \(G\)-stacks over \(Y\):
  \[d: u^* \mathcal{P} \overset{\sim}{\rightarrow} Y \times G\]

Cech data for \(\mathcal{P}\) - tells us how to glue \(G\)-bundles from the trivial bundle on an open over \(Y\).

Note: The first level of gluing data is a bibundle \(\Phi\)

\[Y \times Y \times G \overset{\Phi}{\rightarrow} Y \times Y \times G\]
which we can assume is trivial as an $A$-bundle

- $u : Y \rightarrow X$
- $g : Y \times Y \rightarrow G$ satisfying "cocycle" conditions.
- $\gamma : Y \times Y \times Y \rightarrow A$

**Example**: An $A$-gerbe over $X$ is a principal $\#A$-bundle over $X$.

**Čech data**:

\begin{align*}
u : Y & \rightarrow X \\
\gamma : Y \times Y \times Y & \rightarrow A \text{ a 2-cocycle}.
\end{align*}

**Definition**: An $A$-2-gerbe is determined by

\begin{align*}
u : Y & \rightarrow X \\
\lambda : Y \times Y \times Y \times Y & \rightarrow A \text{ a 3-cocycle}.
\end{align*}

**Observe**: We have a forgetful functor $\pi_c : \text{Bun}_g(X) \rightarrow \text{Bun}_G(X)$ in terms of Čech data: $(u, g, \gamma) \mapsto (u, g)$

**Theorem**: [Bereich-Evans, C., Murray, Nakade, Phillips]

$\pi_c : \text{Bun}_g(X) \rightarrow \text{Bun}_G(X)$ is a torsor over the symmetric monoidal bicategory $\text{Gerbe}_A(X)$.

**Sketch of proof**: The fibre over $(u, g)$ is given by the $\gamma$'s that complete the triple $(u, g, \gamma)$.

- $\gamma$ determines a 3-cocycle $\lambda_{p,x} = \gamma^x : Y_x^3 \rightarrow A$:

\begin{align*}
(y, y_2, y_3, y_4) & \mapsto \alpha(g(y, y_3), g(y_2, y_3), g(y_3, y_4)) \\
i.e. & \text{ a 2-gerbe}. (* \text{Chern- Simons 2-gerbe} *)
\end{align*}

**Claims**: 1) the data of $\gamma$ is equivalent to a trivialisation
of this 2-gerbe \( \lambda_{p, \alpha} \)

2) The bicategory of trivializations of a fixed 2-gerbe is a topos over the symmetric monoidal bicategory of genres.

**Principal 2-group bundles in terms of classifying stacks:**

**Expectation:** \( \mathcal{G} \)-bundles are classified by maps \( X \to \mathcal{B}\mathcal{G} \).

![Diagram](image)

**Definition:** A flat \( \mathcal{G} \)-bundle is a principal \( \mathcal{B}\mathcal{G} \)-bundle with discrete topology.

**Recall:** Flat principal \( \mathcal{G} \)-bundles are classified by homomorphisms \( \pi_1(X) \to \mathcal{G} \).

**Theorem [BCHNP]**

For \( X \) with contractible universal cover:

\[
\begin{array}{ccc}
\Bun_{\mathcal{B}\mathcal{G}}^b(X) & \overset{\sim}{\longrightarrow} & \Hom_{\mathcal{B}\text{Cat}} ( \ast \!/ \pi_1(X), \ast \!/ \mathcal{G} ) \\
\downarrow \pi & & \downarrow \pi \\
\Bun_{\mathcal{G}}^b(X) & \overset{\sim}{\longrightarrow} & \Hom_{\text{Cat}} ( \ast \!/ \pi_1(X), \ast \!/ \mathcal{G} ) \cong \Hom_{\text{Grp}} ( \pi_1(X), \mathcal{G} ) \cong \mathbb{G}
\end{array}
\]

A homomorphism \( \pi_1(X) \to \mathcal{G} \):

* \( g : \pi_1(X) \to G \) a homomorphism
• For all \( a, b \in \Pi(X) \), \( g(a)g(b) \xrightarrow{\gamma(a,b)} g(ab) \in \mathcal{G} \)
\[ \gamma(a,b) \in A \quad \text{+ cocycle condition} \]

A natural transformation \( (\gamma_1, \gamma_2) \Rightarrow (\gamma_3, \gamma_4) \):

• \( t \in G \) s.t. \( \forall a \in G, \ t \gamma_1(a) t^{-1} = \gamma_2(a) \)
• \( \forall a \in G, \ t \gamma_3(a) \xrightarrow{\gamma_4(a)} \gamma_3(a)t \in \mathcal{G} \)

\[ \gamma(a) \in A \quad \text{+ cocycle condition} \]

• 2-morphisms are given by \( \gamma \in A \).

**Theorem [BCMNP]** The action of \( G \) on \( \text{Hom}_{\text{grp}}(\Pi(X), G) \) lifts to an action of \( G \) on the bicategory \( \text{Hom}_{\text{bicat}}(\ast//\Pi(X), \ast//G) \).

This gives \( \mathcal{T} \) the structure of a cloven 2-fibration.

§3. APPLICATIONS

3.1. Freed-Quinn line bundle \( (A = UC_1), X \) a Riemann surface)

• We've seen that \( \mathcal{T} : \text{Bun}^b_G(X) \to \text{Bun}^b_G(X) \) is a cloven 2-fibration

with fibres equivalent to \( \text{Gerbe}_{\text{UC}_1}(X) \)

• We can now take isomorphism classes along fibres.

• Since isomorphism classes of gerbes are given by

\[ H^2(X, UC_1) \cong UC_1, \]

we obtain a principal \( UC_1 \)-bundle on \( \text{Bun}^b_G(X) \).

**Theorem [BCMNP]** The associated line bundle is the Freed-Quinn line bundle \( \mathcal{L}_X \).

(cf. Willerton for \( X \) a torus)
Recent work in progress [Benwick-Evans, C.]

- We study categorical tori $\mathbb{T}^n$ from compact torus $1 \rightarrow \mathbb{C}/\mathbb{C}^\times \rightarrow \mathbb{T}^n \rightarrow \mathbb{T} \rightarrow 1$ (cf. Ganter)

- We have $\text{Fun}_{\text{bicat}} (\mathbb{C}/\mathbb{C}^\times, \mathbb{C}/\mathbb{T}) \\ \downarrow \text{Fun}_{\text{cat}} (\mathbb{C}/\mathbb{Z}, \mathbb{C}/\mathbb{T}) \cong \mathbb{T} \times \mathbb{T}$

Theorem (?) This fibration categorifies the line bundle from Chern-Simons, with curvature equal to the Atiyah-Bott symplectic form.

§3.B string structures

- We work with the string 2-group (not finite!) (cf. Schommer-Pries)

$1 \rightarrow \mathbb{C}/\mathbb{C}^\times \rightarrow \text{String}(n) \rightarrow \text{Spin}(n) \rightarrow 1$.

- Starting from a finite group representation $\rho_0 : \rho_0 \rightarrow \text{SO}(n)$,

we produce a 2-group:

\[
\begin{array}{ccccc}
\mathbb{C}/\mathbb{C}^\times & \rightarrow & \mathbb{C}/\mathbb{T} & \rightarrow & \text{String}(n) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/\mathbb{Z} & \rightarrow & \mathbb{G} & \rightarrow & \text{Spin}(n) \\
\downarrow & & \downarrow & & \downarrow \\
\rho_0 & \rightarrow & \text{SO}(n) \\
\end{array}
\]

Recall Let $P_0 \rightarrow X$ be an oriented vector bundle with structure group $G$.

- A spin structure on $P_0$ is a lift to a $G$-bundle $P$.

Definition [Schommer-Pries] a string structure on $P$ is a lift to a $\mathbb{C}/\mathbb{T}$-bundle $P$.
Alternative definition [Waldorf] a string structure on $P$ is a trivialization of the 2-gerbe $\lambda_{P,\alpha}$.

Theorem [BCHNP] The two definitions coincide.

* Let $P \rightarrow X$ be a principal flat $G$-bundle and consider Chern-Simons theory for $P$, $CS_p$.

Definition [Stolz-Teichner] a geometric string structure on $P$ is a trivialization of $CS_p$.

* in particular, for suitable $f: M^2 \rightarrow X$, $CS_p(f)$ is a line, and we require a non-zero point in this line.
  
i.e. an isomorphism class of flat $G$-bundle over $f^*\mathcal{P}$
* this could be given by $f^*\mathcal{P}$ for $\mathcal{P}$ a flat lift of $P$.

So (part of the data of a trivialization of $CS_p$)

\[
\uparrow
\]

(part of the data of $\mathcal{D} \in \pi^{-1}(P) \subset \text{Bun}_G^b(X)$)

Work in progress: complete this story.

#

Thank you!