Partition functions, BPS states

and abelianization

Lotte Hollands

Edinburgh

HERIOT WATT UNIVERSITY

THE ROYAL SOCIETY

Based on:

2109.14699 (w/ P. Rüter and R. Szabo)

+ previous works w/ A. Neitzke and O. Kidwai

(also see [2105.03777] by Grassi-Hao-Neitzke)

String-Math '17 ; context 4d $N=2$ theories in $\frac{1}{2}\Omega$-background

while pointing out parallels with recent work in the context of topological string theory:

in particular 2109.06878 (Murad Alim, Arpan Saha, Jörg Teschner, Iván Tulli)

Instanton partition function [Nekrasov, Losev-Moore-Nekrasov-Shatashvili]

for four-dim'l $\mathcal{N}=2$ gauge theory:

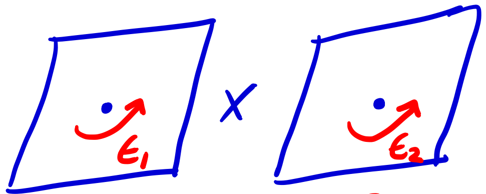
$$Z_{inst} = \sum_k q^k \oint_{\mathcal{M}_k} 1 \qquad \left(Z_{inst} = \sum_k q^k \oint_{\mathcal{M}_k} e(V \otimes \mathcal{L} \otimes M) \right)$$

instanton parameter

moduli space of instantons

matter contributions

$\downarrow$ in Ω-background



$\mathbb{R}^2_{\epsilon_1}$ $\times$ $\mathbb{R}^2_{\epsilon_2}$

$$Z^{Nek}_{4d}(q; a; m; \epsilon_1, \epsilon_2) = Z_{cl}\, Z_{1\text{-loop}}\, Z_{inst}$$

$\exp\left(-\frac{1}{\epsilon_1 \epsilon_2} \tau a^2\right)$

Barnes' double gamma functions $\Gamma_2(x \mid \epsilon_1, \epsilon_2)$

sum over Young diagrams:

$$\sum_{\vec{Y}} q^{|\vec{Y}|} z_{vector}(\vec{a}, \vec{Y})\, z_{matter}(\vec{a}, \vec{Y}, m)$$

$$Z^{Nek}(q;a;m;\epsilon_1,\epsilon_2) = Z_{cl}\, Z_{1\text{-loop}}\, Z_{inst}$$

$\exp\left(-\frac{1}{\epsilon_1\epsilon_2}\tau a^2\right)$

Barnes' double gamma functions $\Gamma_2(x|\epsilon_1,\epsilon_2)$

sum over Young diagrams:

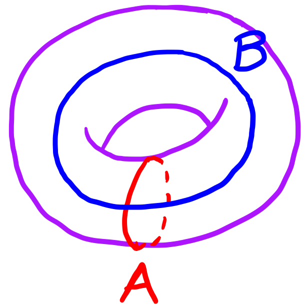
$$\sum_{\vec{Y}} q^{|\vec{Y}|} z_{vector}(\vec{a},\vec{Y})\, z_{matter}(\vec{a},\vec{Y},m)$$

$$\log Z^{Nek} = \frac{1}{\epsilon_1\epsilon_2} F_0(q;a;m) + \text{terms regular in } \epsilon_1,\epsilon_2$$

hol'c prepotential: [Seiberg-Witten]

$$a = \oint_A \lambda\,, \quad a_D = \oint_B \lambda$$

$$a_D = \frac{\partial F_0(q;a;m)}{\partial a}$$



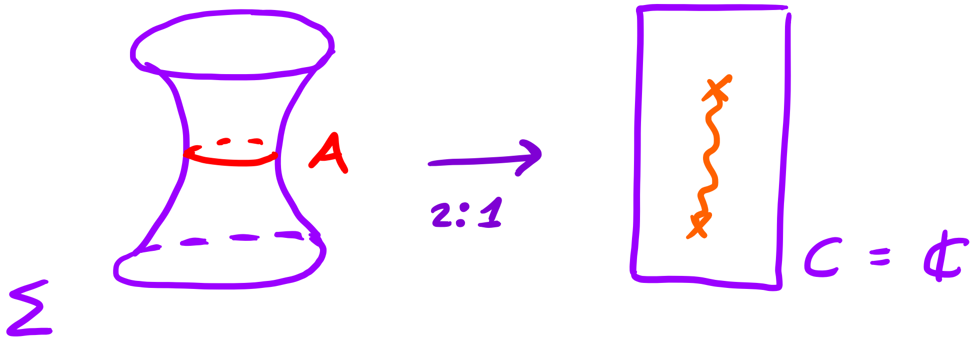
Seiberg-Witten curve Σ

purely algebraic

[Nekrasov-Okounkov, Braverman-Etingof, Nakajima-Yoshioka] consider four-dim'l $\mathcal{N}=2$ theory of class S: [Gaiotto, Gaiotto-Moore-Neitzke]

SW curve Σ is spectral curve of Hitchin integrable system $\mathcal{M}_H(C)$

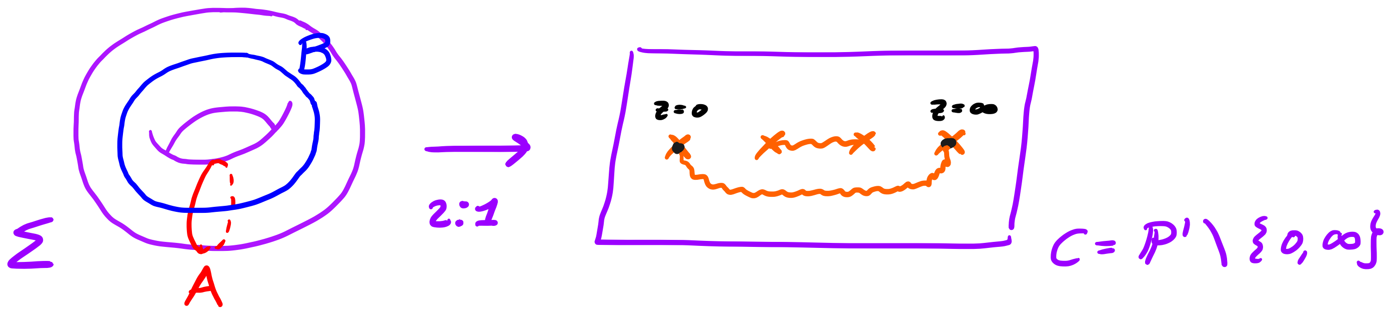
Ex. AD_2 theory



$\lambda = w\,dz$

$$w^2 = z^2 + m$$

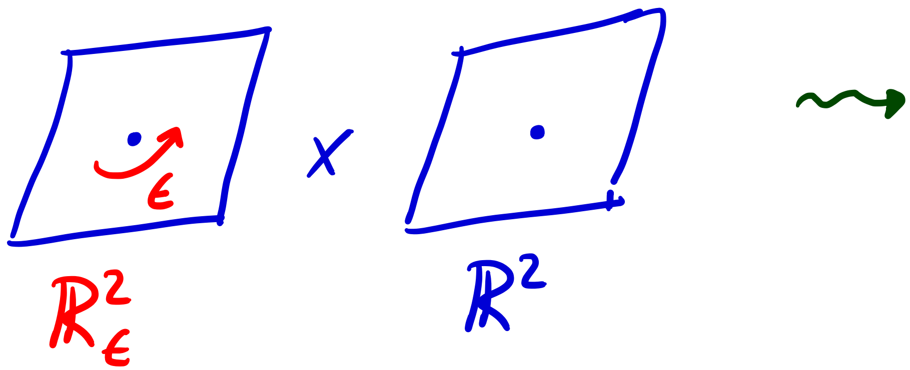
Ex. pure SU(2) theory



$$w^2 = \frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$

Nekrasov-Shatashvili limit $\epsilon_2 \to 0$: [Nekrasov-Shatashvili]

4d $\mathcal{N}=2$ theory on



effective 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{R}^2$ with

$$\boxed{\begin{aligned}\frac{1}{\epsilon}\widetilde{W}^{\text{eff}} &= \lim_{\epsilon_2\to 0} \epsilon_2 \log Z^{\text{Nek}} \\ &= \frac{1}{\epsilon}\mathcal{F}_0 + \dots\end{aligned}}$$

also known as $\mathcal{F}^{4d}_{NS}$

Ex. AD_2 theory: $W^{4d}_{NS} = \frac{\epsilon}{2}\Upsilon\left(\frac{1}{2}+\frac{m}{2\epsilon}\right)$

$$\Upsilon(x) = \int_{1/2}^{x} \log\frac{\Gamma(x')}{\Gamma(1-x')}\,dx'$$

Ex. pure $SU(2)$ theory:

$$W^{4d}_{NS} = \frac{a^2}{\epsilon}\log\left(\frac{\Lambda}{\epsilon}\right) - \frac{\epsilon}{2}\Upsilon\left(-\frac{a}{\epsilon}\right) - \frac{\epsilon}{2}\Upsilon\left(\frac{a}{\epsilon}\right) + \frac{2\Lambda^4}{\epsilon(a^2-\epsilon^2)} + \frac{(5a^2+7\epsilon^2)\Lambda^8}{\epsilon(a^2-4\epsilon^2)(a^2-\epsilon^2)} + O(\Lambda^{12})$$

Nekrasov-Shatashvili limit $\epsilon_2 \to 0$: [Nekrasov-Shatashvili]

effective 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{R}^2$ has geometric description as a quantization of Hitchin integrable system $\hbar = \epsilon$

Ex. pure SU(2) theory:

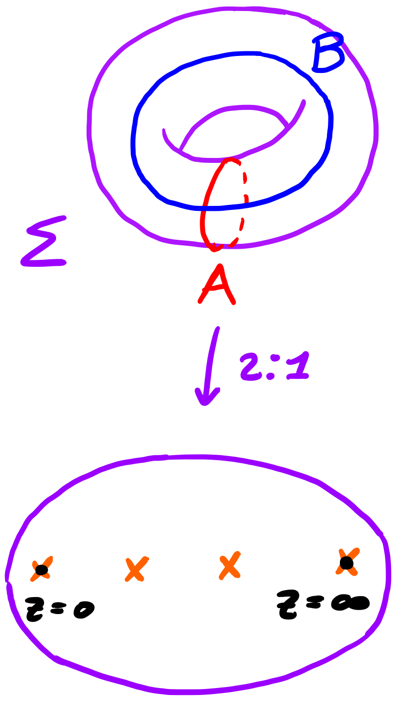
quantum Hamiltonian is sl(2)-oper:

[Beilinson-Drinfeld]

$$D_\epsilon = \epsilon^2 \partial_z^2 - \left(\frac{\Lambda^2}{z^3} - \frac{2u + \epsilon^2/4}{z^2} + \frac{\Lambda^2}{z} \right)$$

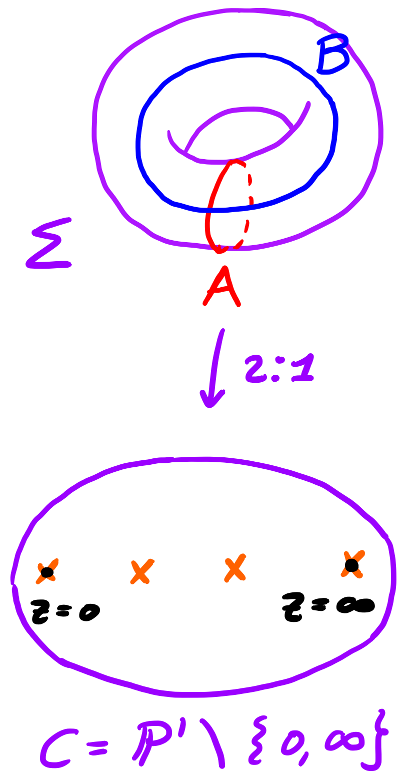
$$\Big\downarrow \; \epsilon \to 0$$

$$\Sigma: \quad w^2 = \frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$



$C = \mathbb{P}^1 \setminus \{0, \infty\}$

Ex. pure SU(2) theory:



spectral problem:

$$\epsilon^2 \frac{\partial^2 \psi(z)}{\partial z^2} - \left(\frac{\Lambda^2}{z^3} - \frac{2u + \epsilon^2/4}{z^2} + \frac{\Lambda^2}{z} \right) \psi(z) = 0$$

with $\psi(z) = e^{\frac{1}{\epsilon} \int S(z') dz'}$

quantum periods: [Mironov - Morozov]

$$a_\epsilon = \frac{1}{\epsilon} \oint_A S(z) dz$$

$$a_{D,\epsilon} = \frac{1}{\epsilon} \oint_B S(z) dz$$

→ ϵ-expansion of $W_{NS}^{4d}(\Lambda; a; \epsilon)$ through

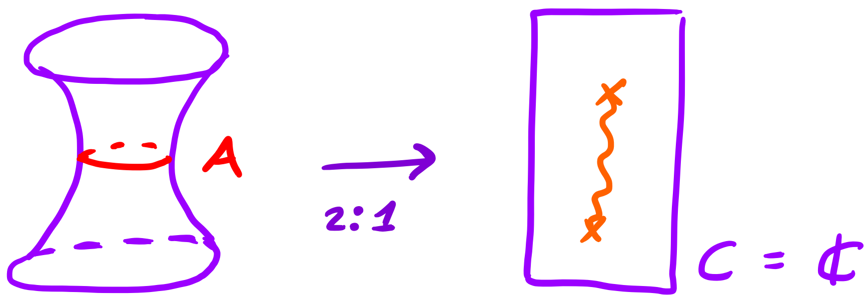
$$\boxed{a_{D,\epsilon} \cong \frac{\partial W_{NS}^{4d}}{\partial a_\epsilon}}$$

$\xrightarrow{\epsilon \to 0}$ $a_D = \frac{\partial F_0}{\partial a}$

Can we obtain exact results as well?

Example: AD_2 theory [M-R-S]

single free hyper of mass m



$$\Sigma : w^2 = z^2 + m$$

oper: $D_\epsilon \psi = \epsilon^2 \psi''(z) - (z^2+m)\psi(z) = 0$

Schrödinger eqn for complex harmonic oscillator

AD_2 oper : $D_\epsilon \psi = \epsilon^2 \psi''(z) - (z^2+m)\psi(z) = 0$

consider the solutions:

$$\psi_1(z) = D_\mu\left(\sqrt{\tfrac{2}{\epsilon}}\, z\right) \qquad \psi_2(z) = e^{-\frac{\pi i}{2}(\mu+1)} D_{-\mu-1}\left(-\sqrt{\tfrac{2}{\epsilon}}\, i z\right)$$

$$\psi_3(z) = e^{\pi i \mu} D_\mu\left(-\sqrt{\tfrac{2}{\epsilon}}\, z\right) \qquad \psi_4(z) = e^{-\frac{\pi i}{2}(3\mu-1)} D_{-\mu-1}\left(\sqrt{\tfrac{2}{\epsilon}}\, i z\right)$$

where $D_\mu(t) = e^{-t^2/4} \frac{1}{\Gamma(-\mu)} \int_0^\infty ds\, e^{-ts-\frac{s^2}{2}} s^{-\mu-1}$ $\qquad \mu = -\frac{m}{2\epsilon} - \frac{1}{2}$

(integral representation of Weber parabolic cylinder function)

Then: $$W_{NS}^{4d} = \frac{1}{2}\int dx \log \sqrt{\frac{[\psi_1,\psi_4][\psi_1,\psi_3]}{[\psi_2,\psi_4][\psi_1,\psi_2]}} \qquad [\psi_1,\psi_4] = Wr(\psi_1,\psi_4)$$

$$= \frac{\epsilon}{2}\, Y\left(\frac{1}{2} + \frac{m}{2\epsilon}\right)$$

Explanation: exact W^{4d}_{NS} is obtained as Borel sum

in direction $\arg(\epsilon) = \frac{\pi}{2}$

$$\boxed{\text{oper}: \quad D_\epsilon \psi = \epsilon^2 \psi''(z) - (z^2+m)\psi(z) = 0}$$

formal solutions:

$$\psi_-^{for}(z) = e^{-t^2/4}\, t^{\mu} \sum_{n=0}^{\infty} (-)^n \frac{\mu(\mu-1)\cdots(\mu-2n+1)}{n!\, 2^n\, t^{2n}}$$

$$\psi_+^{for}(z) = e^{t^2/4}\, t^{-\mu-1} \sum_{n=0}^{\infty} \frac{(\mu+1)(\mu+2)\cdots(\mu+2n)}{n!\, 2^n\, t^{2n}}$$

$t = \sqrt{\frac{2}{\epsilon}}\, z$

ψ_-^{for} decreases fastest along $t \in \mathbb{R}$

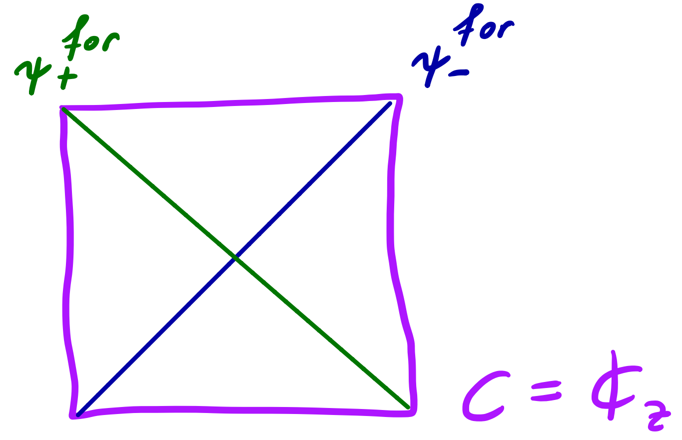
ψ_+^{for} decreases fastest along $t \in i\mathbb{R}$

Borel sum:

$$\left.\begin{aligned} f(t) &= \sum_{k=0}^{\infty} a_k t^k \\ \mathcal{B}f(s) &= \sum_{k=0}^{\infty} \frac{a_k}{k!} s^k \end{aligned}\right\} \quad S_\theta f(t) = \int_0^{\infty e^{i\theta}} e^{-s}\, \mathcal{B}f(ts)$$

Borel transform

in direction θ

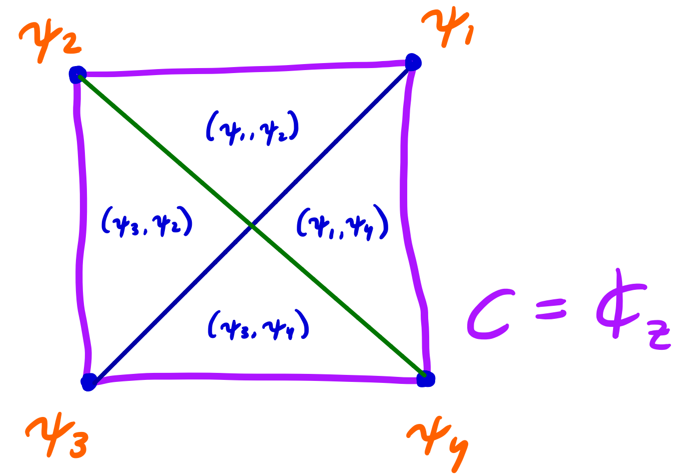


suppose $\theta = \arg(\epsilon) = \frac{\pi}{2}$ and $m > 0$:

ψ_-^{for} decreases fastest along $t \in \mathbb{R}$

ψ_+^{for} decreases fastest along $t \in i\mathbb{R}$

ψ_1 Borel sum of ψ_- for $t > 0$
ψ_2 " " " ψ_+ for $it < 0$
ψ_3 " " " ψ_- for $t < 0$
ψ_4 " " " ψ_+ for $it > 0$

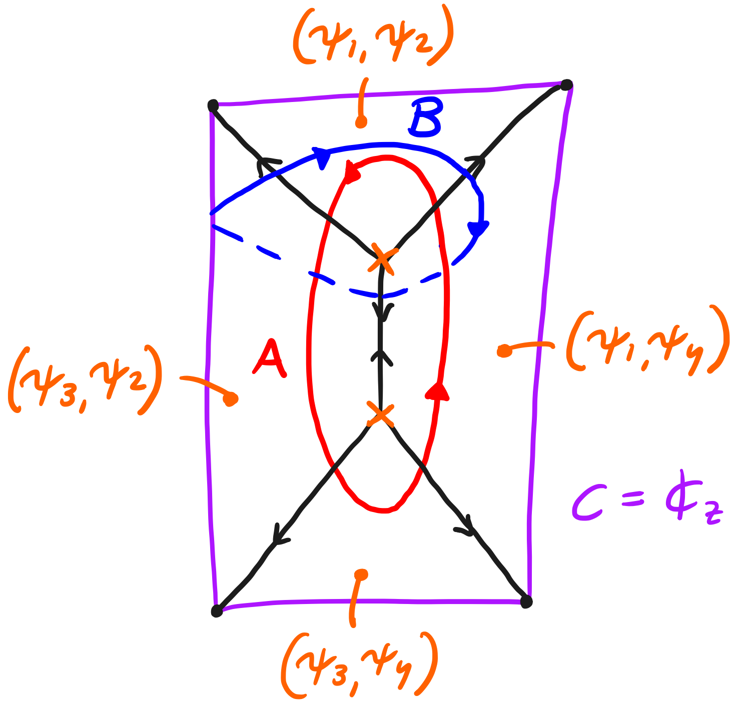


why $\vartheta = \arg(\epsilon) = \frac{\pi}{2}$?

special since $\arg(Z) = \frac{\pi}{2}$ for hyper

spectral network or
Stokes graph at $\vartheta = \frac{\pi}{2}$:

[Gaiotto-Moore-Neitzke]



trajectories $e^{-i\vartheta}\sqrt{z^2+m} \in \mathbb{R}^\times$

exact WKB

$\rightsquigarrow$

abelianization

[H-Neitzke]

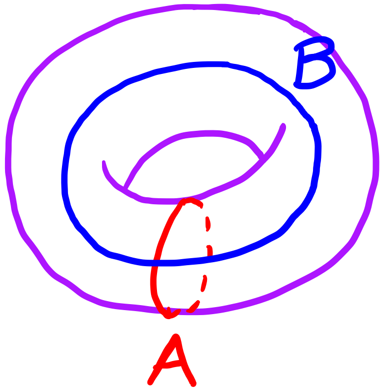
exact quantum periods:

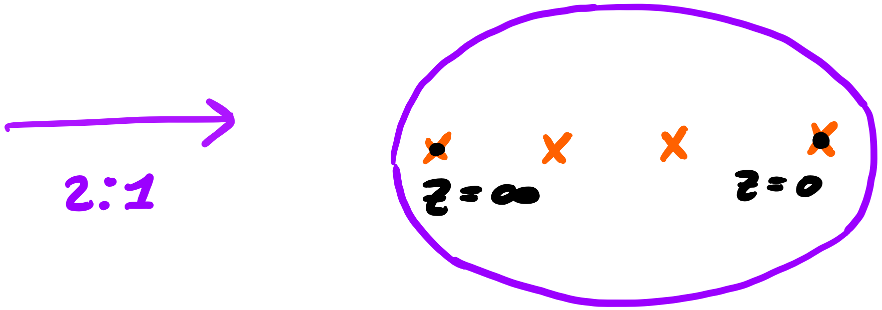
$$\mathcal{X}_A = \frac{Wr[\psi_1,\psi_2]\; Wr[\psi_3,\psi_4]}{Wr[\psi_3,\psi_2]\; Wr[\psi_1,\psi_4]} \equiv -e^{\frac{\pi i a_\epsilon^{ex}}{\epsilon}}$$

$$\mathcal{X}_B = \sqrt{\frac{Wr[\psi_1,\psi_4]\; Wr[\psi_1,\psi_3]}{Wr[\psi_2,\psi_4]\; Wr[\psi_1,\psi_2]}} \equiv e^{2\epsilon\, a_{D,\epsilon}^{ex}}$$

$$\boxed{a_{D,\epsilon}^{ex} = \frac{\partial W_{NS}^{4d}}{\partial a_\epsilon^{ex}}}$$

Example: pure $SU(2)$ theory [H-R-S]





$C = \mathbb{P}^1 \setminus \{0, \infty\}$

$$\Sigma : w^2 = \frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$

u Coulomb par

Λ UV scale

family of opers parametrized by

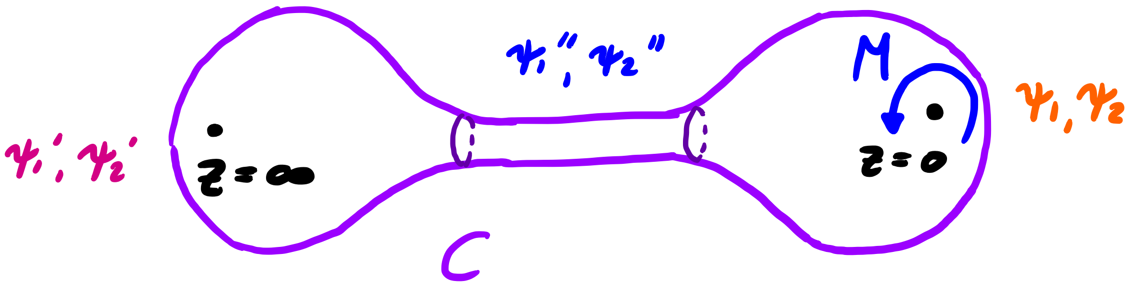
Mathieu differential eqn:

$$D_\epsilon \psi(z) = \epsilon^2 \psi''(z) - \left(\frac{\Lambda^2}{z^3} - \frac{2u + \epsilon^2/4}{z^2} + \frac{\Lambda^2}{z} \right) \psi(z) = 0$$

(note: u just free par here)

Example: pure SU(2) theory

weak coupling $|\Lambda^2/u| \ll 1$:



ψ_1 asympt small (when $z \to 0$ along neg real axis)

ψ_1' " " " (when $z \to \infty$ " ")

ψ_2 small eigenvector of M

for $\vartheta = \arg(\epsilon) = \pi/2$ and $-u \gg \Lambda^2$

$$W_{NS}^{4d} = \frac{1}{2}\int dx \log \sqrt{\frac{[\psi_1', \psi_1''][\psi_1, \psi_2'']}{[\psi_1, \psi_1''][\psi_1', \psi_2']}} = \frac{a^2}{\epsilon}\log\left(\frac{\Lambda}{\epsilon}\right) - \frac{\epsilon}{2}\gamma\left(-\frac{a}{\epsilon}\right) - \frac{\epsilon}{2}\gamma\left(\frac{a}{\epsilon}\right)$$

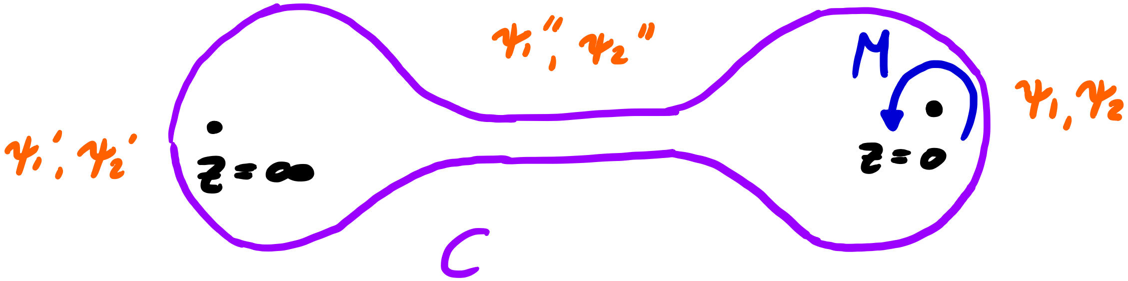
$$+ \frac{2\Lambda^4}{\epsilon(a^2-\epsilon^2)} + \frac{(5a^2+7\epsilon^2)\Lambda^8}{\epsilon(a^2-4\epsilon^2)(a^2-\epsilon^2)}$$

$$+ O(\Lambda^{12})$$

not very easy, but you can verify this in a perturbation in Λ [M-R-S]

Idea of computation: [H-R-S]

weak coupling $|\Lambda^2/u| \ll 1$:



- compute ψ_1'', ψ_2'' as eigenfunctions of M in series in Λ^2

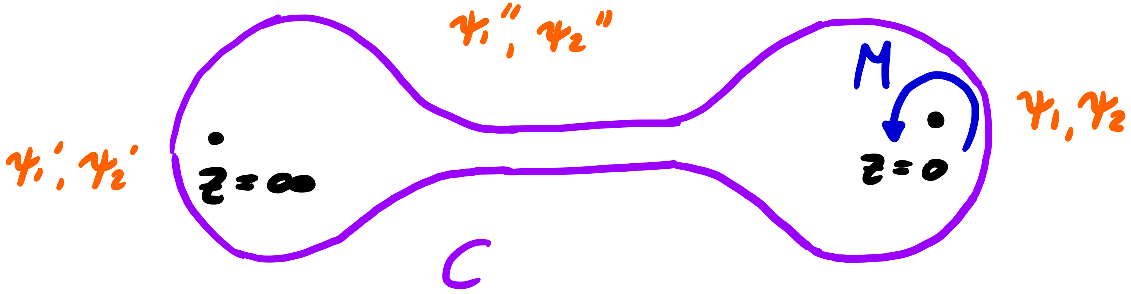
- introduce new coordinate $v = z/\Lambda^2$ near $z=0$ and compute ψ_1, ψ_2 in v in series in Λ^2

- analytically continue ψ_1, ψ_2 to $v \to \infty$ while $z \sim 1$, and reorganize as series in Λ^2

- then compare to ψ_1'', ψ_2''

similar to [H-Kidwai]

Physics interpretation:

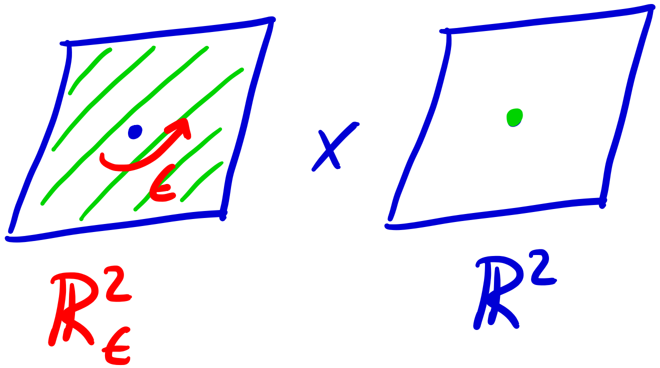


- their moduli space is the surface C
- their vevs ψ obey

$$D_\epsilon \psi(z) = 0$$

[Alday - Gaiotto - Gukov - Tachikawa - Verlinde]

- consider simplest half BPS surface defects:



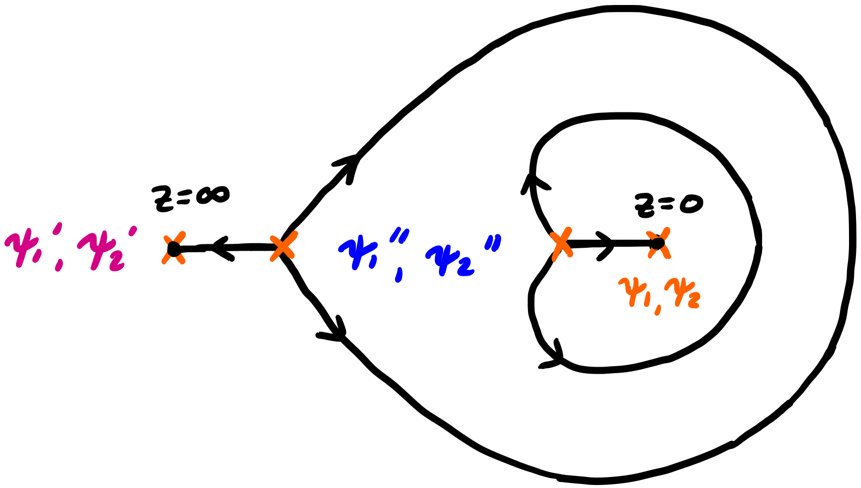
- gauge theory proof: [Nekrasov - Jeong]

analytically continued ψ_i:
corr to orbifold description
of surface defect

why $\vartheta = \arg(\epsilon) = \frac{\pi}{2}$?

special since W-bosons have $\arg(Z) = \frac{\pi}{2}$

spectral network or
Stokes graph at $\vartheta = \frac{\pi}{2}$:



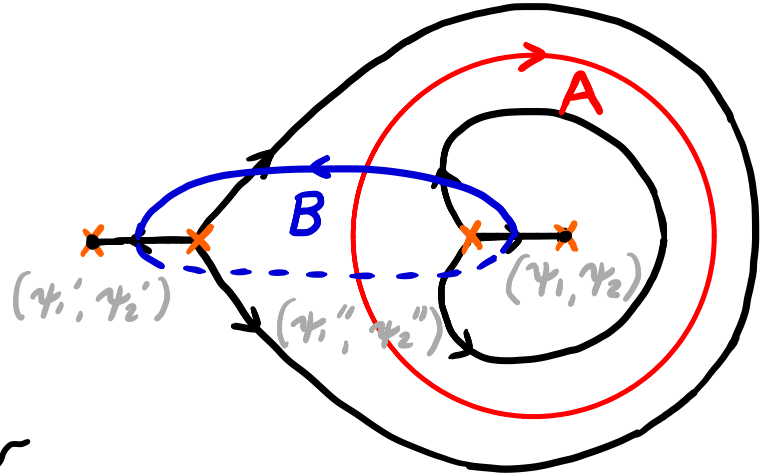
ψ's are Borel resummed solutions to $D_\epsilon \psi = 0$ in direction $\vartheta = \pi/2$

$\|$

ψ's form local bases for abelianization of D_ϵ

trajectories $e^{-i\vartheta}\sqrt{\frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}} \in \mathbb{R}^\times$

spectral network or
Stokes graph at $\vartheta = \frac{\pi}{2}$:



spectral coordinates or

Voros periods or

exact quantum periods:

$$\mathcal{X}_A(\nabla^{oper}) = \text{eigenvalue } M \text{ corr to } \psi_2'' \equiv - e^{\frac{\pi i a^{ex}_{\epsilon}}{\epsilon}}$$

$$\boxed{a^{ex}_{D,\epsilon} = \frac{\partial W^{4d}_{NS}}{\partial a^{ex}_{\epsilon}}}$$

$$\mathcal{X}_B(\nabla^{oper}) = \sqrt{\frac{[\psi_1', \psi_1''][\psi_1, \psi_2'']}{[\psi_1, \psi_1''][\psi_1', \psi_2']}} \equiv e^{2\epsilon a^{ex}_{D,\epsilon}}$$

This can be rephrased as: [Nekrasov-Rosly-Shatashvili]

for 4-dim'l $\mathcal{N}=2$ gauge theories

W_{NS}^{4d} can be computed analytically as a

generating function of opers in terms of

generalized Fenchel-Nielsen coordinates

- FN coord defined using spectral networks at special phases [H-Neitzke]

  =

  analytically continuing the corresponding exact quantum periods [H-Neitzke]

- opers are generalized differential eqns on C [H-Kidwai]

can also be computed using TBA equation [GMN]

Generalized NS partition function

define $W_{NS}^{4d,\vartheta}$ as Borel sum in direction $\vartheta = \arg(\epsilon)$

(and then analytically continue in ϵ)

that is, define $W_{NS}^{4d,\vartheta}$ through

$$\boxed{a_{D,\vartheta}^{ex} = \frac{\partial W_{NS}^{4d,\vartheta}(a_\vartheta^{ex})}{\partial a_\vartheta^{ex}}}$$

where a_ϑ^{ex} and $a_{D,\vartheta}^{ex}$ are the

exact quantum periods / Voros periods / spectral coordinates

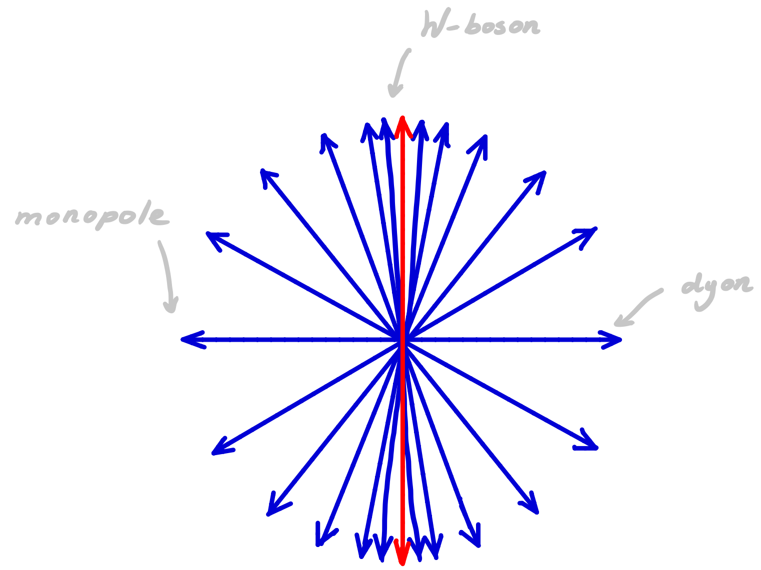
corr to the Stokes graph / spectral network at phase ϑ

Since the spectral network only changes at discrete phases ϑ,

$W^{4d,\vartheta}_{NS}$ is piecewise constant.

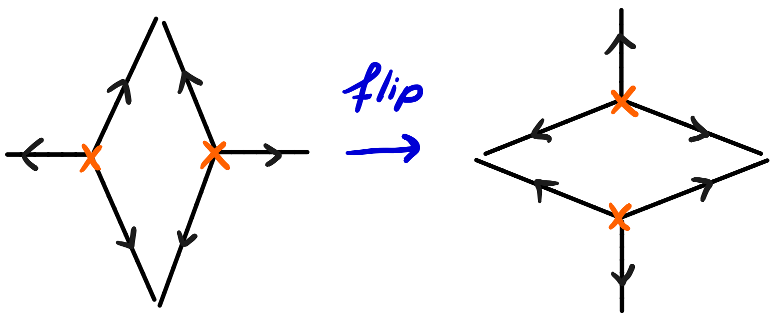
ex: "active rays" for pure $SU(2)$ theory

i.e. $\arg(Z)$ for 4d BPS particles



"Flip" corr to

$$a^{ex}_{\vartheta'} = a^{ex}_{\vartheta}, \quad a^{ex}_{D,\vartheta'} = a^{ex}_{D,\vartheta} + \log\left(1 - e^{\pi i\, a^{ex}_{\vartheta}/\epsilon}\right)$$



i.e. $\Delta W^{4d}_{NS} = -\frac{\epsilon}{2\pi i} \mathrm{Li}_2\left(e^{\pi i\, a^{ex}_{\vartheta}/\epsilon}\right)$

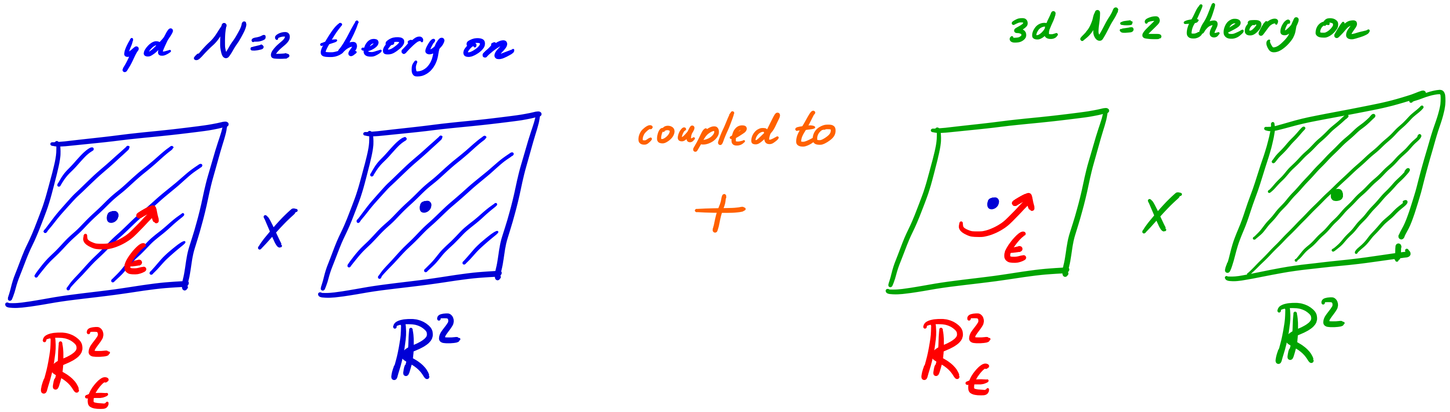
$\rightsquigarrow$ $W^{4d,\vartheta}_{NS}$ encodes all 4d BPS states

- relates to Peacock patterns [Mariño et al]
- as well as to tau functions [Bridgeland et al]

$\rightsquigarrow$ $W^{4d,\vartheta}_{NS}$ transforms as a section of a distinguished line bundle on $\mathbb{C}^*_\epsilon \times \mathcal{M}_u$

- Stokes jumps are transition functions [Neitzke, Alexandrov-Persson-Pioline] [Coman-Longhi-Teschner, Alim-Saha-Teschner-Tulli]

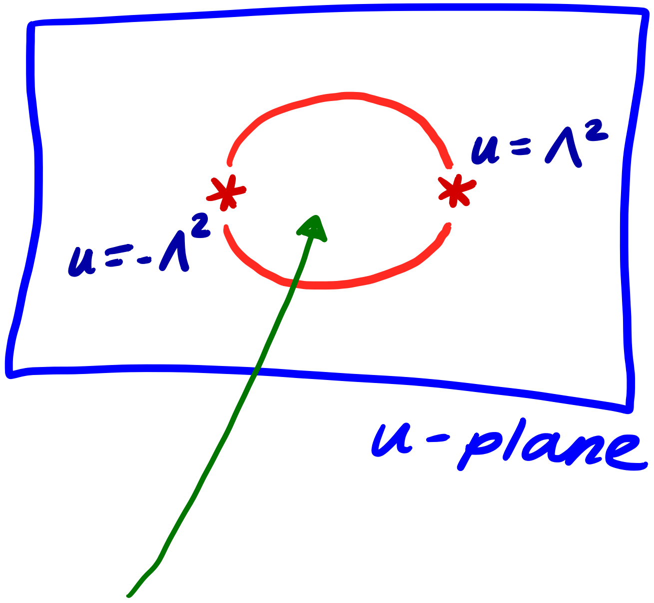
$\rightsquigarrow$ in the 4d gauge theory ϑ labels an IR boundary condition:

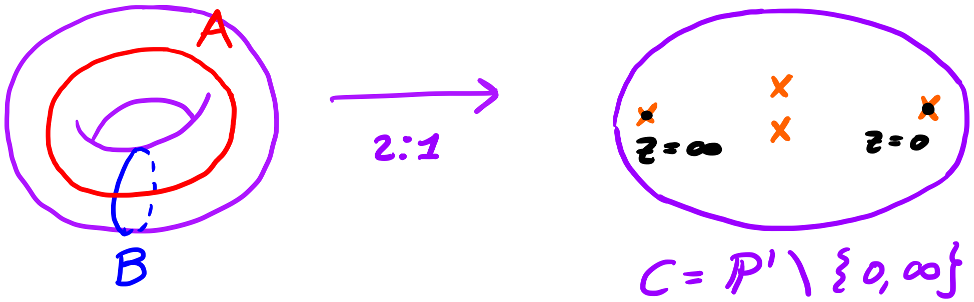


[H-R-S, H-Neitzke] [Dimofte-Gaiotto-Veen]

Example : pure $SU(2)$ theory [wip]

strong coupling $\Lambda^2 \simeq u$





$C = \mathbb{P}^1 \setminus \{0, \infty\}$

~> BPS spectrum different

=> exact quantum periods different

$$\Sigma : w^2 = \frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$

u Coulomb par

Λ UV scale

family of opers parametrized by Mathieu differential eqn:

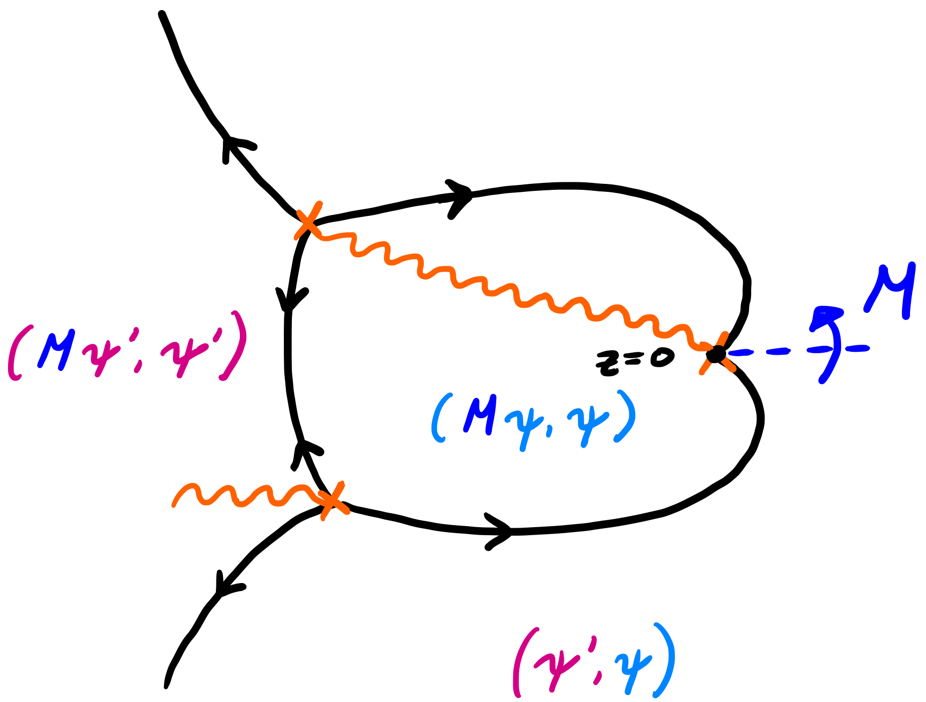
$$D_\epsilon \psi(z) = \epsilon^2 \psi''(z) - \left(\frac{\Lambda^2}{z^3} - \frac{2u + \epsilon^2/4}{z^2} + \frac{\Lambda^2}{z} \right) \psi(z) = 0$$

Example : pure SU(2) theory

dyon point $\Lambda^2 \simeq -u$

$\vartheta = \arg(Z) = 0$ for dyon

spectral network or Stokes graph at $\vartheta = 0$:



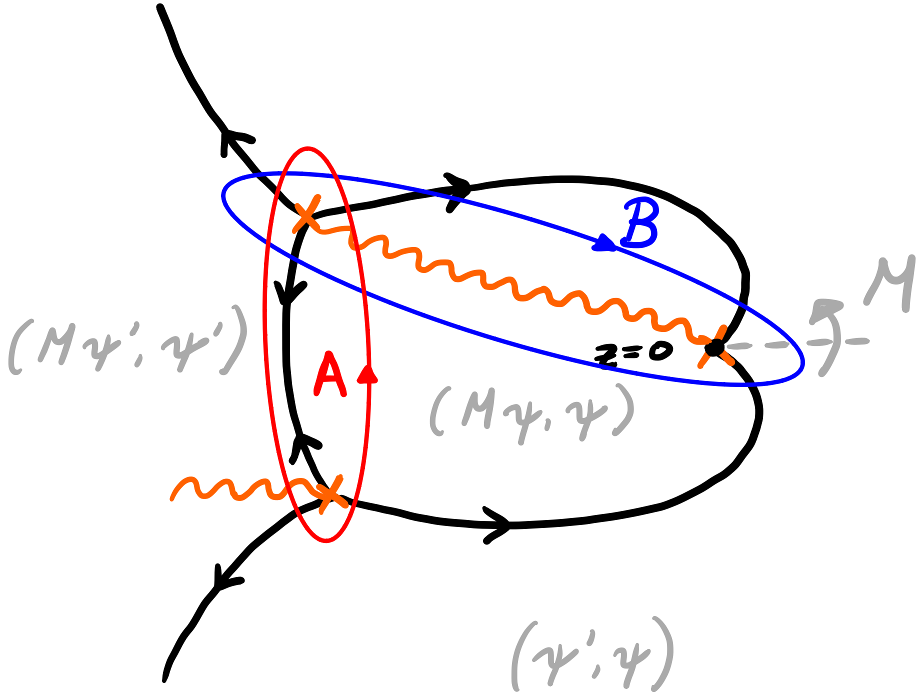
ψ asympt small (when $z \to 0$ along pos real axis)

ψ' " " " (when $z \to \infty$ " ")

for $\vartheta = \arg(\epsilon) = 0$ and $-u \simeq \Lambda^2$

(ψ's are Borel resummed solutions to $D_\epsilon \psi = 0$ in direction $\vartheta = 0$)

spectral network or
Stokes graph at $\vartheta = 0$:



exact quantum periods :

$$\mathcal{X}_A(\nabla^{oper}) = \frac{[\psi,\psi']^2}{[\psi,M\psi][\psi',M\psi']} \equiv -e^{\frac{\pi i \tilde{a}_\epsilon^{ex}}{\epsilon}}$$

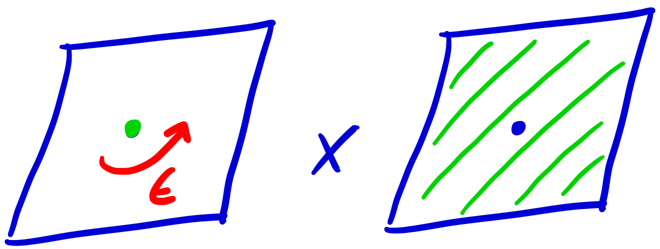
$$\mathcal{X}_B(\nabla^{oper}) = \sqrt{\frac{[M\psi,\psi']}{[M\psi',\psi]}} \equiv e^{2\epsilon \tilde{a}_{D,\epsilon}^{ex}}$$

$$\rightsquigarrow \quad \boxed{\tilde{a}_{D,\epsilon}^{ex} = \frac{\partial W_{NS}^{4d}}{\partial \tilde{a}_\epsilon^{ex}}}$$

Relation to $\mathcal{F}^{4d}_{sd}$ $\quad (\epsilon_2 = -\epsilon_1)$ [wip]

likely: consider opers with apparent singularities

in physics this corr to introducing half-BPS surface defects on $\mathbb{R}^2$:



$\mathbb{R}^2_\epsilon \qquad \mathbb{R}^2$

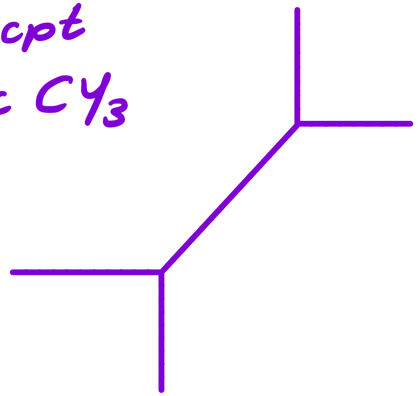
Nakajima-Yoshioka blow-up eqns $\rightsquigarrow$ $\mathcal{F}^{4d}_{sd}$ is generating function of opers with apparent singularities wrt generalized FN coord

see 1.3 of [Bershtein, Gavrylenko, Grassi] for refs

Lift to 5d

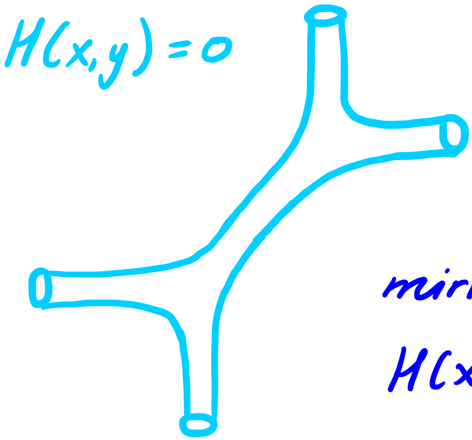
4d $\mathcal{N}=2$ theories T_{4d} of class $\mathcal{S}$ may be lifted to 5d $\mathcal{N}=1$ theories T_{5d}:

non-cpt toric CY_3



$\longleftrightarrow$ mirror symmetry [Hori-Vafa]

$uv + H(x,y) = 0$



mirror curve $H(x,y) \subset \mathbb{C}^*_x \times \mathbb{C}^*_y$

$\updownarrow$ geometric engineering [Katz-Klemm-Vafa]

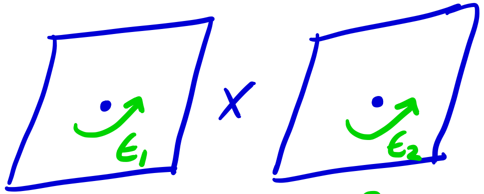
"$\updownarrow$" $\lambda = \log y \, d\log x$

T_{5d} on $\mathbb{R}^4 \times S^1_\beta$ $\longleftrightarrow$ 5d SW curve Σ_{5d}

$\downarrow \beta \to 0$ $\qquad$ $\downarrow \beta \to 0$

T_{4d} on $\mathbb{R}^4$ $\longleftrightarrow$ 4d SW curve Σ_{4d}

K-theoretic instanton counting: [Nekrasov]

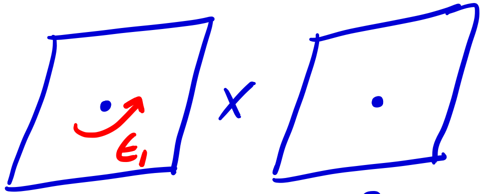
T_{5d} on 

$\mathbb{R}^2_{\epsilon_1}$ $\mathbb{R}^2_{\epsilon_2}$

$$\rightsquigarrow \quad Z^{5d}_{inst} = \sum_{k=0}^{\infty} q^k \, \mathrm{tr}_{\mathcal{H}_k} e^{\beta(\epsilon_1 J_1 + \epsilon_2 J_2 + a_i H_i)}$$

$\mathcal{H}_k$: hol'c functions on k-instanton moduli space

$\downarrow$ NS limit

T_{5d} on 

$\mathbb{R}^2_{\epsilon_1}$ $\mathbb{R}^2$

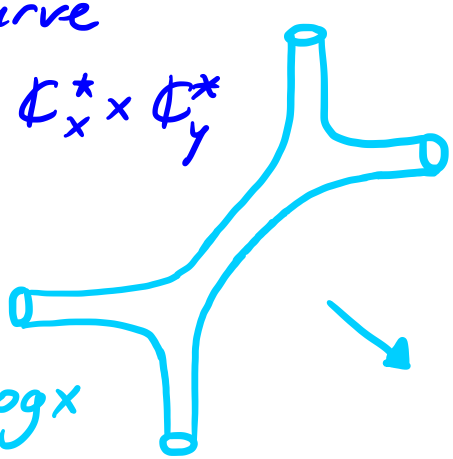
$$\rightsquigarrow \quad \frac{1}{\epsilon} W^{5d}_{NS} = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z^{5d}_{Nek} = \frac{1}{\epsilon} \mathcal{F}^{5d}_0 + \ldots$$

$\updownarrow$

quantized multiplicative Hitchin system

mirror curve

$H(x,y) \subset \mathbb{C}^*_x \times \mathbb{C}^*_y$



$\lambda = \log y \, d \log x$

$\rightarrow \mathbb{C}^*_x$

All ingredients for lift to 5d available:

- multiplicative Hitchin system on $C = \mathbb{C}^*$ has hyperkähler structure:

  I: multiplicative Higgs bundles = periodic monopoles on $C \times S^1$

  other: difference connections $\supset$ q-opers

  [Elliott - Pestun]

- spectral networks $\longrightarrow$ exponential spectral networks

  [Eager - Selmani - Walcher] [Banerjee - Longhi - Romo]

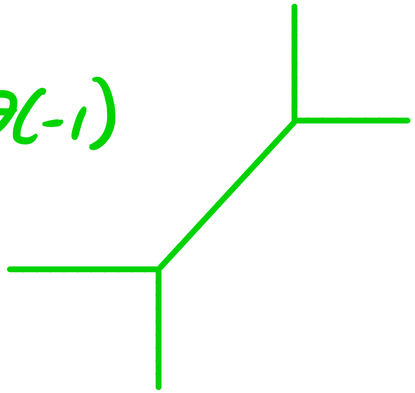
- physics interpretation in terms of surface defects on $\mathbb{R}^2 \times S^1_\beta$

  $\longleftrightarrow$ B-branes ending on mirror curve Σ

  [Dimofte - Gukov - Hollands]

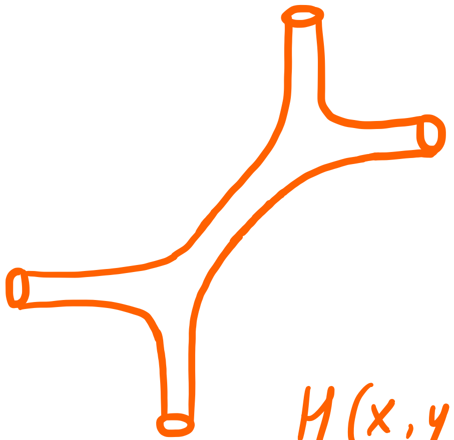
Example: resolved conifold

$$\mathcal{O}(-1)\oplus\mathcal{O}(-1)$$
$$\downarrow$$
$$\mathbb{P}^1$$



$\longleftrightarrow$

mirror symmetry



$$H(x,y) = 1 + x + y + Q^{-1} x y^{-1} = 0 \subset \mathbb{C}^*_x \times \mathbb{C}^*_y$$

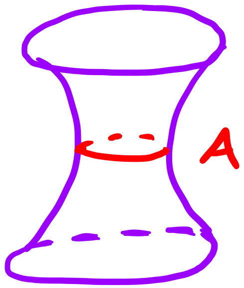
$$\big\downarrow {\scriptstyle 2:1}$$

$$\mathbb{C}^*_x$$

4d limit

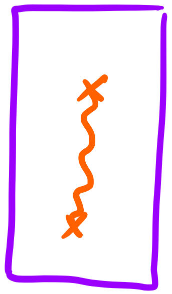
AD_2 theory

single free hyper of mass m



A

$\xrightarrow[2:1]{}$



$C = \mathbb{C}$

$$\Sigma: w^2 = z^2 + m$$

Example: resolved conifold ~ 5d $\mathcal{N}=1$ $U(1)$ gauge theory

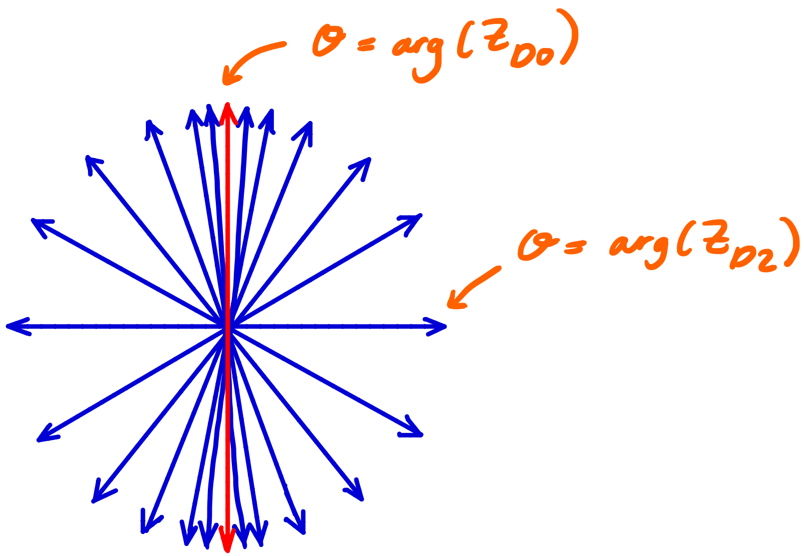
5d BPS spectrum consists of

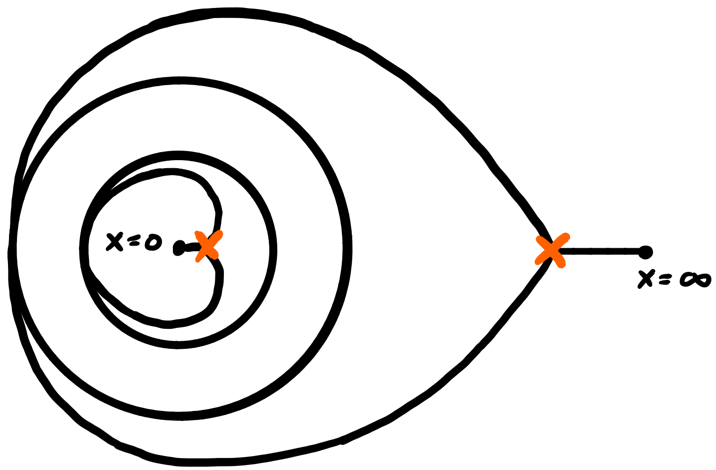
3 BPS particles, corr to 3 KK towers

with D2 charge $0, \pm 1$

$\xrightarrow{\text{4d limit}}$ $\left[\begin{array}{l}\text{KK momentum} \\ \frac{2\pi}{\beta} \to \infty\end{array}\right]$ 4d BPS hyper + anti-hyper

e.g. KK-tower with D0-charge 0 obtained from 5d "Fenchel-Nielsen" network:



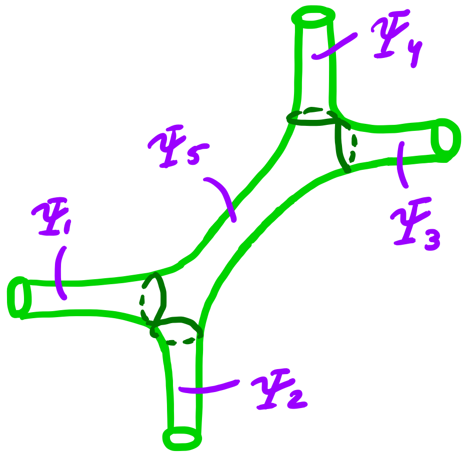


[Banerjee - Longhi - Romo]

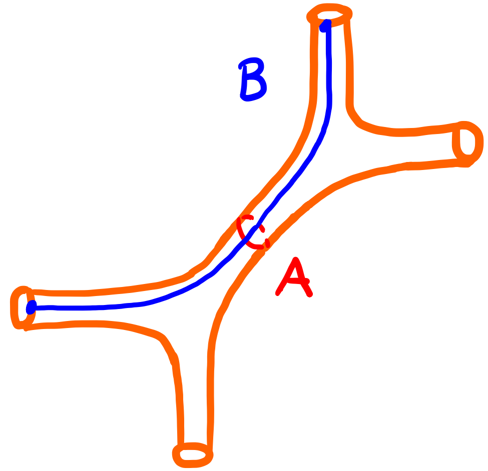
Quantum periods (viewpoint $\Sigma^{sd}_{SW} \xrightarrow{1:1} \mathbb{C}^*_y$) [wip]

At $\Theta = \arg(Z_{D0})$ essentially

5 regions:



$\rightsquigarrow$



$$e^{\frac{1}{\epsilon}\oint_A \lambda} = Q = e^{-t}$$

$$e^{\frac{1}{\epsilon}\int_B \lambda} = \frac{\Psi(t)}{\Psi(\infty)}$$

$\wr$

$$e^{\frac{1}{\epsilon}\int_B \lambda} \equiv e^{\frac{1}{\epsilon}\partial_t W^{sd}_{NS}}$$

$$\Rightarrow e^{\frac{1}{\epsilon}W^{sd}_{NS}} = M^2(1)\, M(Q)^{-1}$$

with $M(Q) = e^{\sum_{n=1}^{\infty} Q^n / n^2 [n]_\epsilon}$

local (bases of) sections given by B-brane wave-functions

$$H(e^X, e^{\epsilon\, \partial/\partial X})\, \Psi(X) = 0$$

[Aganagic - Cheng - Dijkgraaf - Krefl - Vafa]

Changing ϑ across active rays $\Rightarrow$ W^{sd}_{NS} picks up Stokes factors $\sim \epsilon \, \mathrm{Li}_2(e^{-\pi i t/\epsilon})$

[Garoufalidis - Kashaev]

$\updownarrow$ Nakajima-Yoshioka blow-up

Non-pert topological string in self-dual limit

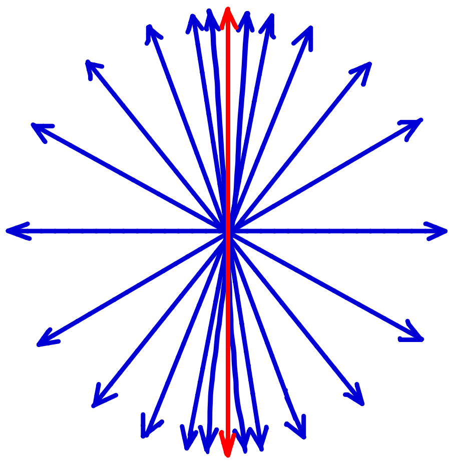
[Alim, Saha, Teschner, Tulli]

$\leftrightarrow$ Riemann-Hilbert problem

[Bridgeland]

$\leftrightarrow$ non-comm DT

[Szendroi, Jafferis-Moore]



$$\mathcal{F}_{GV}(\lambda,t) = \sum_{k=1}^{\infty} \frac{Q^k}{k\left(2\sin\left(\frac{\lambda k}{2}\right)\right)^2}$$

$$\mathcal{F}_{np}(\lambda,t) = \mathcal{F}_{GV}(\lambda,t) + \frac{\partial}{\partial\lambda}\lambda W^{sd}_{NS}\left(\frac{1}{\lambda}, \frac{1}{\lambda}\left(t-\frac{1}{2}\right)\right)$$

[Hatsuda, Mariño, Moriyama, Okuyama]


Thank you!
Hope to see you soon
in Edinburgh!!