Quantum geometric Langlands as a fully extended TFT

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Outline
I) Character varieties, character stacks and their quantizations
II) Recent approach via factorization homology & cobordism hypothesis
III) Applications in quantum topology and representation theory

Broader goals:
- Underscore utility and convenience of stacks and higher categories
- Introduce stratified factorization homology, and its zoo of defects
- Emphasize 4D KW viewpoint vs 3D CS viewpoint on skeins
- Formulate S-duality/Langlands duality in skein theory
Character stacks

Def: A $G$-local system on a space $X$ is a principal $G$-bundle $E$ over $X$, together with a flat connection.

Our primary object of study is the character stack:

$$\text{Ch}(X) = \left\{ \text{G-local systems on } X \right\} \bigg/ \cong \left\{ p: \pi_1(X) \to G \right\} \bigg/ \text{conj by } G$$

Via gauge-fixing, we can present it as a quotient stack:

$$\text{Ch}^{\text{fv}}(X) = \left\{ \text{G-local systems } E \text{ on } X \right\} \bigg/ \cong \left\{ p: \pi_1(X) \to G \right\} \bigg/ \text{conj by } G$$

$$\text{Ch}(X) = \text{Ch}^{\text{fv}}(X) / G$$

$\text{Ch}(X)$ is $(2-\dim(X))$-shifted symplectic (Pantev-Toen-Vaquie-Vezzosi).
- When $\dim(X)=2$, given by the Atiyah-Bott/Goldman/Fock-Rosly bracket.
- When $\dim(X)=3$, given by the Batalin-Vilkovisky structure on the critical locus of the Chern-Simons action functional.
Appearances of quantum character stacks in physics

Kapustin-Witten twist of 4D N=4 theory
- has a dependence on a parameter $\psi$.
- On surfaces, it should attach a category $\text{Fuk}(T^* \text{Bun}_G(\Sigma))$, which is mathematically ill-defined.
- Varying $\psi \leftrightarrow q$-deformation.

4D N=2 "Theories of class S", from a Riemann surface.
- Compactifying on circle gives "Sicilian" 3D N=4 theory.
- Its Coulomb branch is the Hitchin system ($\text{Gaiotto-Moore-Neitzke}$), whose hyper-Kahler rotation is $\text{Ch}(\Sigma)$.
- Omega-background $\leftrightarrow q$-deformation.
Character stacks vs. character varieties

We study a quotient stack through its category of sheaves:
- $O(X)$ gets an action of the group $G$
- $\mathbf{QC}(X/G) = G$-equivariant $O(X)$-modules
- $\mathbf{QC}(\overline{X/G}) = O(X)^G$-modules
- We have a functor $\Gamma : \mathbf{QC}(X/G) \to \mathbf{QC}(\overline{X/G})$, but is destructive.

Who cares about stacks?
- Character stacks glue, their quotient varieties don't!
- Example: Compute $\text{Ch}(S^1)$ and hence $Z(S^1) = \mathbf{QC}(\text{Ch}(S^1))$.
  $\mathbf{QC}(\text{pt}/G) = G$-equivariant vector spaces = $\text{Rep}(G)$.
  $\text{Ch}^{fr}([0,1]) = \text{pt}$.  \hspace{2em} \text{Ch}([0,1]) = \text{pt}/G.  \hspace{2em} \text{Ch}([0,1]) = \text{pt}.

\[
Z(0) = Z(C) \boxtimes Z(D) = \text{Rep}_G \boxtimes \text{Rep}_\mathcal{U} = G\text{-eq. } O(C)^G\text{-modules}
\]
\[
\text{Rep}_G \cong \mathbf{QC}(\frac{G}{G}) \text{ as expected.}
\]

$\text{Ch}^{fr}(S^1) = G$.  \hspace{2em} $\text{Ch}(S^1) = \frac{G}{G}$.  \hspace{2em} $\text{Ch}(S^1) = \text{Spec}(O(C)^G) = H/W$
Quantizations of character varieties of surfaces

Historically, how it was done:

- Skein algebras (Przytycki, Turaev, many others…) for \( G=\text{SL}(2) \) mostly:
  \[ \text{Sk}(M^3) = C \cdot \{ \text{links}^3 \}_{\text{skein rel}^3} \]
  \( \text{Sk}(\Sigma \times I) \) is an algebra via stacking in \( I \) direction.

  \( \hat{X} = q^{2} \hat{\otimes} + q^{-2} \hat{\otimes} \)

  Thm (Bullock–Frohman–Kania–Bartoszynska): At \( q=1 \), \( \text{Sk}(S) = O(\text{Ch}(S)) \), with \( AB/G \) Poisson bracket.

- Moduli algebras (Alekseev–Grosse–Schomerus) – any \( G \), \( S \) punctured
  \[ \text{Ch}^\text{fr}(\Sigma_{g,v}) \cong G^{2g+v-1}, \text{ if } v \geq 1 \]
  \[ \Theta(\text{Ch}^\text{fr}(\Sigma_{g,v})) \cong \Theta(G)^{2g+v-1} \]

- Quantum cluster algebras (Fock–Goncharov):
  - Consider decorated character variety (\( B \)-reductions near boundaries)
  - Triangulations/Ptolemy flips give toric log-canonical charts/mutations.

  \( \Delta \) of \( \Sigma \) \( \rightarrow \) \( U(\Delta) \subseteq \text{Ch}^\text{dec}(\Sigma) \), \( U(\Delta) \cong (C^\times)^r \), \( \{ z_i, z_j \} = a_{ij} z_i z_j \)

  \[ \hat{X}_i \hat{X}_j = q^{\frac{2}{a_{ij}}} \hat{X}_j \hat{X}_i \]
Topological factorization homology

(Ben-Zvi-Francis-Nadler) \( \text{Ch}(X) \) is a (2-shifted) sigma model, \( \text{Ch}(X) = \text{Maps}(X, BG) \), hence it satisfies excision:

\[
\text{Ch}(S_1 \cup_{P_{\text{tI}}} S_2) = \text{Ch}(S_1) \times_{\text{Ch}(P_{\text{tI}})} \text{Ch}(S_2)
\]

\[
\Rightarrow \mathcal{Z}(S_1 \cup_{P_{\text{tI}}} S_2) \cong \mathcal{Z}(S_1) \otimes_{\mathcal{Z}(P_{\text{tI}})} \mathcal{Z}(S_2)
\]

(Lurie, Ayala-Francis) Factorization homology is the universal assignment,

\[
(M, A) \quad \mapsto \quad \mathcal{Z}_A(M) = \int_M A
\]

 categoría

(satisfying excision.)

The \( E_n \)-algebra are the local observables, \( \mathcal{Z}_A(M) \) is the global observables. A symmetric monoidal category like \( \text{Rep}(G) \) is an \( E_n \)-algebra for any \( n \).
Quantum character stacks (BZBJ1,BZBJ2)

In the case $n=2$, $E_2$-algebras are braided tensor categories.

**Defn:** The quantum character stack is:

$$\mathcal{Z}(\Sigma) := \mathcal{Z}_{\text{Rep}_q}(\Sigma) = \int_{\Sigma} \text{Rep}_q G$$

**Computations:**

1) (BZBJ1) For punctured surfaces with a gate, we recover AGS algebras:

$$\mathcal{Z}(\Sigma) \approx \mathcal{U}_q(\mathfrak{g})\text{-equivariant } A(\Sigma)\text{-modules}$$

2) (BZBJ2, BJ, GJV) For $G=\text{GL}(N)$, on the closed torus we recover the $q=t$ specialization of the spherical DAHA, via quantum Hamiltonian reduction.

$$\mathcal{Z}(\bigcirc) \approx \mathcal{Z}(\bigcirc \otimes \bigcirc) \otimes \mathcal{Z}(\bigcirc)$$

3) (Cooke) For any surface, we recover Walker's "skein category":

![Diagram of objects and morphisms representing the skein category.](image)
Stratified factorization homology (Ayala-Francis-Tanaka)

- This allows manifold with defects, retaining functoriality and excision.
- Locality replaced by richer algebraic structures:

**Local data**

- Braided module categories
- Central tensor & bimodule cat’s
- Quantum symmetric pairs & K-matrices

**Example coefficients**

- Conjugacy classes, $G$-Hamiltonian spaces
  - $\text{Rep}_q G$, $\text{Min}(t)$
- Parabolic induction, flag varieties, Stokes phenomena
  - $\text{Rep}_q G$, $\text{Rep}_q T$, $B$, $B_\omega$
- Symmetric spaces, orbifold character stacks
  - $\text{Rep}_q G^\theta$

**Example output**

- Type A DAHA for $q \neq t$. (J,BJ, Varagnolo-Vasserot)
- Fock-Goncharov style quantization (JLSS, next slide)
- Boalch-style quantization (AJP)
- Type C DAHA (JM, Weelinck, JW)
More details on Fock–Goncharov cluster quantization

We consider the decorated character stack, the moduli space of:
- $G$-local systems in white $G$-region
- $T$-local system in yellow $T$-region
- $B$-reduction of the resulting $G \times T$-local system along red defects.

Defn (JLSS): The quantum decorated character stack is the factorization homology with coefficients in the triple $(\text{Rep}_q(G), \text{Rep}_q(B), \text{Rep}_q(T))$.

Thm (JLSS): Let $G=\text{SL}(2)$. Each triangulation $\Delta$ of $S$ determines a subcategory $Z(\Delta)$ of $Z(S)$, an equivalence $Z(\Delta) = A(\Delta)-\text{mod}$, and an isomorphism, $A(\Delta) \cong \mathbb{C}_q[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ (a quantum torus).

Recovers, extends quantum cluster $A$-, $X$-, and $P$- varieties of Fock–Goncharov and Goncharov–Shen.
This page was in case of questions... 😞
Skein theory is intrinsically 4-dimensional $\Rightarrow$ KW, not CS

Most approaches to skein theory focus on 3-dimensional aspects.

But, braided tensor categories naturally form a 4-category (Haugseng, Johnson–Freyd–Scheimbauer)
- Objects are braided tensor categories $A$, $B$, …
- $\text{Hom}(A,B)$ are central tensor categories $C,D$, …
- $\text{Hom}(C,D)$ are central bimodule categories $M,N$, …
- $\text{Hom}(M,N)$ are bimodule functors $F,G$, …
- $\text{Hom}(F,G)$ are natural transformations.

Results: Applying cobordism hypothesis $n$-dualizability $\Rightarrow$ $n$-dim TFT’s.
1) (Calaque–Scheimbauer) All braided tensor categories are 2-dualizable.
2) (BJS) Rigid braided tensor categories are 3-dualizable.
3) (BJS, after Freed–Teleman, Walker) Braided fusion categories are 4-dualizable.
4) (BJSS) (non-semisimple) modular tensor categories are invertible.
   - MTC’s give Crane–Yetter theories, hence 3D WRT/CS theories.
Applications to skein theory

Theorem (GuJS, conj. by Witten) Suppose $q$ is generic, and $M$ is an arbitrary closed, oriented 3-manifold. Then the skein module of $M$ has finite dimension.

Theorem (GaJS, asked by Bonahon-Wong, see Frohman-Kania-Bartoszynska-Le) Suppose instead $q$ is a root of unity. Then the skein algebra of $\Sigma$ is Azumaya (i.e. locally Morita-trivial) over the smooth locus of the character variety.

Main ideas:
- First lift the problem from quantum character variety to quantum character stack (c.f. Walker's skein category TFT).
- Given a Heegard splitting of $M = M_1 \cup M_2$, write $Z(M) = Z(M_1) \circ Z(M_2)$, where $Z(M_1):\text{Vect} \to Z(\Sigma)$, $Z(M_2):Z(\Sigma) \to \text{Vect}$.
- $Z(\Sigma)$ is the quantum Hamiltonian reduction of $Z(\Sigma_0)$. We express $Z(M_1) \circ Z(M_2)$ instead as $Z(M_1) \otimes Z(M_2)$.
- $A(\Sigma)$ is a quantization of a smooth variety $Ch^r(\Sigma^0)$. $Z(M_1)$, $Z(M_2)$ are holonomic modules, i.e. they quantize Lagrangians $G^d \subset G^{2q}$.
- Finite-dimensionality follows from Kashiwara-Schapiro theory of def. quantization and holonomicity.
Langlands/$S$-duality for skein modules

At $q=1$, the TFT $Z(X)$ recovers the character variety:
- For $X=S$, it recovers the monoidal category $QC(\hat{G})=QC(\text{Ch}(S^1))$.
- For $X=\Sigma$ a surface, it recovers the category $QC(\text{Ch}(\Sigma))$.
- For $X=M$ a 3-manifold it recovers the vector space $O(\text{Ch}(M))$.

These are precisely the state spaces of Kapustin–Witten equations in each dimension, for $\psi=\infty$.

Expectation: For generic values of $q$, and corresponding generic values of $\psi$, $Z(M)$ recovers the Kapustin–Witten state space of $M$.

Conj. (BZGuJS): The dimension of skein module at generic values of $q$ coincide, for $G$ and its Langlands dual $\hat{G}$ (e.g. $\text{SL}(2)$ and $\text{PGL}(2)$).

Evidence: computations for $\text{SL}(N)$ and $\text{PGL}(N)$ on $T^3$ and lens spaces.