Borcherds-Kac-Moody algebras, 2d strings, & other curiosities

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Western Hemisphere Colloquium for Geometry and Physics

This talk is based on a series of related works:

1601.05412 w/ D. Persson & R. Volpato
1701.05169 w/ D. Persson & R. Volpato
1803.10798 w/ S. Harrison & R. Volpato
2009.14710 w/ S. Harrison, D. Persson, & R. Volpato
2010.xxxxx w/ S. Harrison, D. Persson, & R. Volpato

The underlying motivation for all these studies is to clarify connections between various subjects that are seemingly related to one another:

A confluence of special mathematical structures in string theory

- (Super) vertex operator algebras
- Lattices
- Moonshine
- (Mock) modular & automorphic forms
- Infinite-dimensional Lie algebras/BKMs
- K3 surfaces, Enriques surfaces, Calabi-Yaus...
- (Refined, motivic,...) Gromov-Witten invariants/enumerative geometry
- ...

If I had to try to synthesize the lessons from these works, I might say something like this:

- String compactifications down to d ≤ 3 enjoy enormous (nonperturbative) duality group symmetries.
- BPS-saturated quantities are often duality-invariant → modular/automorphic objects
- **Automorphic forms on string theory moduli spaces** (different perturbative duality frames leading to different expansions)
- Certain theories @ special points in moduli space (e.g. moonshine modules): denominators of Borcherds-Kac-Moody algebras
- Try to decompactify and track symmetry breaking patterns
- Decompactification limits in special moonshine theories know about genus zero

$$\Phi_{12} : O^+(\Gamma^{2,26})$$

weight 12

$$\Gamma^{2,26} \to \Gamma^{1,1} \oplus \Gamma^{1,1} \oplus N(-1)$$

Niemeier/Leech lattices

$$\Phi_{12}(\tau,\zeta,\sigma) = p^A q^B (e^{2\pi i(\zeta,\vec{C})}) \prod_{m,n,\lambda} (1 - p^m q^n (e^{2\pi i(\zeta,\lambda)}))^{f(mn,\lambda)}$$

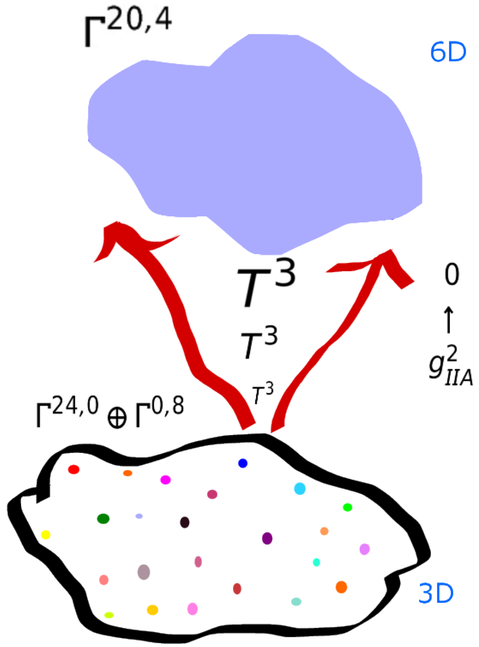
w/ $$\frac{\theta_N(\tau,\zeta)}{\eta^{24}(\tau)} = \sum_{\lambda\in N, n\in\mathbb{Z}} f(n,\lambda) q^n e^{2\pi i(\zeta,\lambda)}$$

Denominator of Fake Monster BKM

Expansions give denominators of affine Lie algebras associated to N

[Gritsenko]

[Harrison-Kachru-NMP-Volpato-Zimet] (2d heterotic string)



The Umbral groups arise as symmetries of string theory near the 23 Niemeier points in the moduli space of 3D vacua. Decompactification to 6D breaks (some of) the Umbral symmetry, leaving the groups known to appear in K3 compactification of IIA string theory.

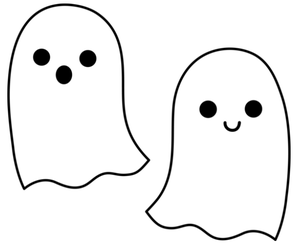
w/ S. Kachru & R. Volpato

also see [Harvey-Moore '17]

The rest of this talk will illustrate this ethos in concrete examples

$$V \otimes \bar{W} \otimes \mathbb{R}^{1,1}$$ $10d$

$\mathfrak{Q}_{BRST}$



Moonshine modules (holomorphic SCFTs/SVOAs)

V, W

e.g. $V = V^\natural$, the Monster module

$$\downarrow$$

$$\mathbb{R}^{1,1}, \mathbb{R} \times S^1, \mathbb{T}^2$$ $2d$

The plan for the rest of the talk:

Caveat emptor: we will emphasize BKM algebras in this talk, and deemphasize many other important things
(i.e. some moonshine modules will serve as nice examples, but I will not say much about moonshine)

- The Monster Lie algebra (what Borcherds did)
- New examples: super-BKMs (following Borcherds & Scheithauer)
- Monstrous heterotic string theory (what we did)
- Musings, confusions, speculations,...

A brief reminder

$\tau \in \mathbb{H}$

$q := e^{2\pi i \tau}$

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + \ldots = \sum_{n=-1}^{\infty} c(n)q^n$$

[McKay, Thompson]

Modular functions: meromorphic fns on compact surface $\overline{\mathbb{H}/\Gamma}$

$\Gamma = SL(2,\mathbb{Z})$

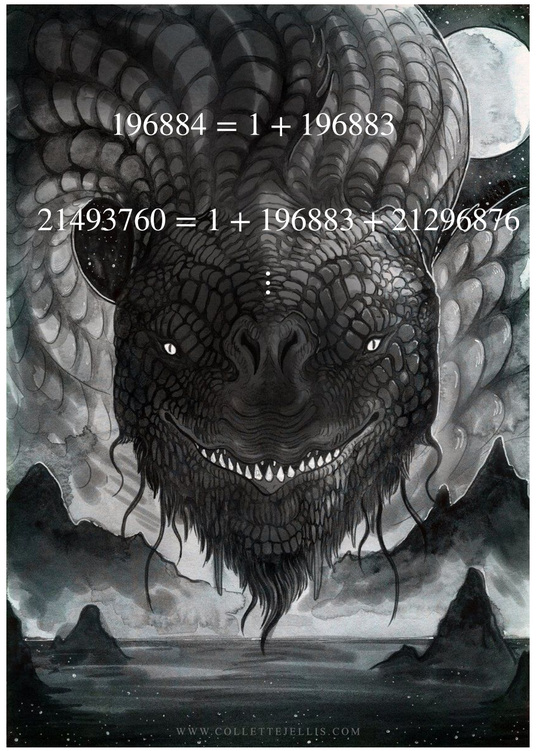
$$J(\tau) = J(\tau+1) = J(-\frac{1}{\tau})$$

$$\overline{\mathbb{H}/\Gamma} \simeq \mathbb{P}^1$$

genus zero groups

"normalized Hauptmodul" for $\Gamma = SL(2,\mathbb{Z})$

(i.e. all modular fns for a genus zero group are rational functions of the Hauptmodul)



$\mathbb{M} \curvearrowright V^\natural$ $\qquad V^\natural = \oplus_{n=-1}^{\infty} V_n$

$$J(\tau) = \sum_n \chi_{V_n}(1) q^n$$

$$T_g(\tau) = \sum_n \chi_{V_n}(g) q^n$$

McKay-Thompson series

There exists an infinite-dimensional $\mathbb{Z}-$graded module $V^\natural$ for the Monster with characters $T_g(\tau)$

It has the structure of a vertex operator algebra of $c = 24$ [Frenkel-Lepowsky-Meurman]

Conjecture: All $T_g(\tau)$ are normalized Hauptmoduln for *genus zero* groups $\Gamma_g \subset SL(2,\mathbb{R})$ [Conway-Norton]

The monstrous moonshine conjectures are a theorem [Borcherds]

Borcherds-Kac-Moody algebras (or Generalized-Kac-Moody algebras)

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$
- Imaginary simple roots allowed: $\alpha \in \mathfrak{h}^*, (\alpha, \alpha) \leq 0$
- Associated Cartan matrix (satisfying some conditions) $A = (a)_{ij}$, abelian Lie algebra $\mathfrak{h}$
- $\mathfrak{h}_{\mathbb{R}}$ w/ elements $h_i, i \in I$ (countably infinite) indexing set
- symmetric, non-degenerate, real bilinear form $(\mid)$
- Can define via Chevalley-Serre generators & relations (reduce to KM if $(a)_{ii} > 0, \forall i \in I$)
- Useful alternative characterizations using bilinear form, various properties of the roots
- Weyl group (reflections wrt real simples), integrable highest weight modules,...

[Borcherds]

(see also [Jurisich])

[Ray]

$$\chi_{L(\lambda)} = \frac{\sum_{w \in W} \epsilon(w) w(S_\lambda)}{e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{mult_\alpha}} \xrightarrow[\text{take trivial rep}]{} e^{-\rho} \sum_{w \in W} \epsilon(w) w(S_\lambda) = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{mult_\alpha}$$

denominator identity

Interesting BKMs have **automorphic denominators**! (Borcherds lift/theta correspondence)

- automorphic denominators e.g. from Lorentzian lattices

[Gritsenko-Nikulin]

BKMs haunt string theory

- Threshold corrections in 4d $\mathcal{N} = 2 \leftrightarrow$ CY 3folds
- Algebras of BPS states?
- M-theory
- **From SVOAs** (next slides)

[Harvey-Moore]

[Henry-Labordere - Julia - Paulot]

BKMs can be super

- $\mathbb{Z}_2$-gradings!
- Supersymmetric bilinear form
- $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$
- h_i can be even or odd, etc.
- Denominator *and super-denominator*

[Scheithauer]

[Ray]

The Monster Lie algebra: the ur-BKM

$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + \ldots = \sum_n c(n)q^n$

- ∞-dim Lie algebra of rank 2: $\Gamma^{1,1} \simeq \mathbb{Z} \oplus \mathbb{Z}$

$$\alpha = (m, n) \qquad (\alpha \,|\, \alpha) = -2mn$$

$$\begin{array}{cccccccc}
\mathfrak{g}_- & & & \mathfrak{h} & \mathfrak{g}_+ & & & \\
\ddots & & & \vdots & & & & \iddots \\
0 & 0 & 0 & 0 & \mathbb{C}^{864299970} & \mathbb{C}^{4252023300096} & \mathbb{C}^{3176440229784420} & \\
0 & 0 & 0 & 0 & \mathbb{C}^{21493760} & \mathbb{C}^{20245856256} & \mathbb{C}^{4252023300096} & \\
0 & 0 & \mathbb{C} & 0 & \mathbb{C}^{196884} & \mathbb{C}^{21493760} & \mathbb{C}^{864299970} & \\
\cdots\ 0 & 0 & 0 & \mathbb{C}^2 & 0 & 0 & 0 & \cdots \\
\mathbb{C}^{864299970} & \mathbb{C}^{21493760} & \mathbb{C}^{196884} & 0 & \mathbb{C} & 0 & 0 & \\
\mathbb{C}^{4252023300096} & \mathbb{C}^{20245856256} & \mathbb{C}^{21493760} & 0 & 0 & 0 & 0 & \\
\mathbb{C}^{3176440229784420} & \mathbb{C}^{4252023300096} & \mathbb{C}^{864299970} & 0 & 0 & 0 & 0 & \\
\iddots & & & \vdots & & & & \ddots
\end{array}$$

real simple

$\mathrm{mult}(\alpha(m,n)) = c(mn)$

$\mathfrak{g}^{(m,n)} \simeq V^{\natural}_{mn}$

Consider Chevalley-Eilenberg complex of trivial rep of $\mathfrak{g}_+$, compute Lie algebra homology:

[Kostant, Garland-Lepowsky]

$$\Lambda^\bullet(\mathfrak{g}_+) = H_\bullet(\mathfrak{g}_+)$$

$$H_0(\mathfrak{g}_+) = \mathbb{C}$$

$$H_1(\mathfrak{g}_+) = \oplus_{n \geq -1} \mathbb{C}^{c(n)} pq^n$$

$$H_2(\mathfrak{g}_+) = \oplus_{m \geq 1} \mathbb{C}^{c(m)} p^m$$

Weyl vector

$$\rho = (-1, 0)$$

$$(\gamma \,|\, \gamma + 2\rho) = 0$$

Hilbert-Poincare gives denominator identity

$$\boxed{J(\sigma) - J(\tau) = p^{-1} \prod_{m>0, n \in \mathbb{Z}} (1 - p^m q^n)^{c(mn)}}$$

$q := e^{2\pi i \tau} = e^{(1,0)}$

$p := e^{2\pi i \sigma} = e^{(0,1)}$

[Koike, Norton, Zagier]

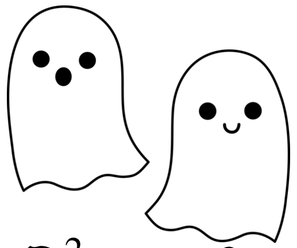
[exposition in Kontsevich (Seminaire Bourbaki)] $\longrightarrow$ $c(4) = c(3) + \frac{c(1)^2 - c(1)}{2}$, ... turns out to imply J is genus 0 (replicability)

From VOA to BKM: a "string theory functor"

(also: $V(\Lambda_{Leech})$ to Fake Monster) [Borcherds]

"Chiral" bosonic string theory

As a rep of $\mathbb{M}$, Monster Lie algebra at deg (m, n) $\simeq V^{\natural}_{mn}$



$\mathfrak{Q}^2_{BRST} = 0$

Tensor in a lattice VA: $V := V^{\natural} \otimes V_{\Gamma^{1,1}}$ $\quad c = 26$

Don't be afraid of no ghosts: $V^{\natural} \otimes V_{\Gamma^{1,1}} \otimes V_{bc}$ $\quad c = 0$

BRST/semi-infinite cohomology: $\mathscr{H}_{phys} = \mathscr{H}^1(V, \mathfrak{Q}_{BRST})$

"physical state condition" & "no-ghost theorem"

[Lian-Zuckerman] One can endow the space of physical states with a natural Lie algebra structure to get the Monster Lie algebra

Study the g-equivariant denominator:

$$\boxed{T_g(\sigma) - T_g(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{\tilde{c}(mn)}}$$

Also known: orbifolds of $V^{\natural}$ to `Monstrous' Lie algebras (used to prove Generalized Moonshine conjectures!) [Carnahan]

Three other natural (and super) examples suggest themselves [Creutzig-Duncan-Riedler]

c=12 SVOAs
(+ their twisted modules, aka Ramond sectors)

Conway moonshine module

[FLM, Duncan]

$\mathbb{Z}_2$-orbifold

E_8 lattice VOA + superpartners

Unique choice of $\mathcal{N} = 1$ supercurrent

24 free chiral fermions

$$G \sim \sum c_{ijk} : \lambda_i \lambda_j \lambda_k :$$

8 choices of $\mathcal{N} = 1$ supercurrent,
semisimple Lie algebras of dim 24

[Harrison-NMP-Persson-Volpato]

A few other natural (and super) examples suggest themselves

Associated BKMS using "string theory functor"

$V^{f\natural}$

[Harrison-NMP-Volpato]

F_{24}

8 different SBKMs

[Harrison-NMP-Persson-Volpato]

[Scheithauer]

V^{fE_8}

These SVOAs are implicated in a number of recent (stringy) developments...

- Conway module controls K3 σ-model symmetries [Gaberdiel-Hohenegger-Volpato, Duncan-Mack-Crane, Cheng-Harrison-Volpato-Zimet, NMP-Volpato-Zimet,...]
- V^{fE_8} plays a similar role for T^4 σ-model symmetries [Volpato, Anagiannis-Cheng-Duncan-Volpato]
- Plays a role in a proposed tmf explanation of Mathieu moonshine[Johnson-Freyd]
- "Reflection procedure": chiral/non-chiral correspondence [Taormina-Wendland, Anagiannis-Cheng-Duncan-Volpato]
- $V^{f\natural}$ produces moonshine modules for mock modular forms [Cheng-Dong-Duncan-Harrison-Kachru-Wrase]

$Co_0 \curvearrowright V^{f\natural}$

A super-BKM with Conway Co_0 action

[Harrison-NMP-Volpato]

$$T_g^s(\tau) := \sum_{n=-1}^{\infty} \mathrm{sTr}_{V^{f\natural}_{n/2}}(g) q^{n/2} = \sum_{n=-1}^{\infty} c_g^s(n/2) q^{n/2}$$

- **Neveu-Schwarz and Ramond sectors (twisted module)**
- **Ghosts and superghosts**
- **Traces and supertraces**

- $\mathscr{H}^m := V^{f\natural} \otimes V^{(1,1),super}$
- In both NS and R sectors, can split into eigenspaces under $(-1)^F$
- $\mathscr{H}_{GSO} = \left(\mathscr{H}^m_{NS+} \otimes \mathscr{H}^{gh}_{NS+}\right) \oplus \left(\mathscr{H}^m_{NS-} \otimes \mathscr{H}^{gh}_{NS-}\right) \oplus (NS \leftrightarrow R)$
- Take relative BRST cohomology on $\mathscr{H}_{GSO}$
- This ``space of physical states'' admits a Lie bracket, ultimately giving a $\Gamma^{1,1}$ -graded super-BKM
- Can use denominator formula to obtain info about (simple) roots, etc.

$$\prod_{m,n\in\mathbb{Z},m>0} (1-p^m q^n)^{C_{NS-}(mn)}(1+p^m q^n)^{-C_{R-}(mn)}$$

$$p^{-1}\prod_{d=1}^{\infty}\prod_{r\in\mathbb{Z}}(1-p^d q^{r/2})^{C_{d,r}(rd/2)} = T^s(2\sigma) - T^s(\tau)$$

related to $c^s(n/2)$ by GSO projection

new (super)denominator identities!
Conway action allows for equivariant versions...
alternative proof of genus 0?

F_{24}

8 super-BKMs with g-action

[Harrison-NMP-Persson-Volpato]

$\mathcal{N} = 1$ supercurrent

$$G(z) = -\frac{i}{6}\sum_{i,j,k} c_{ijk} : \lambda^i \lambda^j \lambda^k :$$

c_{ijk} structure constants of g: semisimple Lie algebras of dim 24

g: $A_1^8, \quad A_2^3, \quad A_3A_1^3, \quad A_4, \quad B_2G_2, \quad B_2A_2A_1^2, \quad B_3A_1, \quad C_3A_1$

∞-dim, rank 2 + rk(g)

$$\Gamma^{1,1} \oplus P^g$$

$$pqe^{-\rho_g} \prod_{w\in\Delta_g^+} (1-e^w)^{c_{NS-}(0,w)} \prod_{m,n\in\mathbb{Z}_{>0},(m,n)\neq(0,0)} \prod_{w\in P_g} \frac{(1-p^nq^ne^w)^{c_{NS-}(mn,w)}}{(1+p^mq^ne^w)^{c_{R+}(mn,w)}}$$

e.g.

$$\frac{\theta_3(\tau,0)^{r/2}\prod_{\alpha\in\Delta_g^+}\theta_3(\tau,(\zeta\,|\,\alpha))}{\eta(\tau)^{12}} = \sum_{n\in\frac{\mathbb{Z}}{2}}\sum_{w\in P_g} c_{NS}(n,w)q^ne^{2\pi i(\zeta|w)}$$

- also super-denominators
- know additive side for special cases... general formulas?

The heterotic Monster

[NMP-Persson-Volpato]

$\mathscr{H}^m$

$$V^{\natural} \otimes \bar{V}^{f\natural} \otimes S^1_R \otimes \mathbb{R}_t$$

$c = 24$ $\bar{c} = 12$

The heterotic Monster

[NMP-Persson-Volpato]

$\mathscr{H}^m$

$$V^{\natural} \otimes \bar{V}^{f\natural} \otimes S^1_R \otimes \mathbb{R}_t$$

`compactification' (asymmetric T^8 orbifold)

$c = 24$ $\bar{c} = 12$

spacetime

The heterotic Monster

[NMP-Persson-Volpato]

$$\mathscr{H}^m \qquad V^\natural \otimes \bar{V}^{f\natural} \otimes S^1_R \otimes \mathbb{R}_t$$

`compactification' (asymmetric T^8 orbifold) $c = 24$ $\bar{c} = 12$ spacetime

$$\mathscr{H}^{gh,tot} = \mathscr{H}^{gh} \otimes \bar{\mathscr{H}}^{gh}$$

$$Q^{BRST} = Q + \bar{Q}$$

Physical states are BRST cohomology classes

$$(k, \bar{k}) \in \Gamma^{1,1},\ k^2 - \bar{k}^2 = 2mn$$

$m, n \in \mathbb{Z}$

Superstring!

- **anti-holo. NS/R sectors**
- **Ghosts and superghosts**
- **GSO projection wrt $F_{tot} = F + \bar{F}$**

$$k = \frac{1}{\sqrt{2}}(m/R - nR)$$

$$\bar{k} = \frac{1}{\sqrt{2}}(m/R + nR)$$

$$\mathcal{N} = (0,24) \text{ SUSY}: \quad \left\{ \mathcal{Q}^i, \mathcal{Q}^j \right\} = 2\delta^{ij}(H - \bar{P})$$

BPS states: $H|BPS\rangle = \bar{P}|BPS\rangle$

States coming from excitations of a single string:

- $\mathcal{Q}^i|BPS\rangle$ are Q^{BRST}-exact: $\bar{Q}$-images of Q-closed states.
- Quotient this space by Q-exact states
- Monster Lie algebra $\mathfrak{g} \simeq$ (BRST-exact states in this space)
- BPS states transform in adjoint of $\mathfrak{g}$

$\mathfrak{g}$ is like a "spontaneously broken" gauge algebra (broken because the BPS states always have nonzero momentum)

[Borcherds] [NMP-Persson-Volpato]

Math	String Theory
tensor in vertex algebra: $V_{\Gamma^{1,1}}$	level-matching + right-moving mass-shell: left-moving momenta $\in \Gamma^{1,1}$
semi-infinite cohomology $Q^{BRST}_{Borcherds}$	Q
"physical state condition": class reps have conformal weight 1	left-moving mass-shell condition
Monstrous Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$	$V_{Monster} e^{ikX}$ (1:1 w/) BPS states
$\mathfrak{g}_+$	Positive-energy BPS states: $\frac{1}{\sqrt{2}}(m/R + nR) > 0$
$\wedge\, \mathfrak{g}_+$	"Second quantized" BPS states: Fock space from many strings
$\kappa(a,b)$ invariant, nondegen bilinear form	2-pt function of hol'c vertex ops On Cartan: Lorentzian spacetime metric $\eta^{\mu\nu}$
$H_\bullet(\mathfrak{g}_+) : (\gamma \,\vert\, \gamma + \rho) = 0$	$2M_{tot}W_{tot} = 0$
Weyl denominator	BPS index: $Tr_{\mathcal{H}}(-1)^F e^{-\beta H} e^{2\pi i m b} e^{2\pi i n v}$
Weyl reflection exchanging real simples $\sim \mathbb{Z}_2$	Phase transition (T-duality, critical radius)
Modularity/automorphy	(T-)dualities

Examples in the type II string

[Harrison-NMP-Persson-Volpato (to appear)]

$\mathscr{H}^m$

$$V \otimes \bar{W} \otimes S^1_R \otimes \mathbb{R}_t$$

$c = 12 \qquad \bar{c} = 12$

$$\mathscr{H}^{gh,tot} = \mathscr{H}^{gh} \otimes \bar{\mathscr{H}}^{gh}$$

$$Q^{BRST} = Q + \bar{Q}$$

Physical states are BRST cohomology classes

Superstring!

- **NS/R sectors (holo/antiholo)**
- **Ghosts and superghosts**
- **GSO projection wrt $F_{tot} = F + \bar{F}$**

These type II models have many qualitative similarities with the Monster construction,
but they differ in numerous details. Different superalgebras, nontrivial states at zero-momentum, choices for BPS indices, etc.

But as before we can do, e.g, a path integral computation via Harvey-Moore-Borcherds theta-lift.

We recover (super)denominators for super-BKMs in expansions at cusps.

Summary of these examples

- Instructive to study string compactifications to 2d/1d/“0d”
- Special examples given by “factorized” internal worldsheet SCFTs (products of SVOAs)
- When moonshine modules, can explore genus-zero properties
- BPS states in these models form representations of super-BKM algebras
- New examples of super-BKMs, concrete realizations in string theory

Some next steps

- More detailed explorations of the type II models
- Additive denominators for F_{24}
- D-branes in these models
- Genus zero study for Conway example
- Type II dual for the heterotic Monster?
- Duality webs for all our models?

Speculations &c.

- BPS algebras & COHA?
- Limits of moduli space in these models (e.g. decompactification limits)... useful for more general low dimensional compactifications?
- Different BKMs at different cusps... `parent' automorphic forms in physics?
- Connections to Gromov-Witten theory, cf. $1/\Phi_{10}$ [Katz-Klemm-Vafa, Oberdieck-Pandharipande]
- Automorphic forms/BPS indices for more general moduli spaces? e.g. K3, ...

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Thank you, and stay healthy!