Non-SUSY Solitons in Gravity

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Motivation ...

Observation of gravitational waves & Emerging field of gravitational spectroscopy for Ultra Compact Objects

What is the spectrum of Ultra Compact Objects in theories of gravity?

Microstates for BPS black holes admit coherent states that can be describe as smooth horizon-less solutions in supergravity (Microstate geometries — Mathur, Bena-Warner)

Can coherent states of Quantum Gravity exist as non-BPS solutions of classical theories of gravity?

On Solitons in gravity

Need a framework for constructing solitonic solutions in theories of gravity

- Asymtotically flat, smooth and horizon-less solutions
- Finite energy and definite charges
- Classically stable, Meta-stable saddles of the gravitational Euclidian action

Hurdles... Opportunity?!?

Without Supersymmetry — Deal with non-linear PDEs of Einstein equations (Good Luck...)

No-Go Theorem: Serini (1918); Einstein, Pauli, Lichnerowitz (1940's); Recently revisited in supergravity by Gibbons-Warner

If a vacuum solution in four dimensions is

- Asymtotically flat
- Topologically trivial
- Globally Stationary (Global timelike killing vector)

It must be flat!

Hurdles… Opportunity?!?

Ways to move ahead (Gibbons-Warner, 2013)

- Solutions with interesting topology
- Add various Maxwell-type fields — support topological structure
- Topological interactions — Chern-Simons terms
- Most importantly — Extra Dimensions

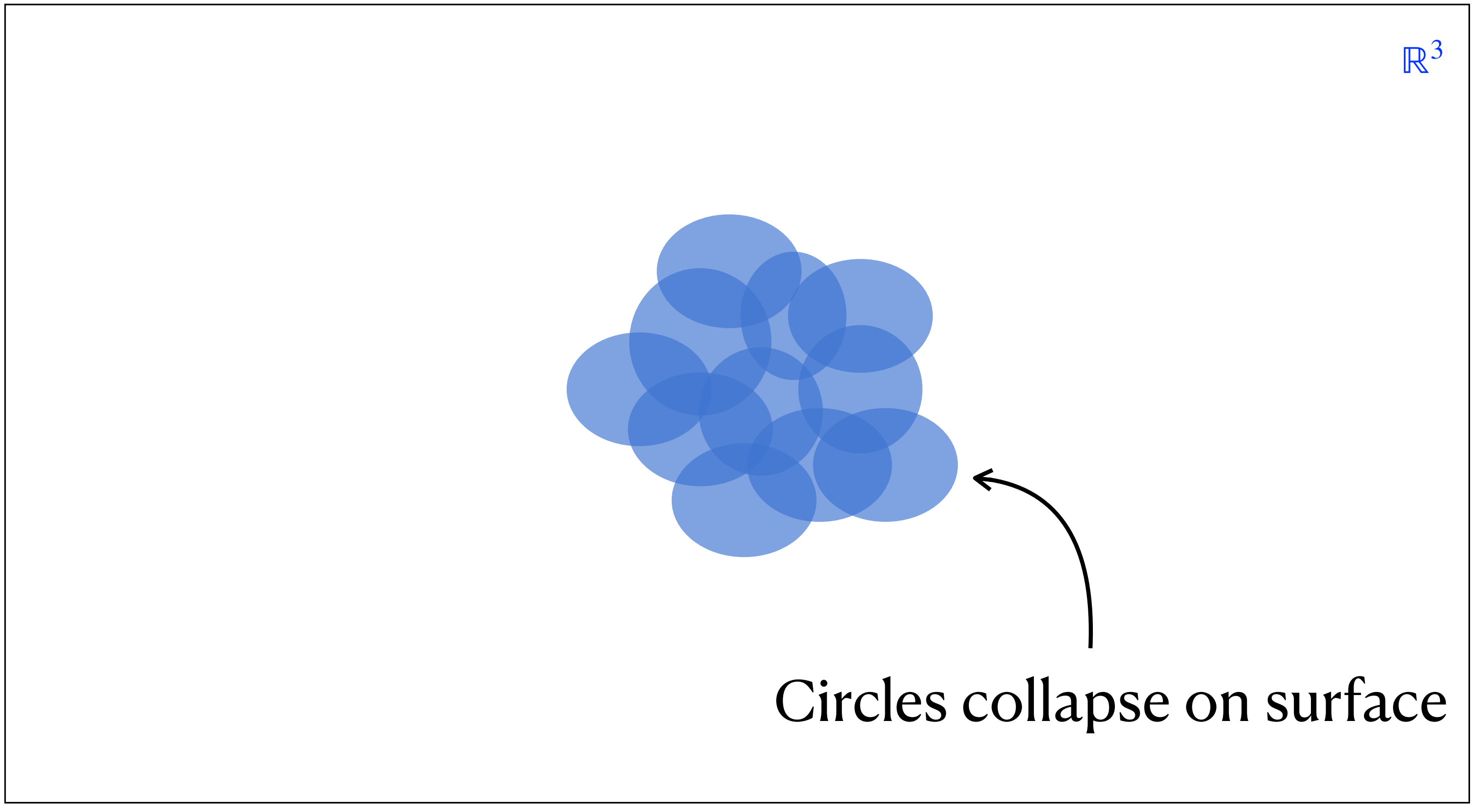
Look for solutions in backgrounds that are Asymptotic to

$M^4 \times T^n$ — 4d Minkowski with extra circles

Topological Star

Look for solutions in backgrounds that are Asymtoptic to

$M^4 \times T^n$ – 4d Minkowski with extra circles

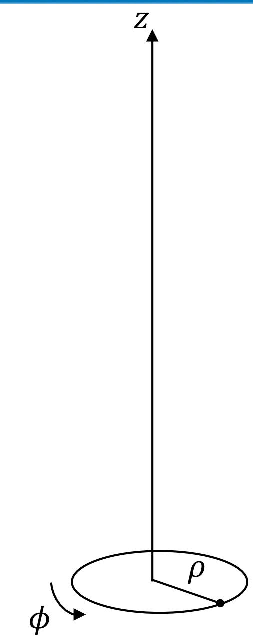


Construct such smooth and regular geometries!?!?

Weyl Systems

Static, axially symmetry spacetimes

4d Weyl Systems



$$ds_4^2 = -\frac{1}{Z}dt^2 + Z\, ds^2(M_3), \qquad ds^2(M_3) = e^{2\nu}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2$$

$$F = dH \wedge d\phi$$

Einstein Equations decoupled into two sectors

Maxwell Sector — Reduced EE

$$\partial^a\left(\frac{1}{\rho Z}\partial_a H\right) = 0, \qquad \left(\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\right) + \partial_z^2\right)\log Z = \frac{c}{\rho Z}\left[\left(\partial_\rho H\right)^2 + \left(\partial_z H\right)^2\right]$$

Base equations

$$\partial_a \nu = P_a\left(H, Z, \partial H, \partial Z\right)$$

Integrability implied by Maxwell Sector

Higher-D Weyl Systems

Generalize the base

$$ds^2(M_3) \;\rightarrow\; ds^2(M_4) = \frac{1}{Z_0}\left(dy_0 + H_0\, d\phi\right)^2 + Z_0\, ds^2(M_3)$$

Generalize the fiber

$$ds^2_{n+5} = \frac{1}{Z_1}\left[-W_0\, dt^2 + \sum_{i=1}^{n} W_i\, dy_i^2\right] + Z_1^n\, ds^2(M_4)$$

$$F = dH_1 \wedge d\phi \wedge dy_0, \qquad W_0 \prod W_i = 1$$

Einstein equations decouple into several sectors

Reduced EE sectors	Vacuum Sector	Base sector
$(Z_1, H_1), \quad (Z_0, H_0)$	$\partial^a\left(\rho\partial_a \log W_i\right) = 0$	$\partial_a \nu = P_a\left(H_I, Z_I, W_i, \partial H_I, \ldots\right)$

For Vacuum Weyl system (Emparan & Reall `01)

Torus Weyl Systems

$$ds_6^2 = \frac{1}{Z_1}\left[-W\,dt^2 + W^{-1}\,dy_1^2\right] + Z_1\,ds^2(M_4)$$

$$ds^2(M_4) = \frac{1}{Z_2}\left(dy_2 + H_2\,d\phi\right)^2 + Z_2\,ds^2(M_3)$$

Find solutions with

Circles shrink at locus

$y_2 : Z_2 \to \infty, \quad Z_1 > 0$

$y_1 : Z_1 W \to \infty, \quad \dfrac{Z_1}{Z_2} > 0$

No Horizon

$Z_1 W^{-1} > 0$

Reduced EE sectors

$(Z_1, H_1), \quad (Z_2, H_2)$

Vacuum Sector

$\partial^a\left(\rho\partial_a \log W\right) = 0$

Base sector

$\partial_a \nu = P_a\left(H_I, Z_I, W, \partial H_I, \ldots\right)$

Constructing Solutions

BPS Solutions

Maxwell Sector — Reduced EE

$$\partial^a \left(\frac{1}{\rho Z} \partial_a H \right) = 0, \qquad \partial^a \left(\rho \partial_a \log Z \right) = - \frac{\kappa^2}{\rho Z^2} \left[\left(\partial_\rho H \right)^2 + \left(\partial_z H \right)^2 \right]$$

Consider the functions (X, Y) — X is a harmonic potential

$$\partial^a \left(\rho \partial_a X \right) = \partial^a \left(\frac{1}{\rho} \partial_a Y \right) = 0, \qquad \partial_a X \partial^a Y = 0, \qquad | \partial_a Y) |^2 = \rho^2 | \partial_a X |^2$$

There is the BPS solutions for multi-center black holes 4d Weyl

$$\kappa \, H = Y, \qquad Z = X + \text{constant}$$

On sources

$$ds_6^2 = \frac{1}{Z_1}\left[-W\, dt^2 + W^{-1}\, dy_1^2\right] + Z_1\, ds^2(M_4)$$

Boundary Conditions

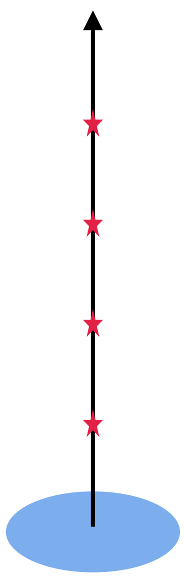
$$y_1 : Z_1 W \to 0 \quad \frac{Z_1}{Z_2} > 0, \quad \frac{Z_1}{W} > 0$$

Point Charges

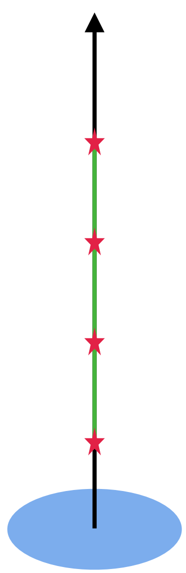
$$\partial^a \left(\rho \partial_a Z_1\right) = 0$$

Line Charges

$$\partial^a \left(\rho \partial_a \log W\right) = 0$$



Sources not compatible and lead to singular geometries when flux is non-trivial



Reduced EE

$$\partial^a \left(\frac{1}{\rho Z} \partial_a H \right) = 0, \qquad \partial^a \left(\rho \partial_a \log Z \right) = -\frac{\kappa^2}{\rho Z^2} \left[\left(\partial_\rho H \right)^2 + \left(\partial_z H \right)^2 \right]$$

Consider the functions (X, Y)

$$\partial^a \left(\rho \partial_a X \right) = \partial^a \left(\frac{1}{\rho} \partial_a Y \right) = 0, \qquad \partial_a X \partial^a Y = 0, \qquad |\partial_a Y)|^2 = \rho^2 |\partial_a X|^2$$

Consider larger class of solutions ?!? $\kappa\, H = Y,$ $\quad Z = F(X)$

$$\boxed{F_1 = \frac{1}{a} \sinh(aX + b),} \qquad F_2 = \frac{i}{a} \cosh(aX + b)$$

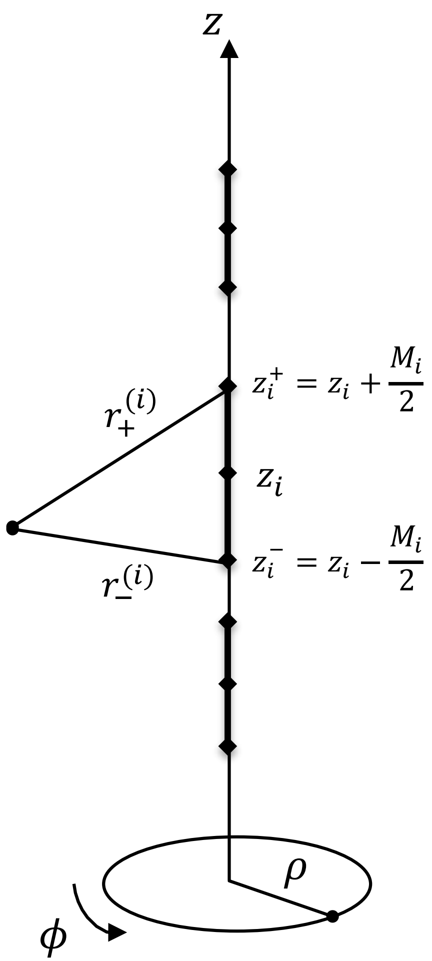
$$(a, b) \in \mathbb{R}$$

$$F_3 = \frac{1}{a} \sin(aX + b), \qquad F_4 = \frac{1}{a} \cos(aX + b)$$

Rod Sources

$$ds_6^2 = \frac{1}{Z_1}\left[-W\,dt^2 + W^{-1}\,dy_1^2\right] + Z_1\,ds^2(M_4)$$

$$F = dH_1 \wedge d\phi \wedge dy_2 \qquad ds^2(M_4) = \frac{1}{Z_2}\left(dy_2 + H_2\,d\phi\right)^2 + Z_2\,ds^2(M_3)$$



For each rod source, construct a solution of the Laplace operator $X_i(\rho, z)$ that diverges at the source

$$Z_I = \frac{1}{2a_I}\left[e^{b_I}\prod_{i=1}^{N} X_i^{a_I P_i^I} - e^{-b_I}\prod_{i=1}^{N} X_i^{-a_I P_i^I}\right]$$

$$W = \prod X_i^{G_i}, \qquad H_I = \sum P_i^I\left(r_+^i - r_-^i\right)$$

$$r_\pm^i = \sqrt{\rho^2 + (z - z_i^\pm)^2}$$

The P_i^I associate a magnetic flux for each rod source

Rod Species

$$ds_6^2 = \frac{1}{Z_1}\left[-W\, dt^2 + W^{-1}\, dy_1^2\right] + Z_1\, ds^2(M_4)$$

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$$W = \prod X_i^{G_i}, \qquad H_I = \sum P_i^I (r_+^i - r_-^i)$$

The P_i^I associates a magnetic flux for each rod

There is a unit magnetic charge for each rod

Asymptotically flat 4d spacetime requires

$$a_I = \sinh(b_I)$$

Black rod — Horizon

$$P_i^I = \frac{1}{2a_I}, \qquad G_i = -\frac{1}{2}$$

Bubble 1 — Circle y_1 shrink

$$P_i^I = \frac{1}{2a_I}, \qquad G_i = \frac{1}{2}$$

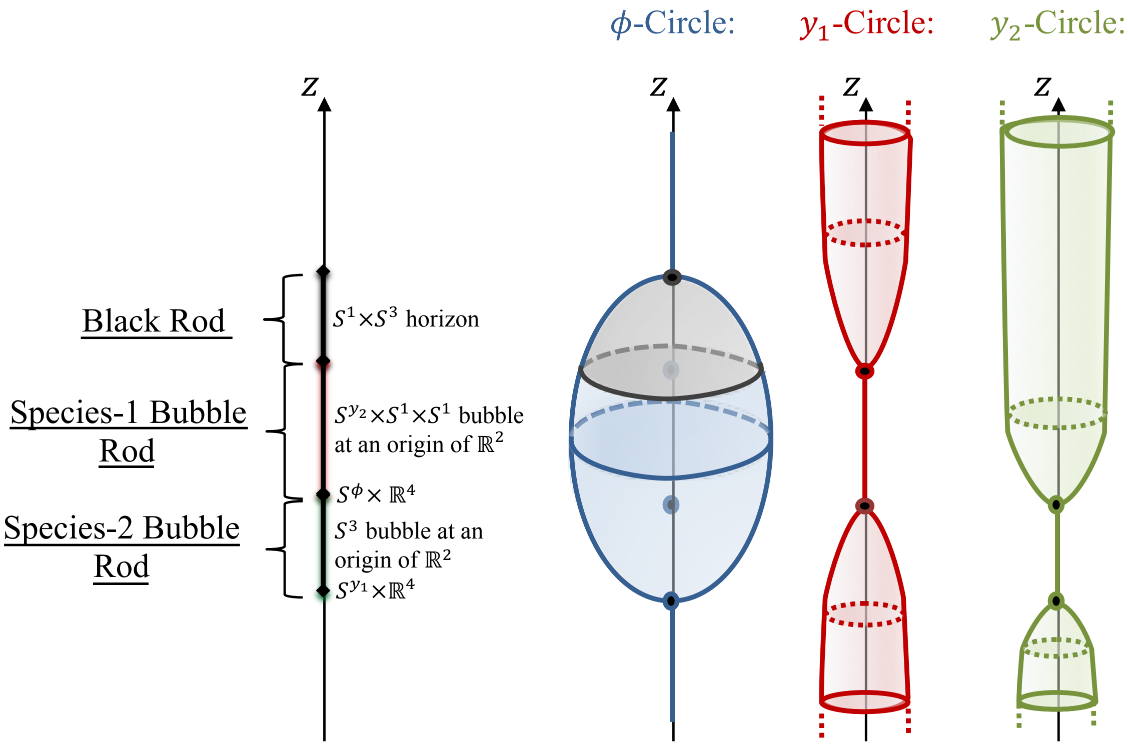
Bubble 2 — Circle y_2 shrink

$$P^2 = \frac{1}{a_2}, \qquad P_i^1 = G_i = 0$$

Rod Species

$$ds_6^2 = \frac{1}{Z_1}\left[-W\, dt^2 + W^{-1}\, dy_1^2\right] + Z_1\, ds^2(M_4)$$

$$F = dH_1 \wedge d\phi \wedge dy_2 \qquad ds^2(M_4) = \frac{1}{Z_2}\left(dy_2 + H_2\, d\phi\right)^2 + Z_2\, ds^2(M_3)$$



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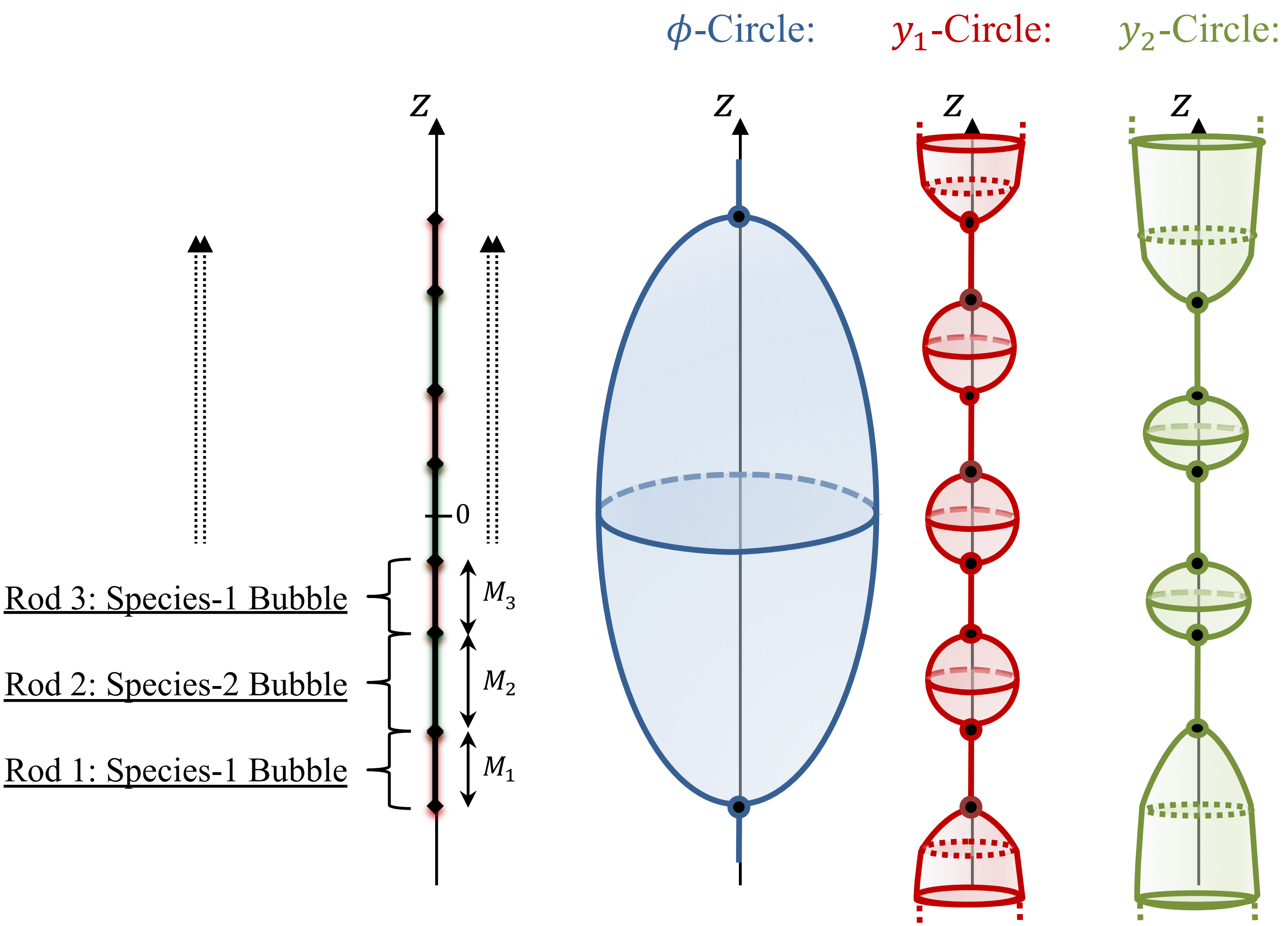
N-Rod System

$$ds_6^2 = \frac{1}{Z_1}\left[-W\, dt^2 + W^{-1}\, dy_1^2\right] + Z_1\, ds^2(M_4)$$

$$F = dH_1 \wedge d\phi \wedge dy_2 \qquad ds^2(M_4) = \frac{1}{Z_2}\left(dy_2 + H_2\, d\phi\right)^2 + Z_2\, ds^2(M_3)$$

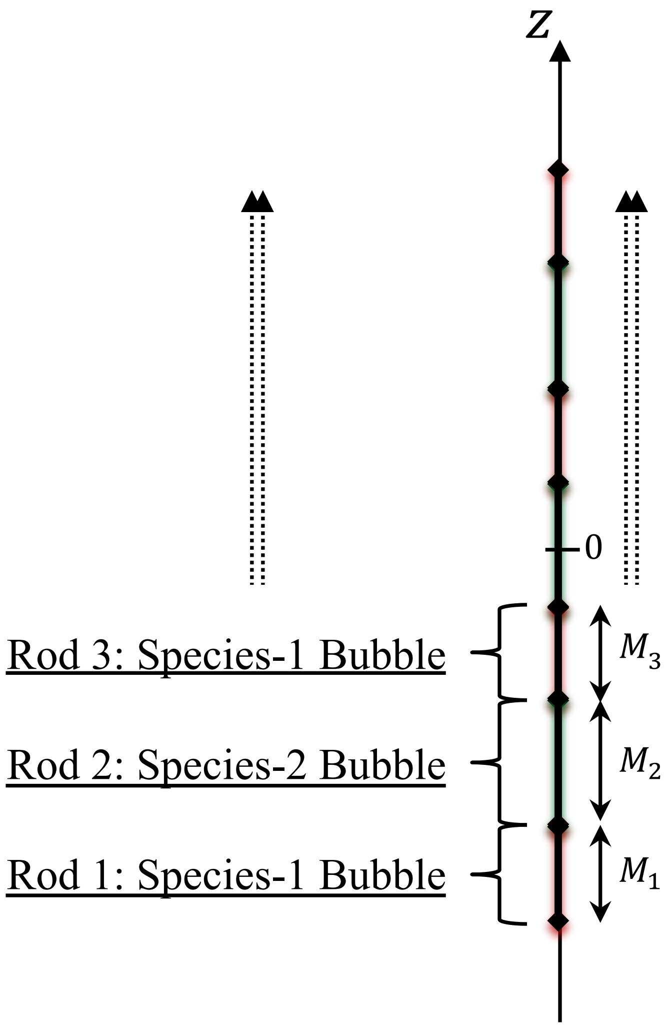


The geometries constructed by the chain of magnetic rods are free from curvature singularities and of horizons!

Regularity of metric leads to N algebraic conditions that relate the mass parameters M_i With asymptotic data (R_{y_1}, R_{y_2}, a_I)

Asymptotic Data

Asymptotic data can be expressed in terms of the mass parameters



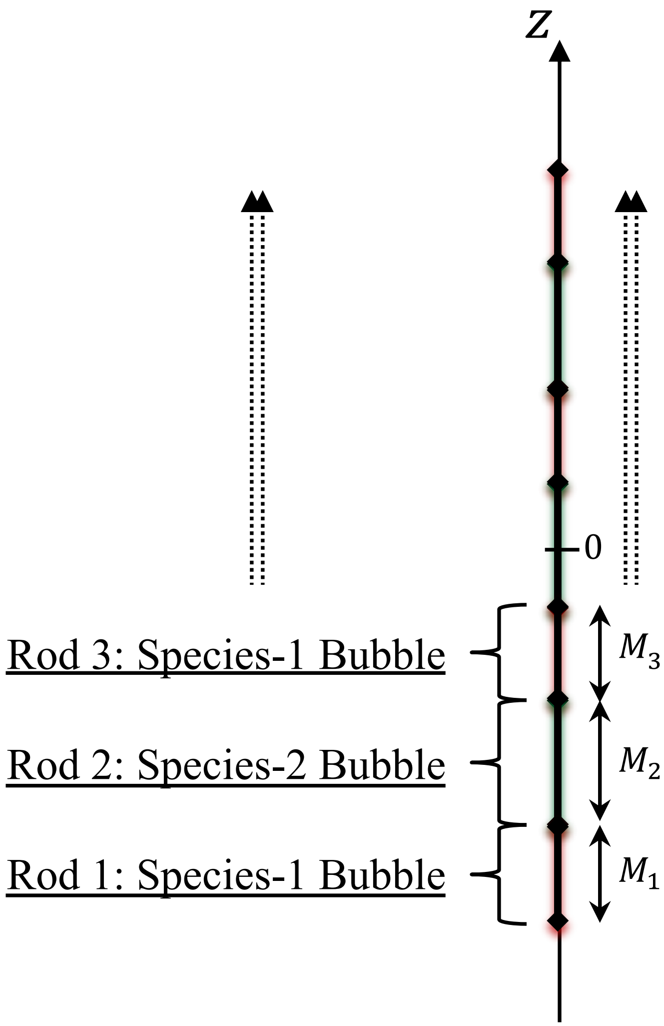
$$M_{ADM} = \frac{1}{4G_4}\left[\coth b_2 \sum_{i=1}^{N} M_{2i} + \frac{\coth b_2 + 2\coth b_1 - 1}{2} \sum_{i=1}^{N+1} M_{2i-1}\right]$$

$$Q_m^1 = \frac{1}{8\sqrt{\pi G_4}\sinh b_1} \sum_{i=1}^{N+1} M_{2i-1}$$

$$Q_m^2 = \frac{1}{8\sqrt{\pi G_4}\sinh b_2}\left[\sum_{i=1}^{N+1} M_{2i-1} + 2\sum_{i=1}^{N} M_{2i}\right]$$

Regularity and Force balance

The mass parameters are constrained by regularity conditions near each Rod



$$L_{2i}(M_1, M_2, \ldots) = \frac{4\sinh^2 b_1}{e^{2b_1}} \frac{2\sinh b_2}{e^{b_2}} R_{y_1}^2$$

$$L_{2i-1}(M_1, M_2, \ldots) = \frac{4\sinh^2 b_2}{e^{2b_2}} R_{y_2}^2$$

The regularity conditions are force balance conditions for the rods. They are ratio of polynomials of degree $> N$.

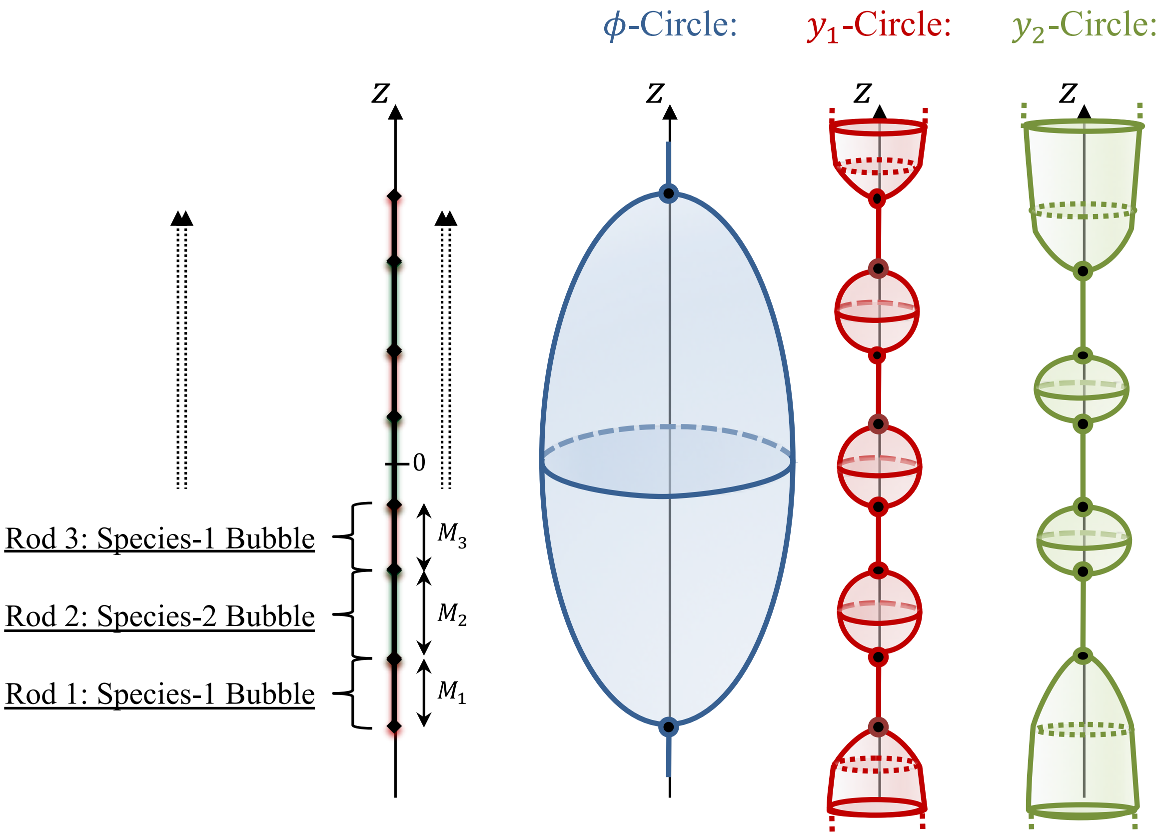
Bubble Bag-End Spacetimes

Solution at large N

N-Rod System

$$ds_6^2 = \frac{1}{Z_1}\left[-W\,dt^2 + W^{-1}\,dy_1^2\right] + Z_1\,ds^2(M_4)$$

$$F = dH_1 \wedge d\phi \wedge dy_2 \qquad ds^2(M_4) = \frac{1}{Z_2}\left(dy_2 + H_2\,d\phi\right)^2 + Z_2\,ds^2(M_3)$$

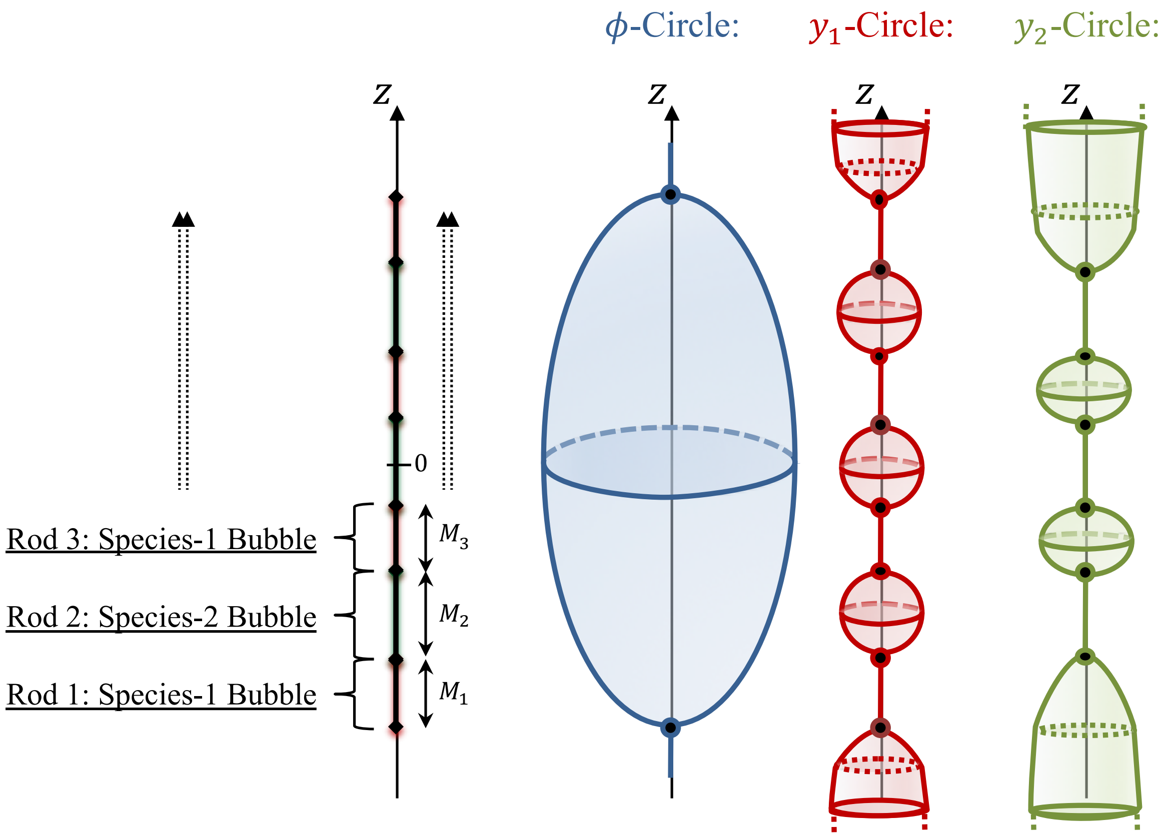


In the large N limit, we look for solutions to the regularity conditions with

$$R_{y_1} \sim R_{y_2} \sim R_y$$

$$M_{2i} \sim M, \qquad M_{2i-1} \sim M$$

N-Rod System solution



Rod length: $M_i \sim N^{-3/4} R_y$

Total mass and Charges

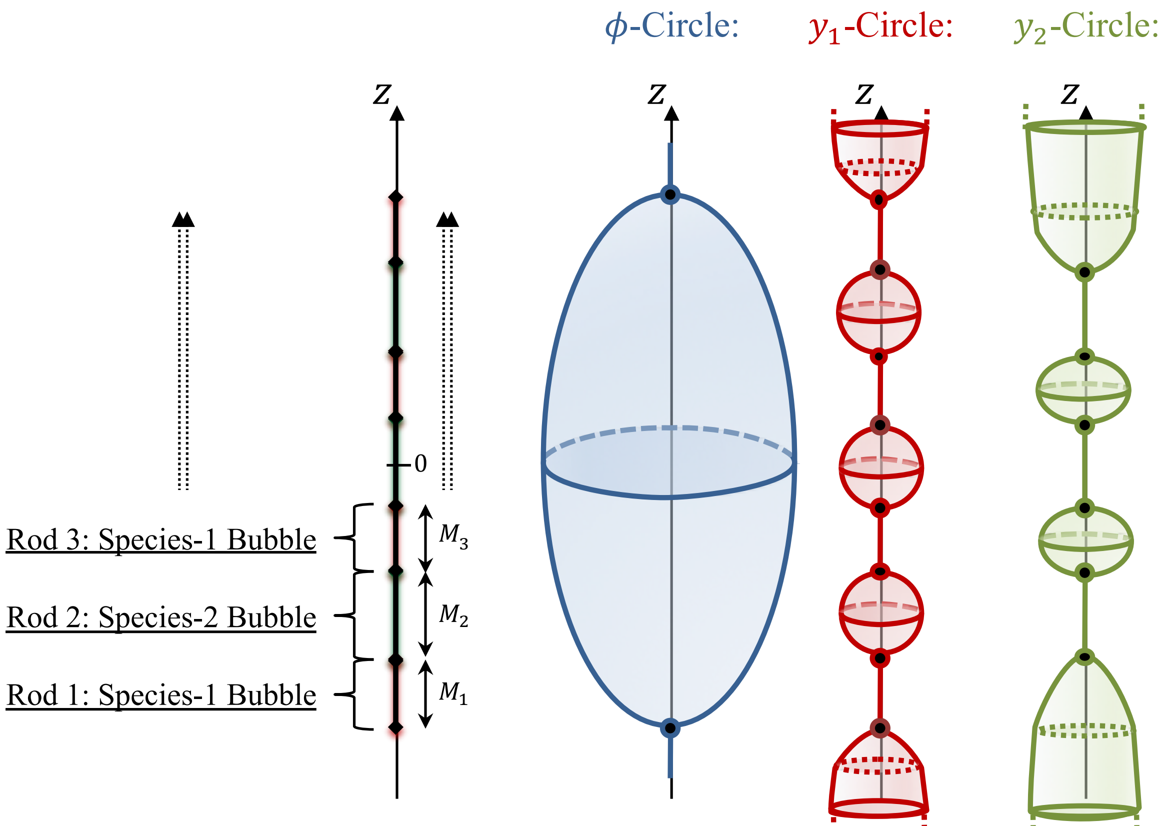
$$M_{ADM} \sim N^{1/4} R_y$$

$$Q_m^1 \sim N^{1/4} R_y \qquad Q_m^2 \sim N^{1/4} R_y$$

Proper length of the system

$$\Delta_{N-S} \sim N^{1/2} R_y \qquad \Delta_{E-S} \sim N^{1/4} R_y$$

Outside Spacetime



Metric dramatically simplify right outside of structure

$$M = \frac{1}{2}\sum M_i \sim N^{1/4} R_y$$

$$r > 2M\left(1 + \mathcal{O}(N^{-1/2})\right)$$

The metric in zero charge limit $r > 2M\left(1 + \mathcal{O}(N^{-1/2})\right)$

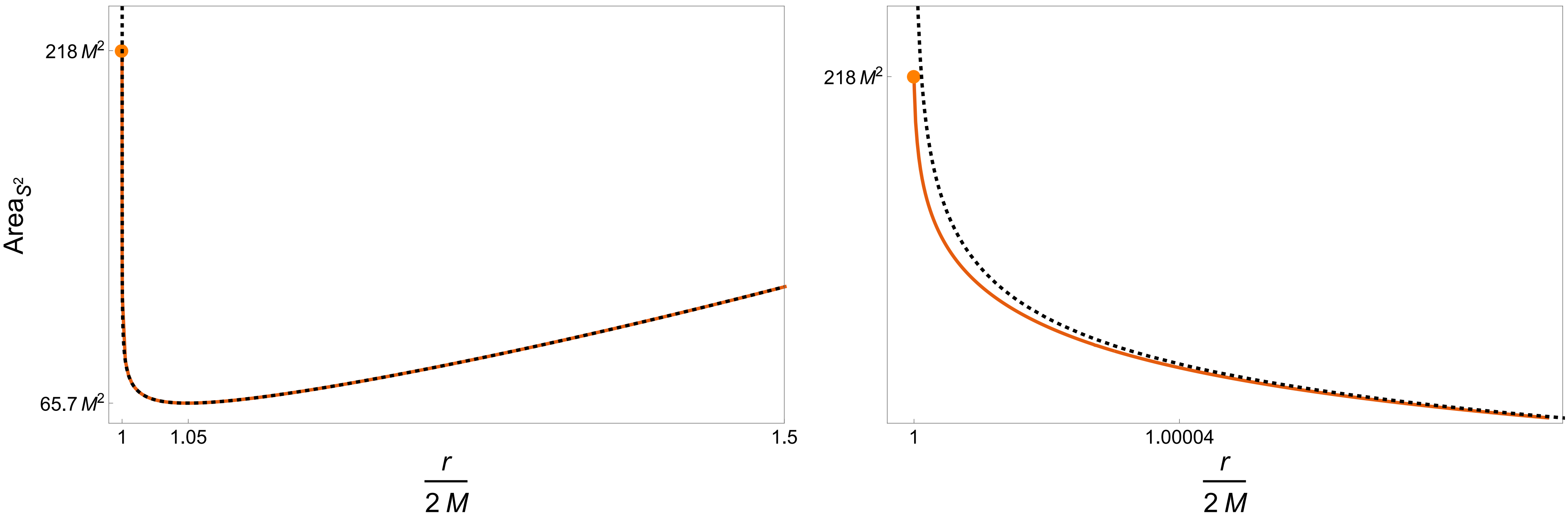
$$ds_6^2 = -dt^2 + \sqrt{1 - \frac{2M}{r}}\left(dy_1^2 + dy_2^2\right) + \left(\frac{(r-M)^2 - M^2\cos^2\theta}{r(r-M)}\right)^{1/4}\left(\frac{rdr^2}{r-2M} + r^2 d\theta^2\right) + r^2\sin^2\theta\, d\theta^2$$

Bag-End structure

Consider co-centric spheres around structure with area

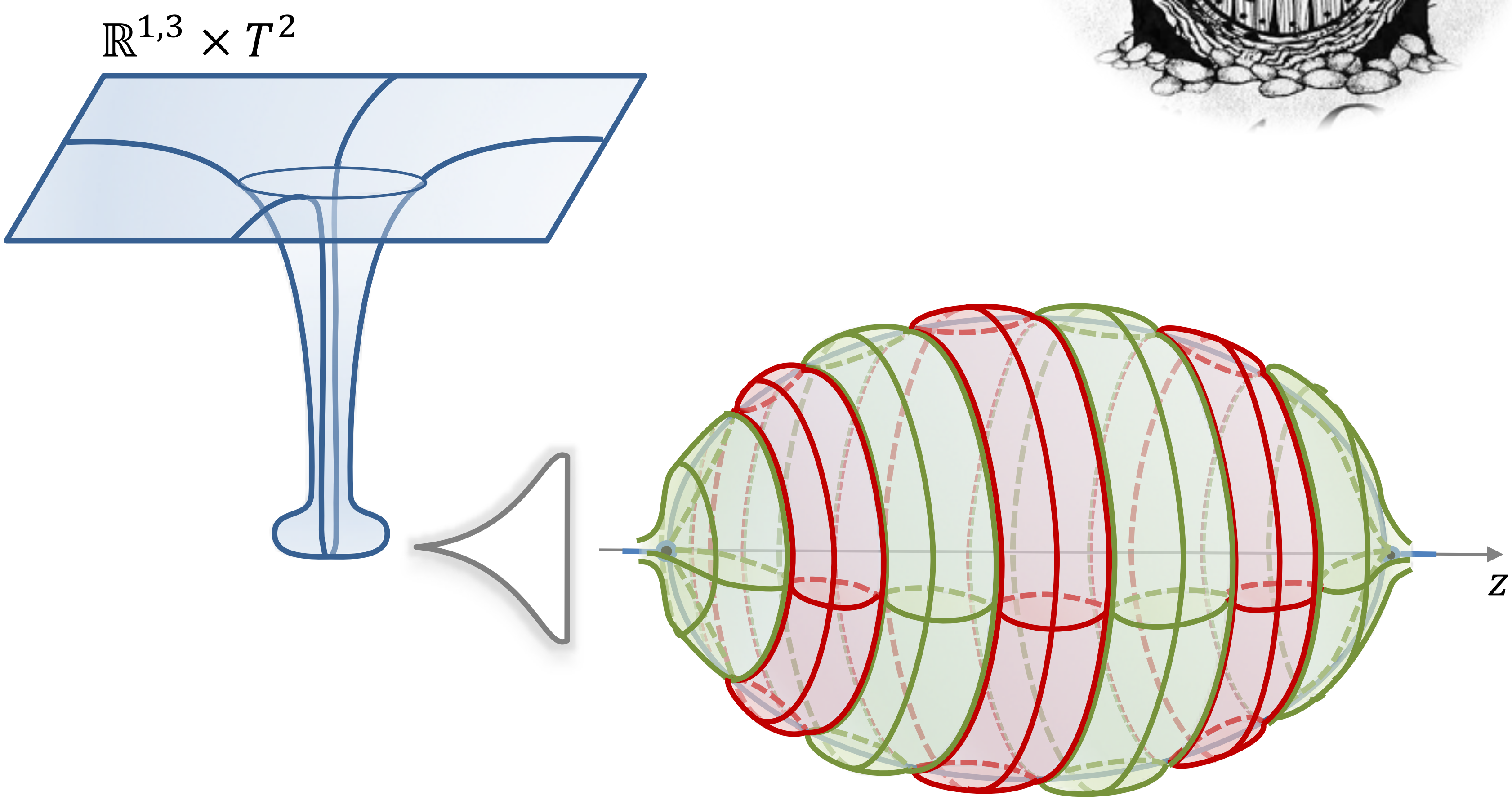
$$\mathrm{Area}_{S^2} = \int \sqrt{g_{\theta\theta} g_{\phi\phi}}$$

Plots for $N = 20$. The area of the spheres shrink, reach a minimal surface and grow again close to the structure



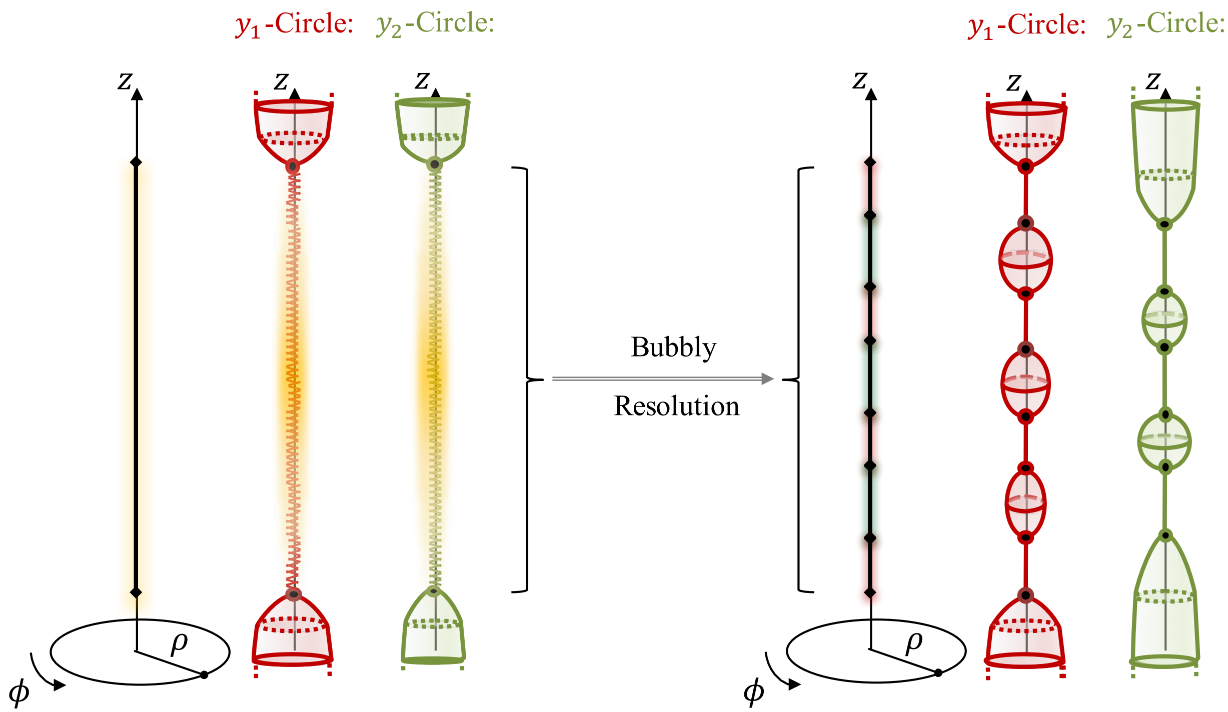
$$ds_6^2 = -dt^2 + \sqrt{1 - \frac{2M}{r}}\,(dy_1^2 + dy_2^2) + \left(\frac{(r-M)^2 - M^2\cos^2\theta}{r(r-M}\right)^{1/4}\left(\frac{r dr^2}{r-2M} + r^2 d\theta^2\right) + r^2 \sin^2\theta\ d\theta^2$$

Bubble Bag-End structure



Resolution of singularity

We can consider a system where the rod sources are smeared to obtain singular solutions

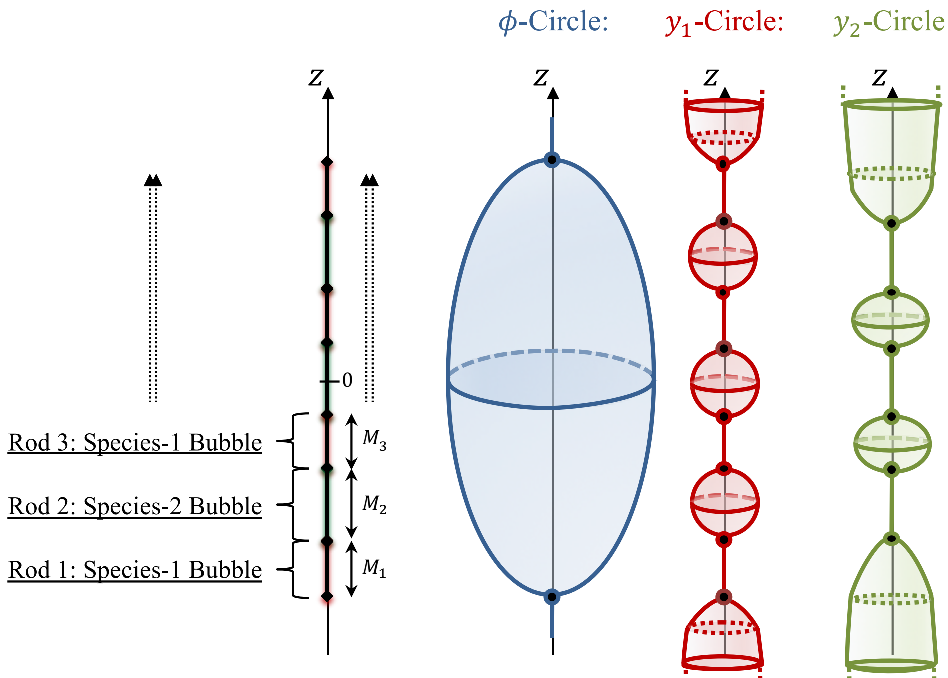


The singular solution is labeled by two parameters (M, D) in addition to charges (Q_m^1, Q_m^2)

(M, D) will be related to the asymptotic radii (R_{y_1}, R_{y_2})

$$ds_6^2 = \frac{1}{Z_1}\left(-dt^2 + \left(1 - \frac{2M}{r}\right)^D dy_1^2\right) + \frac{Z_1}{Z_2}\left(dy + \frac{(2-D)M}{\sinh b_0}\right)^2$$
$$+Z_1 Z_2\left[\left(\frac{(2-M)^2 - M^2\cos^2\theta}{r(r-2M)}\right)^{D(1-D)}\left(\frac{r dr^2}{r-2M} + r^2 d\theta^2\right) + r^2\sin^2\theta d\phi^2\right]$$

Some Outlook



- What is the general space of non-SUSY solitons? Adding Gibbons-Hawking centers? Embedding in string theory!
- Exploit integrability structure in GR, inverse scattering methods? Backlung transformations?
- Physical observables in the sky? Phenomenological realizations and implications?
- General aspect of stability and existence?