S-duality and mirror symmetry in $T\overline{T}$ deformed CFT

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Energy spectrum of a $T\overline{T}$ deformed CFT

Consider a CFT with a discrete spectrum with energies and momenta

$$E_n = h_n + \bar{h}_n - \frac{c}{12} \qquad j_n = h_n - \bar{h}_n.$$

Zamolodchikov and Smirnov showed that the deformed CFT defined by

$$S_{T\overline{T}} \; = \; S_{\rm CFT} - \int d^2z \, O_{T\overline{T}}(\lambda), \qquad \partial_\lambda O_{T\overline{T}}(\lambda) \; = \; T\overline{T} - \Theta^2$$

with Θ the trace of the stress tensor is integrable and that the deformed energy spectrum is exactly given by

$$\mathcal{E}_n(\lambda) \; = \; \frac{1}{\lambda}\big(-1 + \sqrt{1 + 2E_n\lambda + j_n^2\lambda^2}\big).$$

The deformed energies are real when $c \leq 6$ and $\lambda < 1$.

Derivation of the $T\overline{T}$ deformed spectrum

$$\begin{aligned}\langle n|\, T\overline{T}\,|n\rangle &= \langle n|\, T\,|n\rangle\,\langle n|\,\overline{T}\,|n\rangle - \langle n|\,\Theta\,|n\rangle\,\langle n|\,\Theta\,|n\rangle \\ &= -\frac{1}{4}\Big(\langle n|\, T_{\tau\tau}\,|n\rangle\,\langle n|\, T_{xx}\,|n\rangle - \langle n|\, T_{\tau x}\,|n\rangle\,\langle n|\, T_{\tau x}\,|n\rangle\Big)\,.\end{aligned}$$

The left-hand side of represents the λ dependence of the energy E_n

$$\frac{\partial \mathcal{E}_n}{\partial \lambda} = \langle n|\, T\overline{T}\,|n\rangle\,. \tag{1}$$

The stress tensor components have physical meaning as the energy density, pressure and momentum density.

$$\langle n|\, T_{\tau\tau}\,|n\rangle = \frac{\mathcal{E}_n}{L}\,,\quad \langle n|\, T_{xx}\,|n\rangle = \frac{\partial \mathcal{E}_n}{\partial L}\,,\quad \langle n|\, T_{\tau x}\,|n\rangle = \frac{iJ_n}{L}\,.$$

The above relation thus combines into

$$0 = 4\frac{\partial \mathcal{E}_n}{\partial \lambda} + \mathcal{E}_n\frac{\partial \mathcal{E}_n}{\partial L} + \frac{J_n^2}{L}\,. \tag{2}$$

Partition function of $T\overline{T}$ deformed CFT

The torus partition function of the $T\overline{T}$ deformed CFT reads

$$Z_1(\lambda,\sigma) \;=\; \sum_n \exp(2\pi i(\sigma_1 j_n + i\sigma_2 \mathcal{E}_n(\lambda))) \tag{3}$$

with $\sigma = \sigma_1 + i\sigma_2 =$ the modular shape parameter of the torus.

- Z_1 is real and finite provided that $c \leq 6$ and $\lambda \leq 1$.
- Z_1 is invariant under $SL(2,\mathbb{Z})$ modular transformations $\sigma \to \frac{a\sigma+b}{c\sigma+d}$.

Z_1 can be expressed as the following integral transform of the undeformed CFT partition function

$$Z_1(\lambda,\sigma) \;=\; 2\rho_2 \int \frac{d^2\tau}{\tau_2^2}\, e^{-\frac{\pi\rho_2}{\sigma_2\tau_2}|\sigma-\tau|^2}\, Z_{\rm CFT}(\tau) \qquad\qquad \rho_2 \equiv \frac{\sigma_2}{\lambda}$$

Here the integral runs over the full upper half-plane $\mathbb{H}_2$.

Partition function of $T\overline{T}$ deformed CFT

We can rewrite the deformed partition function as an integral over the modular domain

$$Z_1(\lambda,\sigma) \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} K_1(\tau;\rho,\sigma)\, Z_{\rm CFT}(\tau)$$

with

$$K_1(\tau;\rho,\sigma) \;=\; 2\rho_2 \sum_{\gamma\,\in\,{\rm PSL}(2,\,\mathbb{Z})} e^{\frac{i\pi\rho}{2\sigma_2(\gamma\tau)_2}|\sigma-\gamma\tau|^2}$$

Here $\rho = b + i\frac{\sigma_2}{\lambda}$ with b tuned such that $\bar{\rho} = 0$. This kernel K_1 equals the classical partition function of a complex scalar field $X(z,\bar{z})$ that maps the worldsheet torus with modulus τ into the target space torus with metric

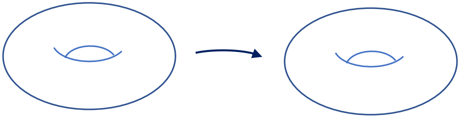
$$ds^2 \;=\; \frac{\rho_2}{\sigma_2}\,\big|\, dx_1 + \sigma\, dx_2\,\big|^2 \tag{4}$$

and with a B-field turned on and tuned such that $\sqrt{g} + iB = 0$. The map $X(z,\bar{z})$ from $\mathbb{T}^2$ to $\mathbb{T}^2$ is restricted to have wrapping 1.

Partition function of $T\overline{T}$ deformed CFT

$K_1(\tau;\rho,\sigma)$ equals the restricted $\Gamma_{2,2}$ Narain sum with wrapping number 1

$$K_1(\tau;\rho_2,\sigma) \;=\; \sum_{n,m} e^{\,iS[X_{nm}]}$$



Maps from from $\mathbb{T}^2$ to $\mathbb{T}^2$ with winding number 1 are labeled by two pairs of winding numbers $w_1 = (m_1, n_1)$ and $w_0 = (m_0, n_0)$ with

$$\gcd(n_0, m_0) = \gcd(n_1, m_1) = 1, \qquad\qquad n_0 m_1 - m_0 n_1 = 1.$$

The corresponding classical solution and classical action take the form

$$X_{nm}(z,\bar{z}) \;=\; \frac{1}{2i\tau_2}\big((n - m\bar{\tau})\, z \,-\, (n - m\tau)\,\bar{z}\big),$$

$$S[X_{nm}] \;=\; \frac{\rho_2}{\tau_2\,\sigma_2}\big|n_1 + n_0\sigma - m_1\tau - m_0\sigma\tau\big|^2.$$

Partition function of a symmetric product CFT

Define the grand canonical partition function of a symmetric product CFT by the weighted sum

$$Z(\rho_1, \sigma) \;=\; \sum_N p^N Z_N(\sigma), \qquad\qquad p \equiv e^{2\pi i \rho_1} \tag{5}$$

with $Z_N(\sigma)$ the partition function of the N-fold symmetric product

$$S^N \mathrm{CFT} = \mathrm{CFT}/S_N \tag{6}$$

of some given seed CFT with partition function $Z_{\mathrm{CFT}}(\sigma) = Z_1(\sigma)$.

$Z(\rho_1, \sigma)$ can be interpreted as the BPS partition function of a free 2nd quantized string theory with a $\mathbb{T}^2 = S^1 \times S^1$ target space.

We will extend this correspondence to a non-chiral setting and introduce a finite string scale set by the $T\overline{T}$ coupling.

The DMVV formula for the grand canonical chiral partition function

$$
\begin{aligned}
Z_{\mathrm{DMVV}}(p,\sigma) &= \prod_{d>0}\prod_{m\geq 0}\frac{1}{(1-p^d q^m)^{c(md)}} \\
&= \prod_{d>0}\prod_{\substack{n,m\geq 0\\ j_n=E_n=md}}\frac{1}{1-p^d\, e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 E_n)}},
\end{aligned}
$$

- Here $c(n)$ counts the degeneracy of states with dimension $h_n = j_n = E_n - \frac{c}{12} = n$ in the seed CFT.
- $\mathcal{H}_N =$ sum over twisted sectors labeled by conjugacy classes of the permutation group S_N. Each sector factorizes into a tensor product of long string sectors, labeled by cyclic permutation of order d.
- The DMVV formula has the following non-chiral generalization

$$
Z_{\mathrm{DMVV}}(p,\sigma) = \left|\prod_{d>0}\prod_{\substack{n,m\geq 0\\ j_n=md}}\frac{1}{1-p^d\, e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 E_n)}}\right|^2.
$$

DMVV free energy as a sum over Hecke operators

The free energy $F_{\rm DMVV} = -\log Z_{\rm DMVV}$ can be expressed as a sum over positive N of Hecke operators T_N acting on the seed partition function

$$F_{\rm DMVV}(\rho_1, \sigma) \; = \; \sum_{N=1}^{\infty} T_N Z_1(\rho_1, \sigma)$$

where $Z_1(\rho_1, \sigma) \; = \; e^{2\pi i \rho_1} Z_{\rm CFT}(\sigma)$ and where T_N is defined via

$$T_N \phi(\rho, \sigma) = \frac{1}{N} \sum_{\substack{ad=N,\, d>0 \\ b \bmod d}} \phi\Big(N\rho, \frac{a\sigma + b}{d}\Big).$$

Geometrically, T_N represents the modular invariant sum $T_N \phi = \frac{1}{N} \sum_f f^* \phi$ over the pullbacks of holomorphic linear maps $f : \mathbb{T}^2 \to \mathbb{T}^2$ of degree N.

This geometric representation points to an interpretation of $F_{\rm DMVV}$ as the one-loop partition function of a second quantized string theory on $\mathbb{T}^2$.

Partition function of $T\overline{T}$ deformed symmetric product CFT

Our object of study is the grand canonical partition function of a $T\overline{T}$ deformed symmetric product CFT given by the weighted sum

$$Z_{T\overline{T}}(\rho,\sigma) \;=\; \sum_N p^N Z_N(\lambda,\sigma), \qquad\qquad p \equiv e^{2\pi i \rho_1},\;\; \lambda \equiv \frac{\sigma_2}{\rho_2}$$

with $Z_N(\lambda,\sigma)$ the deformed partition function of S^NCFT. It is obtained by plugging in the deformed energies into the undeformed DMVV formula

$$Z_{T\overline{T}}(\rho,\sigma) \;=\; \prod_{d>0} \prod_{\substack{n,m\in\mathbb{Z}\\ j_n=md}} \left| \frac{1}{1-p^d e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 \mathcal{E}_n(\lambda/d^2))}} \right|^2$$

The rescaling $\lambda \to \lambda/d^2$ of the dimensionless $T\overline{T}$ coupling λ ensures that the dimensionful $T\overline{T}$ coupling is identical across all long string sectors. The deformed energies are all real provided that $c \leq 6$ and $\lambda \leq 1$.

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Free energy of $T\overline{T}$ deformed symmetric product CFT

The free energy $F_{T\overline{T}} = -\log Z_{T\overline{T}}$ of the deformed symmetric product CFT can be expressed as the integral

$$F_{T\overline{T}}(\rho, \sigma) \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, K_{T\overline{T}}(\tau; \rho, \sigma)\, Z_1(\tau)$$

given by the sum over all non-zero integers N of Hecke operators

$$K_{T\overline{T}}(\tau; \rho, \sigma) \;=\; \sum_{N \neq 0} T_N K_1(\tau; \rho, \sigma)$$

acting on the diffusion kernel in the $N\!=\!1$ wrapping sector

$$K_1(\tau; \rho, \sigma) \;=\; 2\rho_2 \sum_{\gamma \in \mathrm{PSL}(2,\mathbb{Z})} e^{\frac{i\pi}{2}\left(\frac{\rho}{\sigma_2(\gamma\tau)_2}|\sigma - \gamma\tau|^2 - \frac{\bar{\rho}}{\sigma_2(\gamma\tau)_2}|\sigma - \gamma\bar{\tau}|^2\right)}.$$

The integral is convergent for $c \leq 6$ and $\lambda < 1$.

Heat kernel and functional determinant

The heat kernel $K(t,x,y) =$ the unique solution to the heat equation

$$(\partial_t + \Delta_x)K_0(t,x,y) = 0, \qquad K_0(0,x,y) = \delta^{(2)}(x-y)$$

The one-loop determinant on $\mathbb{T}^2$ can be expressed in terms of the trace of the heat kernel as

$$\log\det(\Delta + m^2) \; = \; \rho_2 \int_0^\infty \frac{dt}{t}\, K_0(t,x,x)\, e^{-m^2 t}.$$

We can write $K_0(t,x,x)$ as a sum over all $\mathbb{Z} \oplus \mathbb{Z}\,\sigma$ images

$$K_0(t,x,x) \; = \; \sum_{c,d\in\mathbb{Z}} \frac{1}{4\pi t}\, e^{-\frac{\pi\rho_2}{4t\sigma_2}|d+c\sigma|^2}$$

or as a Poincaré series

$$K_0(t,x,x) \; = \; \frac{1}{4\pi t} + 2\sum_{r=1}^{\infty} \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{PSL}(2,\mathbb{Z})} \frac{1}{4\pi t}\, e^{-\frac{\pi\rho_2 r^2}{4t\,(\gamma\sigma)_2}}.$$

S-duality invariant extension of the integration kernel

The diffusion kernel $K_{T\overline{T}}$ has a natural extension $\hat{K}_{T\overline{T}}$ obtained by including the zero wrapping number sector contribution

$$\hat{K}_{T\overline{T}} \;=\; K_0 + K_{T\overline{T}} \;=\; K_0 + \sum_{N\neq 0} T_N K_1$$

This extended kernel $\hat{K}$ is equal to the full $\Gamma_{2,2}$ Narain partition sum of the gaussian sigma model with a $\mathbb{T}^2$ target space with metric (4) and general B-field modulus $b = \rho_1$.

$$\hat{K}(\tau;\rho,\sigma) \;=\; \rho_2 \sum_{\vec{n},\vec{w}\in\mathbb{Z}^2} e^{\frac{i\pi}{2\tau_2\sigma_2}\left(\rho\,|n_2 + n_1\sigma - \tau(w_1 + w_2\sigma)|^2\right)} + c.c.$$

This $\Gamma_{2,2}$ Narain partition sum is a key player in our story.

Symmetry and spectral properties of the $\Gamma_{2,2}$ Narain sum

The Narain partition sum $\hat{K}(\tau;\rho,\sigma)$ has some remarkable properties.

- $\hat{K}(\tau;\rho,\sigma)$ is triality symmetric under interchanging τ, ρ and σ.
- $\hat{K}(\tau;\rho,\sigma)$ is invariant under the extended T-duality group

$$\mathrm{O}(2,2;\mathbb{Z}) \simeq \mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^2 \tag{7}$$

or in words:

- $\mathrm{PSL}(2,\mathbb{Z})$ = modular group acting on complex structure modulus σ,
- $\mathrm{PSL}(2,\mathbb{Z})$ = T-duality group acting on complex Kähler modulus ρ via

$$\rho \;\to\; \tilde{\rho} = \frac{a\rho + b}{c\rho + d}, \qquad\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}), \tag{8}$$

- $\mathbb{Z}_2$ = symmetry that exchanges the complex and Kähler moduli σ and ρ,
- $\mathbb{Z}_2$ = symmetry that flips the sign of the real part of both σ and ρ.

Spectral decomposition of the $\Gamma_{2,2}$ Narain partition sum

$\hat{K}(\tau;\rho,\sigma)$ admits the following spectral decomposition:

$$\begin{aligned}\hat{K}(\tau,\rho,\sigma) \;=\;& \alpha + \hat{E}_1(\rho) + \hat{E}_1(\sigma) + \hat{E}_1(\tau) \\ & + \frac{1}{4\pi i}\int_{\Re s = \frac{1}{2}} ds \, \frac{2\Lambda(s)^2}{\Lambda(1-s)} E_s(\rho) E_s(\sigma) E_s(\tau) \\ & + 8 \sum_{\epsilon=\pm} \sum_{n=1}^{\infty} \delta_\epsilon \, \frac{\nu_n^\epsilon(\rho)\,\nu_n^\epsilon(\sigma)\,\nu_n^\epsilon(\tau)}{(\nu_n^\epsilon, \nu_n^\epsilon)}.\end{aligned}$$

Here $\epsilon = \pm$ labels the parity of the cusp forms, $\delta_+ = 1, \delta_- = -i$, and

$$\Lambda(s) \;=\; \pi^{-s}\Gamma(s)\zeta(2s), \qquad \hat{E}_1(\sigma) = \lim_{s\to 1}\Big(E_s(\sigma) - \tfrac{3}{\pi(s-1)}\Big),$$

$$\alpha \;=\; \tfrac{3}{\pi}(\gamma_E + 3\log(4\pi) + 48\zeta'(-1) - 4).$$

Spectral properties of the $\Gamma_{2,2}$ Narain partition sum

- In the spectral decomposition expression, the triality symmetry among three moduli parameters ρ, σ, τ is manifest.
- The Eisenstein series, cusp forms are eigenfunctions of Δ_τ and the Hecke operators T_j.
- Hence, by virtue of the above spectral decomposition, we deduce the $\hat{K}$ satisfies the following identities:

$$\Delta_\tau \hat{K}(\tau, \rho, \sigma) = \Delta_\rho \hat{K}(\tau, \rho, \sigma) = \Delta_\sigma \hat{K}(\tau, \rho, \sigma),$$

$$T_j^\tau \hat{K}(\tau, \rho, \sigma) = T_j^\rho \hat{K}(\tau, \rho, \sigma) = T_j^\sigma \hat{K}(\tau, \rho, \sigma).$$

Eisenstein series and cusp forms

The Eisenstein series E_s is the real-analytic modular form

$$E_s(\tau) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{PSL}(2,\mathbb{Z})} \mathrm{Im}(\gamma\tau)^s.$$

Eigenfunction of the Laplacian Δ_τ on $\mathbb{H}^2$ and of the Hecke operators T_j

$$\Delta_\tau E_s(\tau) \;=\; s(1-s)E_s(\tau),$$

$$T_j^\tau E_s(\tau) \;=\; j^{\frac{1}{2}-s}\sigma_{2s-1}(j)\, E_s(\tau), \qquad \sigma_n(j) = \sum_{d|j} d^n$$

Eisenstein series and cusp forms

The cusp form (here $\tau = x + iy$ and $\epsilon = \pm$ labels the parity)

$$\nu_n^\epsilon(\tau) = \sum_{j=1}^{\infty} a_j^{n,\epsilon} \cos(2\pi j x) \sqrt{y} K_{iR_n^\epsilon}(2\pi j y),$$

are also the eigenfunction of Δ_τ and T_j

$$\begin{aligned} \Delta_\tau \nu_n^\epsilon(\tau) &= (\frac{1}{4} + (R_n^\epsilon)^2)\nu_n^\epsilon(\tau), \\ T_j \nu_n^\epsilon(\tau) &= a_j^{n,\epsilon} \nu_n^\epsilon(\tau), \end{aligned}$$

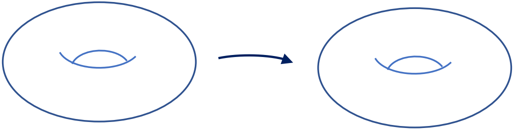
Partition function of a $T\overline{T}$ deformed Symmetric Product CFT

Now let us consider the partition function defined by:

$$\hat{Z}(\rho,\sigma) = e^{-\hat{F}(\rho,\sigma)} \qquad \hat{F}(\rho,\sigma) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \hat{K}(\tau;\rho,\sigma)\, Z_{\rm CFT}(\tau)$$

with $\hat{K}(\tau;\rho,\sigma)$ given by the full unrestricted $\Gamma_{2,2}$ Narain partition sum:

$$\hat{K}(\tau;\rho,\sigma) \;=\; \rho_2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^2} e^{\frac{i\pi}{2\tau_2\sigma_2}(\rho|n_2+n_1\sigma-\tau(m_1+m_2\sigma)|^2)} + c.c.$$



S-duality invariant $T\overline{T}$ deformed partition function

$$\hat{Z}(\rho,\sigma) = e^{-F_0(\rho_2,\sigma)} \left| \prod_{d>0} \prod_{\substack{n,m\in\mathbb{Z} \\ j_n = md}} \frac{1}{1 - p^d e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 \mathcal{E}_n(\lambda/d^2))}} \right|^2$$

$$F_0(\rho_2,\sigma) = A\rho_2 + \sum_{n\in\mathcal{S}} \log\det(\Delta + \rho_2 E_n)$$

$\hat{Z}(\rho,\sigma)$ has many remarkable properties:

- Mirror symmetry: $\hat{Z}(\rho,\sigma) = \hat{Z}(\sigma,\rho)$
- S-duality symmetry: $\hat{Z}(\rho,\sigma) = \hat{Z}(\tilde{\rho},\sigma)$, $\tilde{\rho} = \frac{a\rho+b}{c\rho+d}$,
- Spectral symmetry: $\Delta_\rho \hat{F}(\rho,\sigma) = \Delta_\sigma \hat{F}(\rho,\sigma)$
- Hecke symmetry: $T^\rho_j \hat{F}(\rho,\sigma) = T^\sigma_j \hat{F}(\rho,\sigma)$
- U-duality: $\mathrm{O}(2,2;\mathbb{Z}) \simeq \mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^2$

S-duality invariant $T\overline{T}$ deformed CFT partition function

We introduce a regularization designed to preserve $\mathrm{O}(2,2;\mathbb{Z})$ symmetry.

Consider the $T\overline{T}$-deformed free energy

$$F^c_{T\overline{T}}(\rho,\sigma) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\hat{K}(\tau,\rho,\sigma) Z^c_{\mathrm{CFT}}(\tau),$$

$$Z^c_{\mathrm{CFT}}(\tau) = \Phi(\tau)^{c_0-c} Z^{c_0}_{\mathrm{CFT}}(\tau),$$

where c_0 is the central charge of the seed CFT and $\Phi(\tau)$ is a modular function with $c=1$ growth at the cusp. E.g. $\Phi(\tau) = \sqrt{\tau_2}|\eta(q)|^2$.

We assume that $F^c_{T\overline{T}}(\rho,\sigma)$ is an analytic functions of c and can be analytically continued to $c=0$.

The extra contribution $F_0(\rho_2, \sigma)$ to the free energy has the following explicit form

$$F_0(\rho_2, \sigma) \;=\; A\rho_2 + \frac{\sigma_2}{\pi} \sum_{\Delta \in \mathcal{S}} \sum_{(m,n)'} \sqrt{\frac{8\pi^2 \Delta \rho_2}{\sigma_2 |m\sigma + n|^2}}\, K_1 \left(\sqrt{\frac{8\pi^2 \Delta \rho_2}{\sigma_2} |m\sigma + n|^2} \right)$$

The second term on the right-hand side is the sum of the log of the functional determinants. Recent work by N. Benjamin and C.Chang on scalar modular bootstrap leads to the following conjectured identity for the constant term

$$A = \sum_{\Delta \in \mathcal{S}} 4\sqrt{2\Delta} \sum_{m=1}^{\infty} \left[\frac{1}{m} K_1(2\pi\sqrt{2\Delta}m) + 2\pi\sqrt{2\Delta} K_0(2\pi\sqrt{2\Delta}m) \right]$$

We have checked numerically that this yields a finite answer in the regime $c_0 \leq 6$ and $\lambda = \sigma_2/\rho_2 < 1$.

The derivation makes use of the spectral decomposition of the scalar partition function

$$\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} \;=\; \frac{\Lambda(\frac{c-1}{2})}{\Lambda(\frac{c}{2})} y^{1-c} + \varepsilon_c(\rho,\sigma) y^{-\frac{c}{2}} + \frac{1}{2\pi i}\int_{\Re s=\frac{1}{2}} ds \;\; \pi^{s-\frac{c}{2}}\Gamma(\tfrac{c}{2}-s)\,\mathcal{E}^{c}_{\frac{c}{2}-s}(\rho,\sigma)\, y^{s-\frac{c}{2}},$$

$$\mathcal{E}^c_s(\rho,\sigma) \coloneqq \sum_{\Delta\in\mathcal{S}} (2\Delta)^{-s}, \tag{9}$$

where we have defined

$$\varepsilon_c(\rho,\sigma) \coloneqq 3\pi^{-\frac{c}{2}}\Gamma(\tfrac{c}{2}-1)\mathcal{E}^{c}_{\frac{c}{2}-1}(\mu) \tag{10}$$

S-duality invariant $T\overline{T}$ deformed partition function

$$\hat{Z}(\rho,\sigma) = e^{-F_0(\rho_2,\sigma)}\left|\prod_{d>0}\prod_{\substack{n,m\in\mathbb{Z}\\ j_n=md}} \frac{1}{1-p^d e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2\mathcal{E}_n(\lambda/d^2))}}\right|^2$$

$$F_0(\rho_2,\sigma) = A\rho_2 + \sum_{\Delta\in\mathcal{S}} \log\det(\Delta + \rho_2 E_n)$$

$\hat{Z}(\rho,\sigma)$ has many remarkable properties:

- Mirror symmetry: $\hat{Z}(\rho,\sigma) = \hat{Z}(\sigma,\rho)$
- S-duality symmetry: $\hat{Z}(\rho,\sigma) = \hat{Z}(\tilde{\rho},\sigma)$, $\tilde{\rho} = \frac{a\rho+b}{c\rho+d}$,
- Spectral symmetry: $\Delta_\rho\hat{F}(\rho,\sigma) = \Delta_\sigma\hat{F}(\rho,\sigma)$
- Hecke symmetry: $T_j^\rho\hat{F}(\rho,\sigma) = T_j^\sigma\hat{F}(\rho,\sigma)$
- U-duality: $\mathrm{O}(2,2;\mathbb{Z}) \simeq \mathrm{PSL}(2,\mathbb{Z})\times\mathrm{PSL}(2,\mathbb{Z})\rtimes\mathbb{Z}_2^2$

Spectral decomposition of $T\overline{T}$-deformed partition function

Assuming that $Z^c_{\rm CFT}$ admits the Roelcke-Selberg spectral decomposition

$$Z^c_{\rm CFT}(\tau) \;=\; D_c + \frac{1}{4\pi i}\int\limits_{{\rm Re} s=\frac{1}{2}} ds\, \beta_c(s)\, E_s(\tau) + \sum_{\epsilon=\pm}\sum_{n=1}^{\infty} \alpha^{\epsilon}_{c,n}\, \nu^{\epsilon}_n(\tau)$$

we can write the spectral decomposition of the $T\overline{T}$-deformed free energy as

$$F^c_{T\overline{T}}(\rho,\sigma) \;=\; \gamma_c D_c + \epsilon_c(\alpha + \hat{E}_1(\rho) + \hat{E}_1(\sigma))$$

$$+\frac{1}{4\pi i}\int_{\Re s=\frac{1}{2}} ds\; 2\Lambda(s)\beta_c(s)E_s(\rho)E_s(\sigma) \;+\; 8\sum_{\epsilon=\pm}\delta_\epsilon \sum_{n=1}^{\infty}\alpha^{\epsilon}_{c,n}\; \nu^{\epsilon}_n(\rho)\nu^{\epsilon}_n(\sigma).$$

$$\gamma_c = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\hat{E}_1(\tau), \qquad \epsilon_c = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2} Z^c(\tau)$$

We assume that all coefficients can be analytically continued to $c = 0$.