Automorphic Spectra and the Conformal Bootstrap

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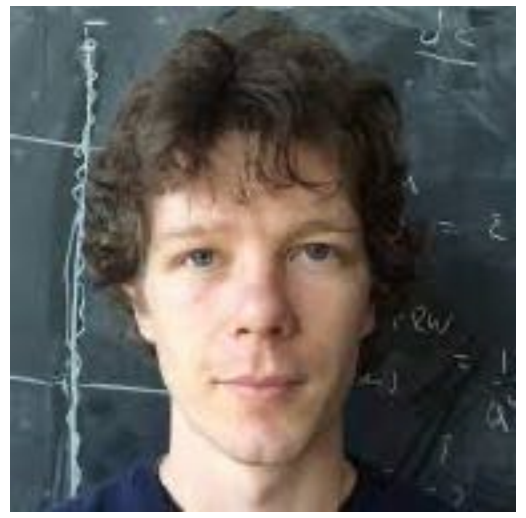
Based on **arXiv:2111.12716** with **Petr Kravchuk** and **Sridip Pal**.

Related results appeared in **arXiv:2111.13215** by **James Bonifacio**.



I will also report on some ongoing work with all of the above.

What Is Quantum Field Theory?

1. How do we rigorously define quantum field theory?

2. How do we compute observables?

Situation better for **conformal field theories**:

- Precise axiomatic formulation in any number of dimensions.
- Effective for computations, even leading to new predictions.

} conformal bootstrap

CFT Axioms

1. V = a unitary representation of the (Lorentzian) conformal group in d dimensions.

   - V = space of states = space of local operators.
   - Decompose into irreducible representations: $V = \bigoplus_i V_{\Delta_i,\rho_i}$.
   - Local operators: $\mathcal{O}_i(x)V_{0,0}$ with $x \in B^d$ generates V_{Δ_i,ρ_i}.

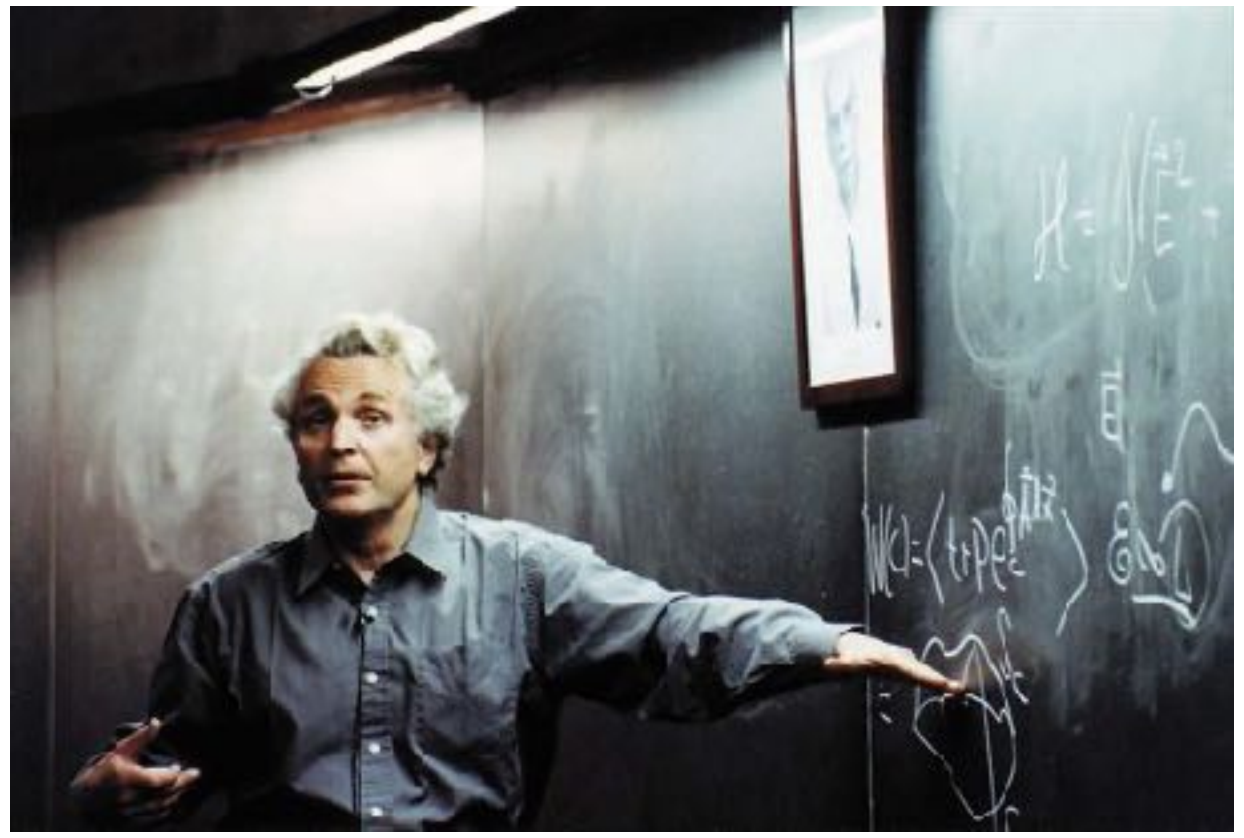
2. Operator product expansion: $\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k c_{ijk} |x-y|^{-\Delta_i-\Delta_j+\Delta_k} \mathcal{O}_k(y)$.

3. Associativity: $\mathcal{O}_i(x)(\mathcal{O}_j(y)\mathcal{O}_k(z)) = (\mathcal{O}_i(x)\mathcal{O}_j(y))\mathcal{O}_k(z)$

$\Rightarrow$ stringent constraints on the spectrum Δ_i, ρ_i and structure constants c_{ijk} .

Long term goal: Solve and classify CFTs in general dimension starting from these axioms.

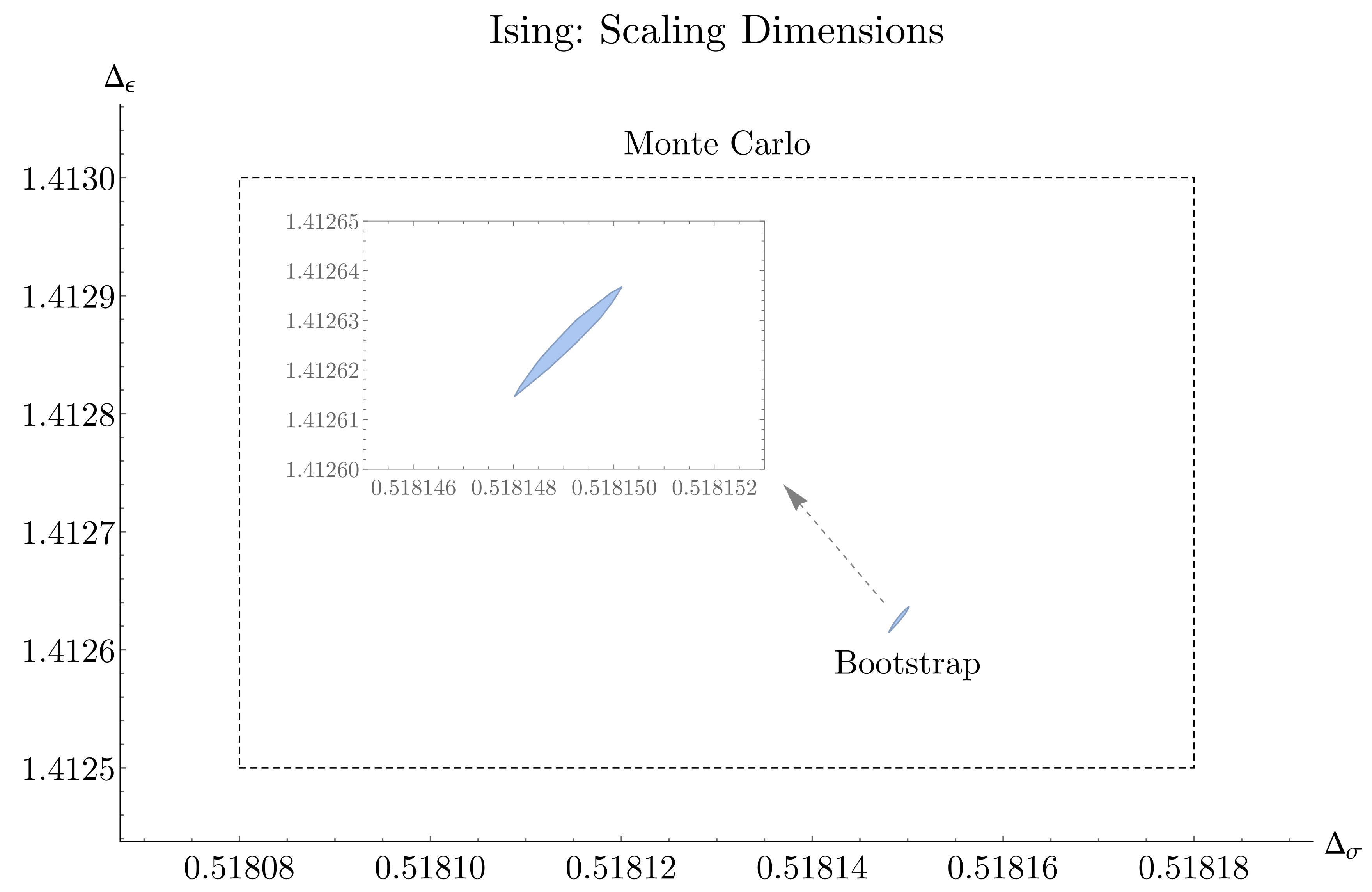
A. Polyakov: “I was dreaming in the 1970s to have a classification of fixed points based on the operator product expansion. The program was successful in two dimensions, and I think it is not excluded that in three dimensions something like that is still possible.”



Current status:

- $d = 2$: Partial progress (rational theories, Liouville theory).
- $d > 2$: The only solved examples are free theories, but infinitely many interacting examples surely exist ($d = 3$ Ising model, $d = 4$ $\mathcal{N} = 4$ Yang-Mills, $d = 6$ $\mathcal{N} = (2,0)$ SCFTs).

The CFT axioms seem capable of isolating interacting CFTs in $d > 2$: nearly sharp bounds on the spectral data from **linear and semidefinite programming**.



[Kos, Poland Simmons-Dufin, Vichi '16]

Speculation: Can we solve an interacting CFT in d>2, such as the 3d Ising model?

Possible strategy: Identify a mathematical structure which produces spectral data satisfying the CFT axioms.

[Moore, Seiberg '89] [Gadde '17]

Today: Hyperbolic manifolds provide a very good toy model for such structure.

Hyperbolic Manifolds

Definition: A hyperbolic d-manifold is a Riemannian d-manifold of constant sectional curvature -1.

The simplest example: Hyperbolic space $\mathbb{H}^d$.

- $-x_0^2 + x_1^2 + \ldots + x_d^2 = -1, x_0 > 0$
  $ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_d^2$
- Isometry group of $\mathbb{H}^d$ = $\mathrm{SO}^+(1, d)$

Fact: Every (closed, connected, orientable) hyperbolic d-manifold is of the form $\Gamma \backslash \mathbb{H}^d$, where Γ is a discrete subgroup of $\mathrm{SO}^+(1, d)$.

The Analogy

conformal field theories in d-1 dimensions $\longleftrightarrow$ **hyperbolic d-manifolds**

Underlying reason:

conformal group of Euclidean $\mathbb{R}^{d-1}$ $=$ $SO^+(1,d)$ $=$ isometry group of $\mathbb{H}^d$

The Dictionary

Notation: $G = \mathrm{SO}^+(1,d)$

a conformal field theory	$\longleftrightarrow$	a hyperbolic manifold $\Gamma \backslash \mathbb{H}^d$
Hilbert space	$\longleftrightarrow$	function space $L^2(\Gamma \backslash G)$
local operators	$\longleftrightarrow$	automorphic functions $F_i \in L^2(\Gamma \backslash G)$
correlation functions	$\longleftrightarrow$	$\langle F_1 \ldots F_n \rangle = \int_{\Gamma \backslash G} dg F_1(g) \ldots F_n(g)$
conformal Casimir eigenvalue	$\longleftrightarrow$	Laplacian eigenvalue $\lambda_i = \Delta_i(d-1-\Delta_i)$
operator product expansion	$\longleftrightarrow$	point-wise product $F_i(g)F_j(g) = \sum_k c_{ijk} F_k(g)$
structure constants	$\longleftrightarrow$	triple product integrals $c_{ijk} = \langle F_i F_j F_k \rangle$

Today: Adapt well-developed techniques used in the conformal bootstrap to prove new results about the spectra of hyperbolic manifolds.

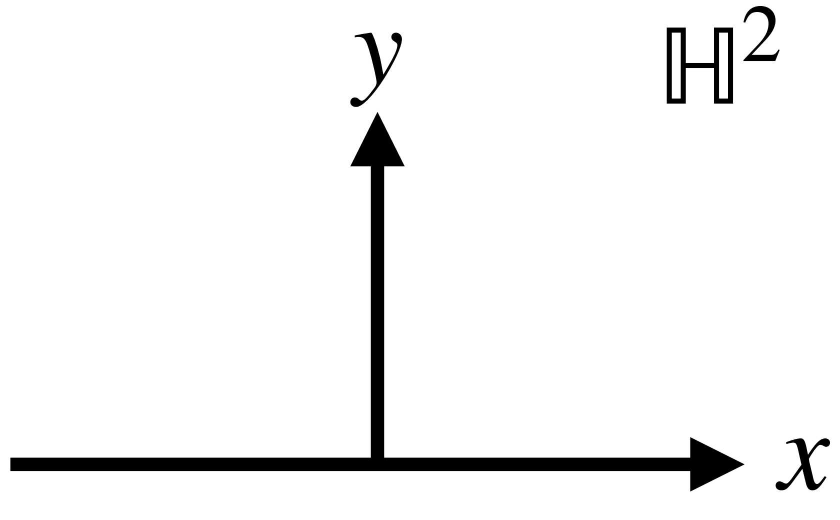
Previous Work

Bonifacio+Hinterbichler (2020): Einstein manifolds $R_{ab} = \frac{R}{d} g_{ab}$

Bonifacio (2021): Hyperbolic manifolds $R_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}$

Kravchuk, DM, Pal (2021): Pointed out the role played by $\mathrm{SO}(1,d)$ in the case of hyperbolic manifolds, and systematized the ideas using its representation theory.

2D Hyperbolic Orbifolds

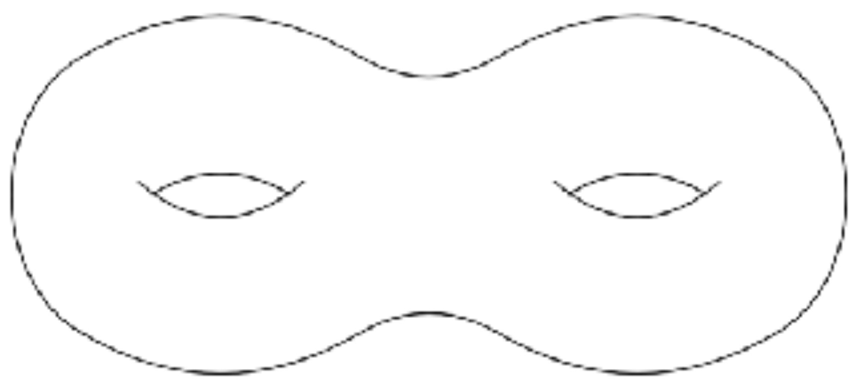


1. Upper half-plane with the hyperbolic metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$

   - $G = \mathrm{PSL}_2(\mathbb{R})$ acts on $z = x + iy \in \mathbb{H}^2$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : \; z \mapsto \dfrac{az+b}{cz+d}$

2. Γ = discrete subgroup of $\mathrm{PSL}_2(\mathbb{R}) \; \Leftrightarrow \; \Gamma \backslash \mathbb{H}^2$ = a hyperbolic orbifold.

   $\Gamma \backslash \mathbb{H}^2$

   - Will assume $\Gamma \backslash \mathbb{H}^2$ has finite volume.
   - Γ only has hyperbolic elements $\Leftrightarrow \Gamma \backslash \mathbb{H}^2$ is a compact surface.
   - Γ only has hyperbolic and elliptic elements $\Leftrightarrow \Gamma \backslash \mathbb{H}^2$ is a compact orbifold.

Laplacian Spectrum of $\Gamma \backslash \mathbb{H}^2$

The Laplacian on $\mathbb{H}^2$: $\nabla^2 = y^2(\partial_x^2 + \partial_y^2)$

$$-\nabla^2 h(x, y) = \lambda\, h(x, y)$$

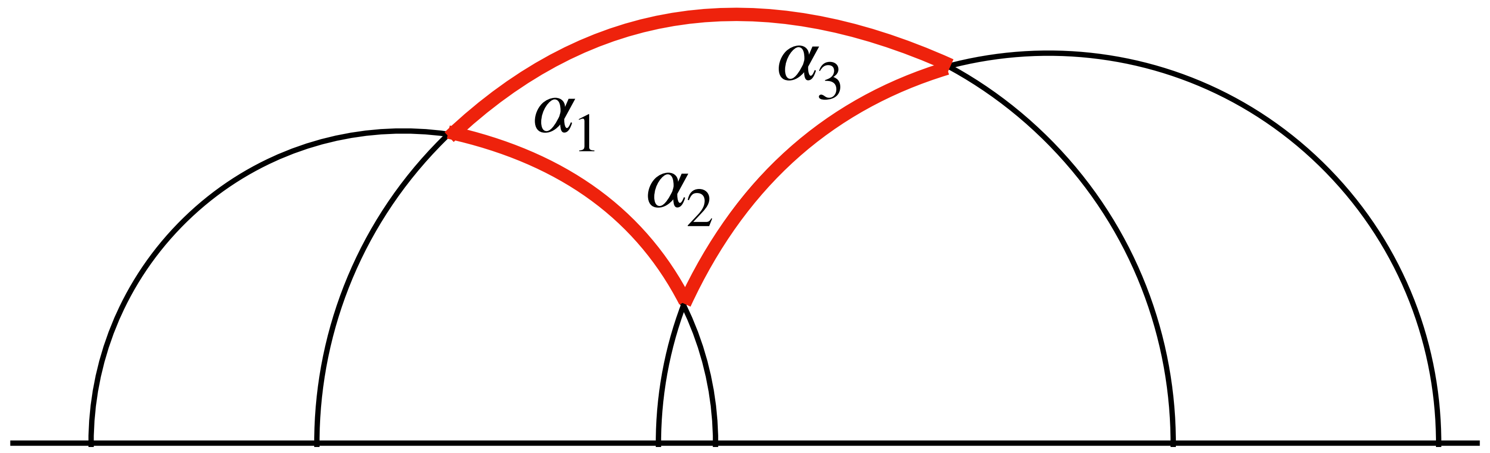
$h(x, y)$: a smooth real function on $\mathbb{H}^2$ satisfying $h(\gamma \cdot (x, y)) = h(x, y)$ for all $\gamma \in \Gamma$.

Spectrum: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

- no closed expression for λ_i
- a useful model for quantum chaos

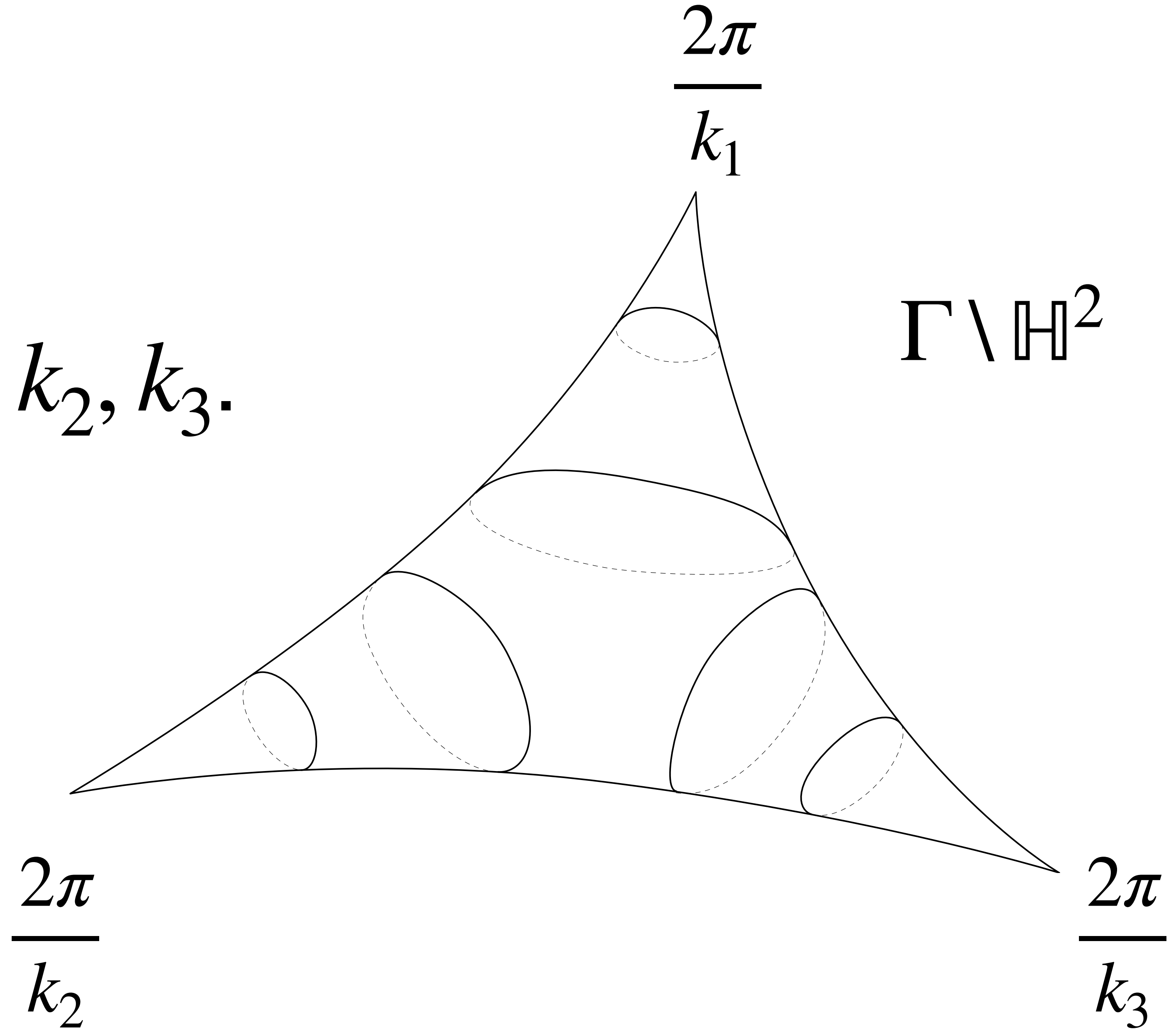
Today: New upper bounds on λ_1.

Example 1: Hyperbolic Triangle Groups



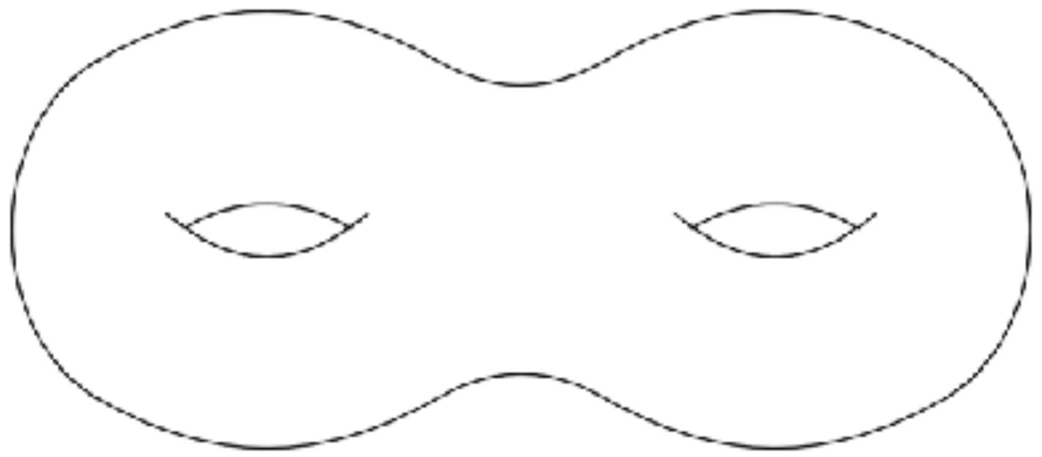
$$\alpha_i = \frac{\pi}{k_i} \qquad k_i \in \mathbb{N}_{\geq 2}$$

- Γ generated by rotations around vertices by angles $\dfrac{2\pi}{k_i}$.
- A fundamental domain of Γ consists of two adjacent triangles.
- $\Gamma \backslash \mathbb{H}^2$ is an orbifold of genus 0 with 3 orbifold points of orders k_1, k_2, k_3.
- Orbifold of minimal area: $[k_1, k_2, k_3] = [2,3,7]$.



Example 2: The Bolza Surface

A hyperbolic surface without orbifold points must have genus ≥ 2.

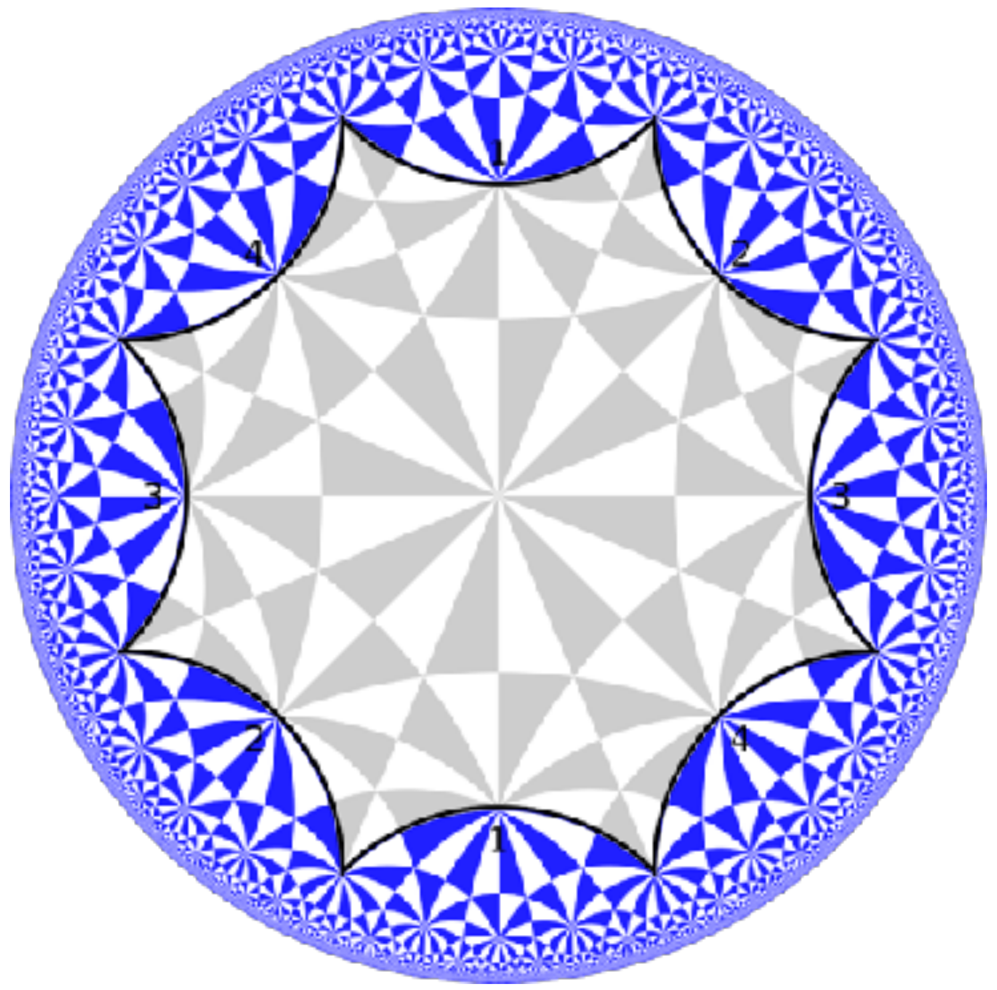


- Genus $= 2$: six-dimensional moduli space.

Bolza surface: the genus-two surface with the largest group of isometries.

- Isometries(Bolza) = $\mathrm{GL}_2(\mathbb{F}_3)$, a group of order 48.

- Bolza = $\Gamma \backslash \mathbb{H}^2$, where Γ is a normal subgroup of index 48 of the [2,3,8] triangle group.



General Orbifolds

Topological type of $\Gamma \backslash \mathbb{H}^2$: $[g; k_1, \ldots, k_r]$ $\Leftrightarrow$ isomorphism type of Γ

genus   orders of orbifold points

Main results

[Kravchuk, DM, Pal '21] [Bonifacio '21]

Theorem:

1. The $[2,3,7]$ triangle orbifold maximizes λ_1 among all hyperbolic orbifolds

$$[2,3,7] \text{ triangle}: \lambda_1 \approx 44.88835$$

2. Every hyperbolic orbifold of genus two satisfies: $\lambda_1 \leq 3.8388977.$

$$\text{Bolza surface: } \lambda_1 \approx 3.838887258$$

$$\text{previous bound: } \lambda_1 \leq 4$$

[Yang, Yau '80] [Soufi, Ilias '83]

3. Every hyperbolic orbifold of genus three satisfies: $\lambda_1 \leq 2.6784824.$

$$\text{Klein quartic: } \lambda_1 \approx 2.6779$$

$$\text{previous bound: } \lambda_1 \leq 2(4-\sqrt{7}) \approx 2.7085$$

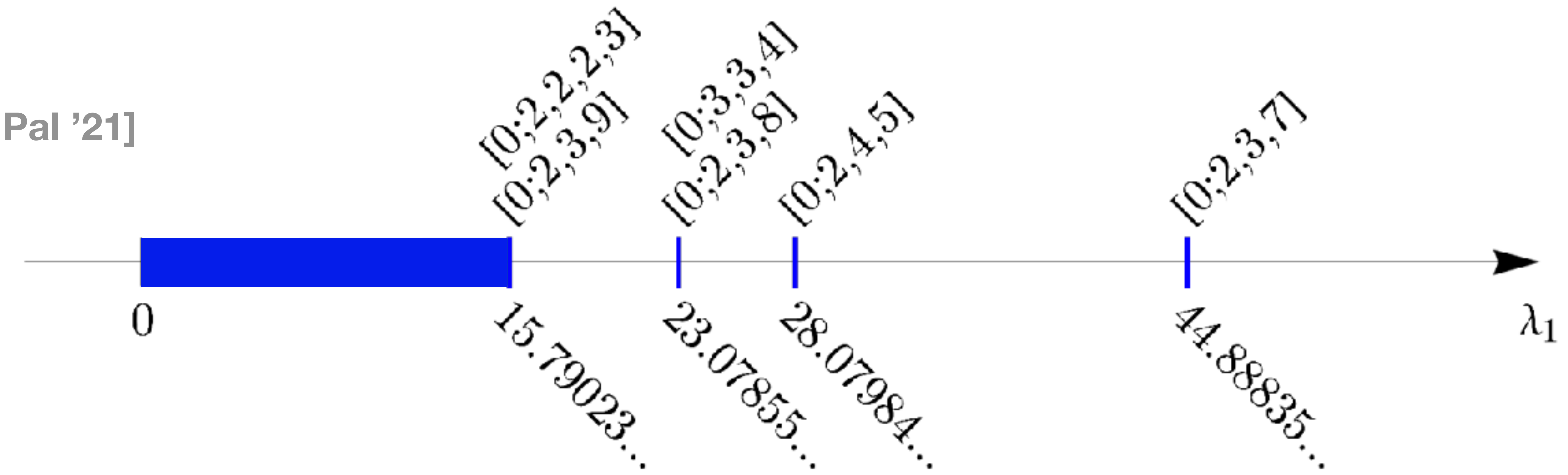
[Ros '20]

Spectrum of the Spectrum

Question: What values does $\lambda_1(X)$ take when X ranges over **all** hyperbolic orbifolds?

Answer:

[Kravchuk, DM, Pal ’21]



Conjecture (Selberg 1965): If Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then $\lambda_1 = 1/4$.

The Method

1. The Hilbert space and local operators

2. Operator product expansion

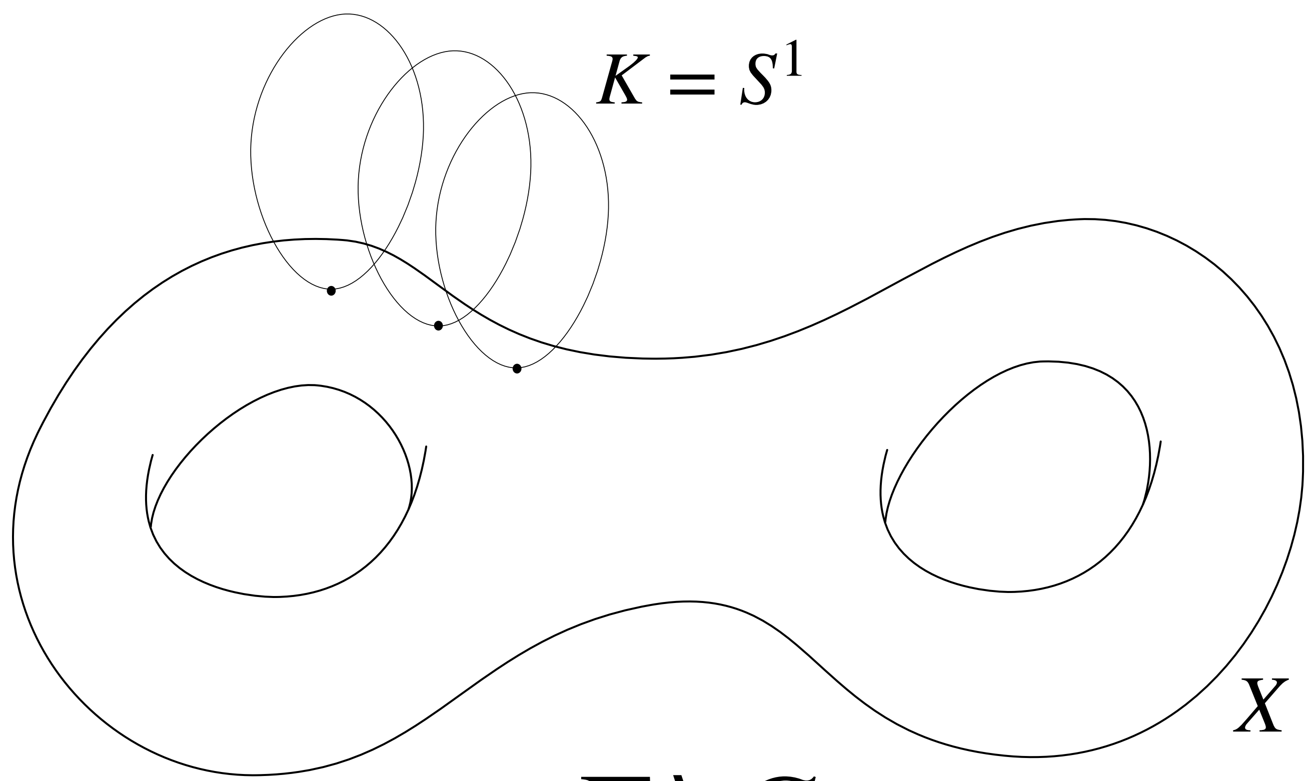
3. Associativity

4. Bounds from linear programming

The Hilbert Space: $L^2(\Gamma \backslash G)$

Consider the space $L^2(\Gamma \backslash G)$

- a representation of G: $F(g) \overset{\tilde{g}}{\mapsto} F(g\tilde{g})$
- unitary, with inner product: $\|F(g)\|^2 = \int_{\Gamma \backslash G} dg \, |F(g)|^2$



Decomposition under K: $L^2(\Gamma \backslash G) = \bigoplus_{n \in \mathbb{Z}} V_n$

- $V_0 = L^2(X)$
- $V_n = L^2(n\text{-forms})$: $f(x, y)\, dz^n$ such that $\forall \gamma \in \Gamma$: $f(z) = (cz + d)^{-2n} f\left(\frac{az + b}{cz + d}\right)$
- Generators of G act as follows: $L_0|_{V_n} = n\,\mathrm{id}$, $L_{\pm 1} : V_n \to V_{n \mp 1}$

The Spectral Decomposition

Decompose $L^2(\Gamma \backslash G)$ into irreducible representations of $G = \mathrm{PSL}_2(\mathbb{R})$:

$$L^2(\Gamma \backslash G) = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} P_{\lambda_i} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j})$$

1. Trivial representation $\mathbb{C}$: constant functions.

2. Principal and complementary series P_λ $\Leftrightarrow$ Laplace eigenfunctions with eigenvalue λ.

3. Holomorphic discrete series D_n $\Leftrightarrow$ holomorphic modular forms of weight $2n \in \mathbb{N}_{>0}$.

$$L^2(\Gamma \backslash G) = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} P_{\lambda_i} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j})$$

Question: What are the constraints on the set of representations on the RHS?

Ingredients:

1. Riemann-Roch theorem: The topology of Γ determines the spectrum of holomorphic forms = discrete series. Namely, for $[g; k_1, \ldots, k_r]$, we have

$$\text{multiplicity}(D_n) = (2n-1)(g-1) + \sum_{i=1}^{r} \left\lfloor n \frac{k_i - 1}{k_i} \right\rfloor + \delta_{n,1}$$

$\Rightarrow$ Focus on specific topology by making simple assumptions about the spectrum of D_n.

2. Consider the pointwise product $C^\infty(\Gamma \backslash G) \times C^\infty(\Gamma \backslash G) \to C^\infty(\Gamma \backslash G)$

$$(F_1(g), F_2(g)) \mapsto F_1(g) F_2(g)$$

Associativity and G-invariance $\Rightarrow$ bounds on the Laplacian spectrum.

Local Operators

Definition (local operator):

Let $F(g) \in L^2(\Gamma \backslash G)$ be a holomorphic modular form of weight n. Define

$$\mathcal{O}(w) = e^{wL_{-1}} \cdot F(g) = F(g) + wL_{-1} \cdot F(g) + \frac{w^2}{2} L_{-1}^2 \cdot F(g) + \dots$$

Properties:

- $\mathcal{O}(w) \in L^2(\Gamma \backslash G) \cap D_n$ for $|w| < 1$.
- As w ranges over the unit disk, $\mathcal{O}(w)$ generates $L^2(\Gamma \backslash G) \cap D_n$.
- $\mathcal{O}(w)$ transforms like a conformal primary operator of scaling dimension n.

$$L_m \cdot \mathcal{O}(w) = [w^{m+1}\partial_z + (m+1)nw^m]\mathcal{O}(w)$$

Similarly, define the conjugate operator $\overline{\mathcal{O}}(w) = w^{-2n}e^{-L_1/w} \cdot \overline{F(g)}$.

- $\overline{\mathcal{O}}(w) \in L^2(\Gamma \backslash G) \cap \overline{D}_n$ for $|w| > 1$.

Correlation Functions

Definition (correlation function):

Given $F_1, \ldots, F_N \in C^\infty(\Gamma \backslash G)$, their correlation function is given by

$$\langle F_1 \ldots F_N \rangle = \frac{1}{\mathrm{vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} d\mu \, F_1(g) \ldots F_2(g)$$

Since μ is G-invariant, so are the correlation functions.

Properties:

- one-point functions: $\langle 1 \rangle = 1, \langle \mathcal{O}_i(w) \rangle = \langle \overline{\mathcal{O}}_i(w) \rangle = 0$
- two-point functions: $\langle \mathcal{O}_i(w_1) \overline{\mathcal{O}}_j(w_2) \rangle = \dfrac{\delta_{ij}}{(w_1 - w_2)^{2n}}$
- three-point functions: $\langle \mathcal{O}_i(w_1) \mathcal{O}_j(w_2) \overline{\mathcal{O}}_k(w_3) \rangle = \dfrac{c_{ijk}}{w_{12}^{n_i+n_j-n_k} w_{23}^{n_j+n_k-n_i} w_{13}^{n_i+n_k-n_j}}$

The Operator Product Expansion

Express products $\mathcal{O}(w_1)\overline{\mathcal{O}}(w_2)$, $\mathcal{O}(w_1)\mathcal{O}(w_2)$ using the spectral decomposition of $L^2(\Gamma \backslash G)$.

- $\mathcal{O}(w_1)\overline{\mathcal{O}}(w_2) = \dfrac{1}{(w_1 - w_2)^{2n}} + \sum_i c_i\, K_i(w_1, w_2)$, where $K_i(w_1, w_2) \in P_{\lambda_i}$.
- $\mathcal{O}(w_1)\mathcal{O}(w_2) = \sum_j \widetilde{c}_j\, \widetilde{K}_j(w_1, w_2)$, where $\widetilde{K}_j(w_1, w_2) \in D_{n_j}$.

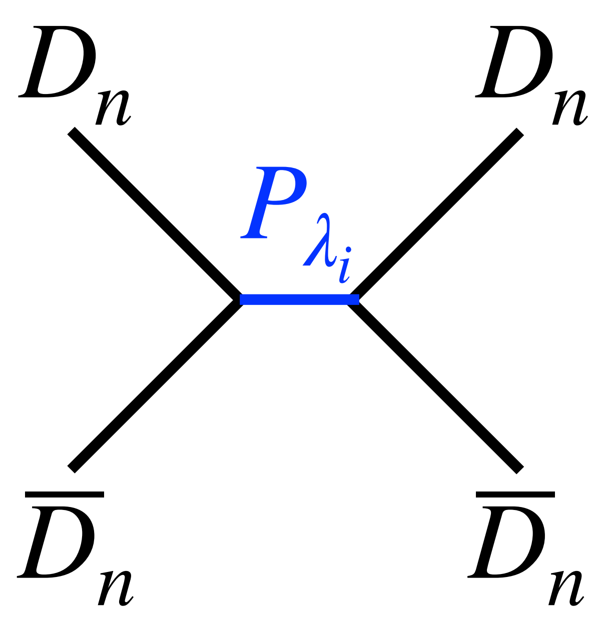
Crucial fact: $K_i(w_1, w_2)$ and $\widetilde{K}_j(w_1, w_2)$ are universal = fixed by G-invariance.

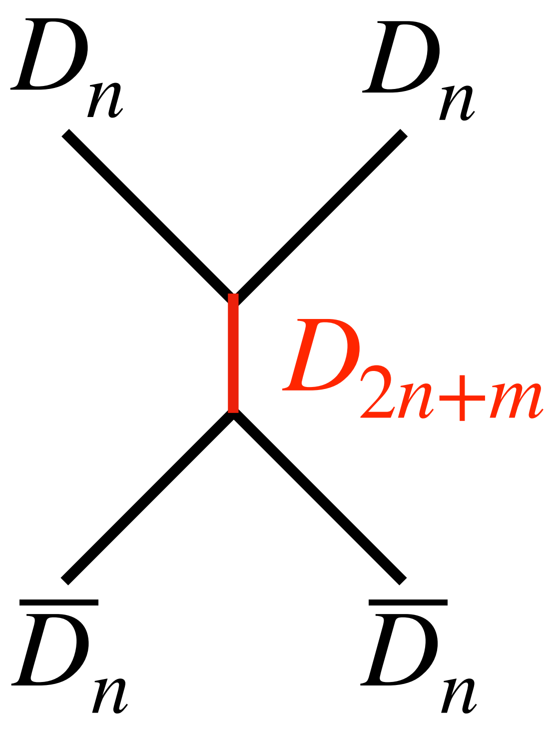
- $c_i \sim \langle f \bar{f} h_i \rangle$, $\widetilde{c}_j \sim \langle f f \bar{f}_j \rangle$, integrals of triple products of automorphic forms.

Imposing Associativity

Suppose $L^2(\Gamma\backslash G)$ contains D_n and let $\mathcal{O}_n(w)$ be the corresponding local operator.

$$\langle \mathcal{O}_n(w_1)\mathcal{O}_n(w_2)\overline{\mathcal{O}}_n(w_3)\overline{\mathcal{O}}_n(w_4)\rangle$$

$\sum$ 

$=$ $\sum$ 

Laplace eigenfunctions

modular forms

$$(1-\chi)^{-2n}\sum_i |c_i|^2 k_{\lambda_i}(\chi) = \chi^{-2n}\sum_{\substack{m\geq 0\\ m\text{ even}}} |\widetilde{c}_m|^2 \widetilde{k}_{2n+m}(\chi)$$

$$k_{s(1-s)}(\chi) = {}_2F_1(s,1-s;1;\tfrac{\chi}{\chi-1})$$

$$\widetilde{k}_m(\chi) = \chi^m {}_2F_1(m,m;2m;\chi)$$

$\Rightarrow$ Get an infinite number of spectral identities by expanding around $\chi = 0$.

Spectral identities

Summary so far:

- Suppose $\Gamma \backslash \mathbb{H}$ has a holomorphic modular form f of weight $2n$.
- Let h_i be the complete set of Laplace eigenfunctions and let $c_i = \int_{\Gamma \backslash \mathbb{H}} dxdy\, y^{2n-2} |f|^2 h_i$.

Then the spectral data λ_i, c_i satisfies the spectral identities

$$\sum_i c_i^2 P_{n,m}(\lambda_i) = d_m^2 \quad \text{for even} \quad m \geq 0,$$
$$\sum_i c_i^2 P_{n,m}(\lambda_i) = 0 \quad \text{for odd} \quad m > 0.$$

Here $P_{n,m}(\lambda)$ are explicit polynomials of degree m.

Crucial point: Both c_i^2 and d_m^2 are non-negative.

Bounds from linear programming

Spectral identities:

$$\sum_i c_i^2 P_{n,m}(\lambda_i) = d_m^2 \quad \text{for even} \quad m \geq 0,$$
$$\sum_i c_i^2 P_{n,m}(\lambda_i) = 0 \quad \text{for odd} \quad m > 0$$

Proposition: Fix $M \in \mathbb{N}$ and suppose $Q(\lambda) = \sum_{m=0}^{M} x_m P_{n,m}(\lambda)$ with $x_m \in \mathbb{R}$, such that

1. $x_m \leq 0$ for all even m
2. $Q(0) = 1$
3. $Q(\lambda) \geq 0$ for all $\lambda \geq \lambda_*$.

Then there is an upper bound on the Laplace spectral gap $\lambda_1 < \lambda_*$ for every hyperbolic orbifold with a holomorphic form of weight $2n$.

Proof: Use the spectral identities to show that $\sum_i c_i^2 Q(\lambda_i) \leq 0$. ◻

Strategy: Minimize λ_* by optimizing over x_m satisfying 1.-3. Increase M to improve the bound. We used the semidefinite programming solver SDPB. [Simmons-Duffin ’15] [Simmons-Duffin, Landry ’19]

Results

Fact: Every hyperbolic orbifold has a modular form with $n \in \{1, 2, 3, 4, 6\}$.

n	our bound on λ_1	largest known λ_1	orbifold
1	8.47032	8.46776	$[1; 2]$ at the $\mathbb{Z}_6$-symmetric point
2	15.79144	15.79023	$[0; 2,2,2,3]$ at the $\mathbb{Z}_3$-symmetric point
3	23.07917	23.07855	$[0; 3,3,4]$
4	30.35432	28.07984	$[0; 2,4,5]$
6	44.8883537	44.88835	$[0; 2,3,7]$

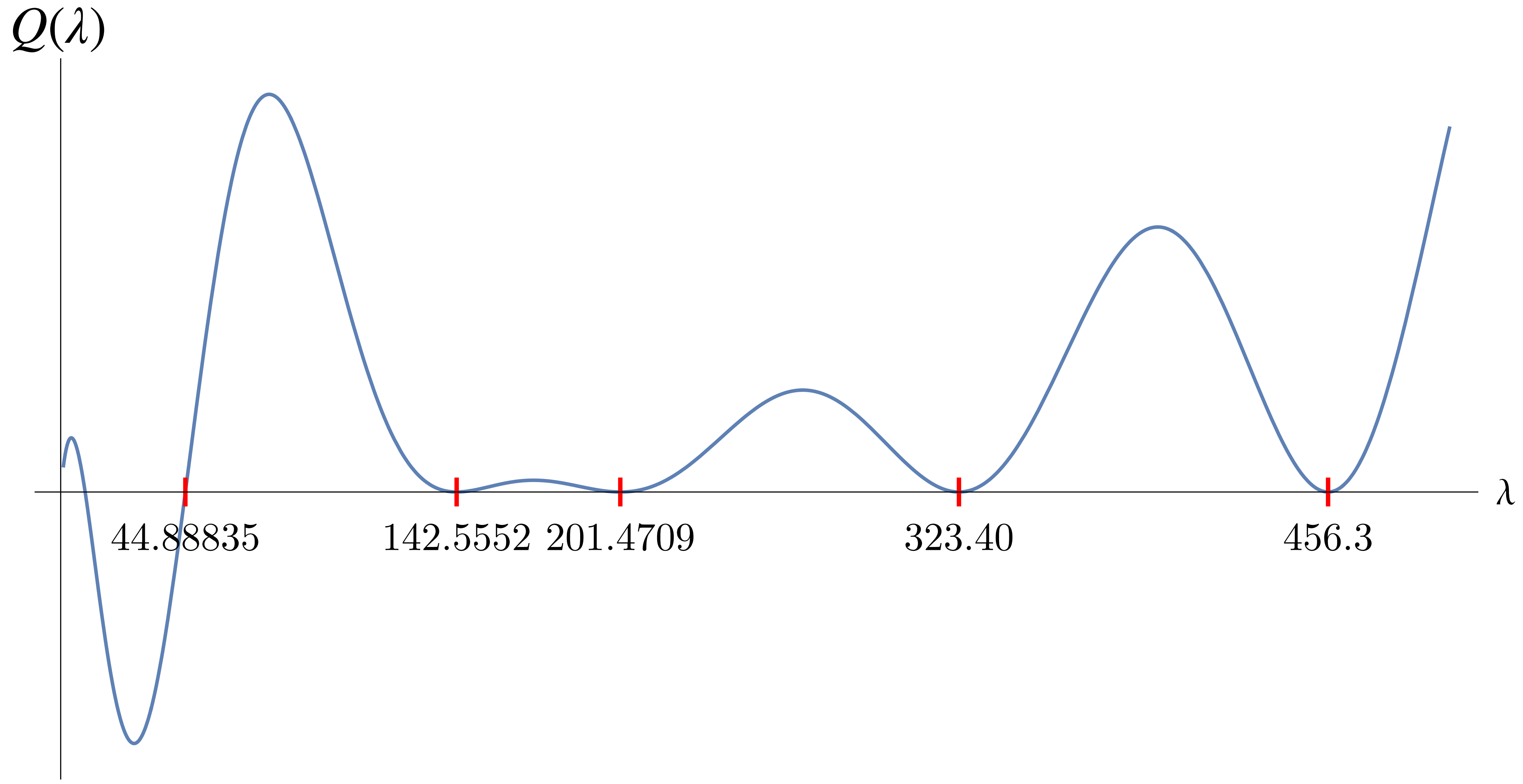
Corrolary: Every hyperbolic orbifold satisfies: $\lambda_1 \leq 44.8883537$.

Sharp bounds?

Question: Is the linear-programming upper bound on λ_1 sharp for $M \to \infty$?

If yes, the linear program must reconstruct the full Laplace spectrum of the $[0;2,3,7]$ orbifold!

- Must have $Q(\lambda_i) = 0$ for all $\lambda_i \in$ spectrum.
- Output of the linear program for $M = 41$ →



- Zeros agree with the $[0;2,3,7]$ spectrum to several decimal places!
- Proof of sharpness = an explicit construction of $Q(\lambda)$ for $M = \infty$.

Optimality in ∞-dimensional linear programs

- A similar ∞-dimensional linear program can be used to prove upper bounds on sphere packing density in $\mathbb{R}^d$ [Cohn+Elkies (2001)].
- The role of the spectral identities is played by the Poisson summation formula for a lattice in $\mathbb{R}^d$.
- Viazovska (2016): The bound is sharp for $d = 8$, proof by direct construction of optimal $Q(\lambda)$.
- Cohn+Kumar+Miller+Radchenko+Viazovska (2016): The bound is sharp in $d = 24$.

- Yet another similar linear program features prominently in the conformal field theory literature, proving upper bounds on the spectral gap of a CFT [Rattazzi+Rychkov+Tonni+Vichi (2008)].
- DM (2016): Proof of sharpness of the bound in 1D conformal field theory.
- Hartman+DM+Rastelli (2019): Precise mapping between Viazovska (2016) and DM (2016).

Challenge: Is the linear-programming upper bound on λ_1 of hyperbolic 2-orbifolds sharp?

Bounds at Fixed Genus

Bounds on λ_1 of genus-g orbifolds: Use g linearly independent holomorphic 1-forms.

Associativity implemented by the system of coupled equations:

$$\langle \mathcal{O}_i(w_1)\mathcal{O}_j(w_2)\overline{\mathcal{O}}_k(w_3)\overline{\mathcal{O}}_l(w_4)\rangle \qquad n_i = n_j = n_k = n_l = 1 \qquad i, j, k, l = 1,\ldots, g$$

A natural matrix generalization of the original problem = mixed-correlator bootstrap

[Kos, Poland, Simmons-Duffin ’14]

genus	our bound on λ_1	largest known λ_1	orbifold
1	8.47032	8.46776	$[1;2]$ at the $\mathbb{Z}_6$-symmetric point
2	3.83890	3.83889	Bolza surface
3	2.67849	2.67793	Klein quartic

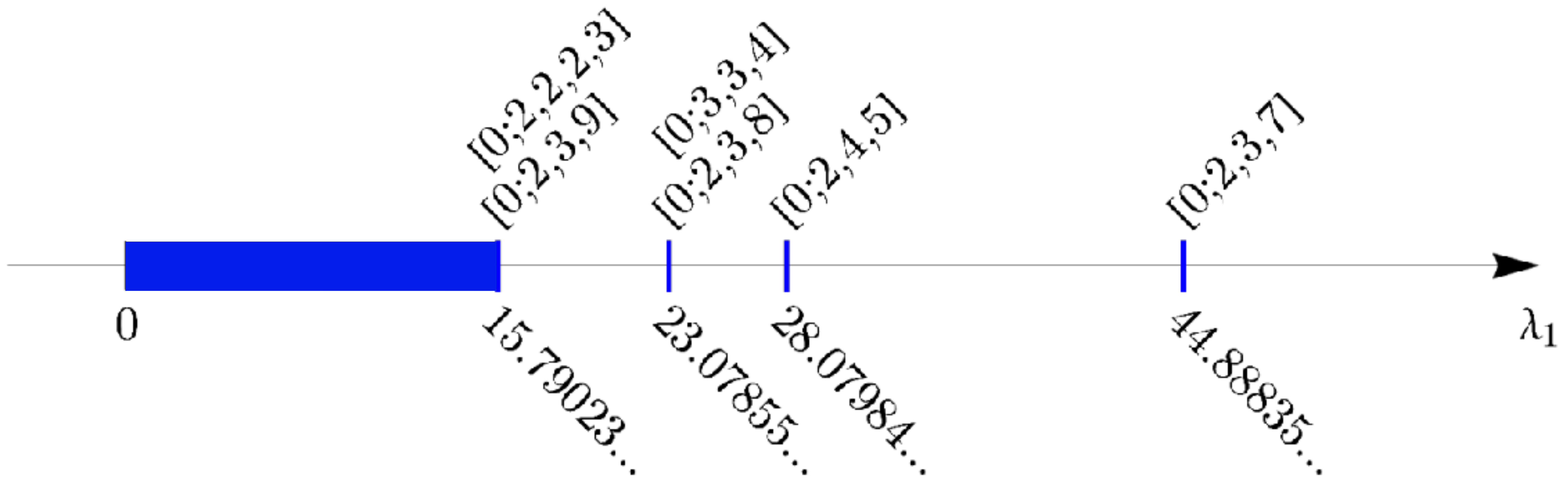
Values of λ_1 Attained by All Orbifolds

Idea: Prove upper bound on everything except for $[0;2,3,7]$, etc.

Study associativity for **two** holomorphic forms of minimal weight $1 \le n_1 < n_2$

$$\langle \mathcal{O}_{n_1}(w_1)\mathcal{O}_{n_2}(w_2)\overline{\mathcal{O}}_{n_1}(w_3)\overline{\mathcal{O}}_{n_2}(w_4)\rangle$$

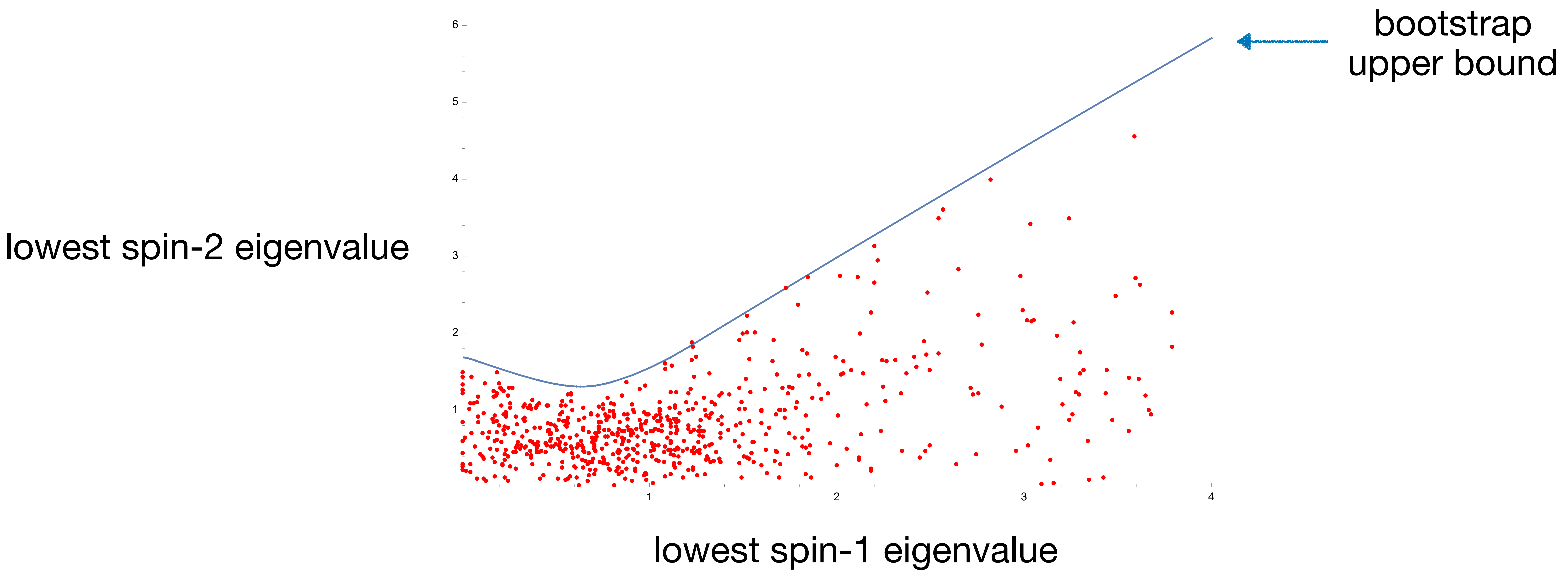
theorem: If X ranges over all orbifolds, $\lambda_1(X)$ takes the following values:



Example: $n_1 = 6, n_2 = 8 \Rightarrow \lambda_1 \le 23.0997$ unless the orbifold is $[0;2,3,7]$ or $n_1 \le 4$.

Hyperbolic three-manifolds

to appear (with J. Bonifacio, P. Kravchuk and S. Pal)



Future Directions

- Bounds on triple overlaps $c_{ijk} = \int h_i h_j h_k \Leftrightarrow$ bounds on L-functions.

[Sarnak]
[Bernstein, Reznikov]
[Michel, Venkatesh]
[Nelson]

- Non-compact orbifolds, and the role of arithmeticity (Hecke operators), $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

- Extract lessons about the conformal bootstrap of CFTs.

Thank you!

What do CFTs and hyperbolic manifolds have in common?

$$G = SO(1, d+1)$$

CFT$_d$	**Hyperbolic $(d+1)$-manifold**
space of field configurations on S^d	coset space $\Gamma \backslash G$
path integral measure	Haar measure
G acts by conformal transformations	G acts by right multiplication
L^2(field configurations on S^d): unitary representation of $SO(1, d+1)$	$L^2(\Gamma \backslash G)$

- All the ingredients we used for hyperbolic manifolds are also present for CFTs.
- Checked the spectral identites are valid for 2D Ising CFT and generalized free theory.

Why are CFTs and hyperbolic manifolds distinct?

CFT$_d$	**Hyperbolic $(d+1)$-manifold**
space of field configurations on S^d	$\Gamma \backslash G$
path integral measure	Haar measure
G acts by conformal transformations	G acts by right multiplication
L^2(field configurations on S^d): unitary representation of $SO(1, d+1)$	$L^2(\Gamma \backslash G)$
reflection positivity	❌
Hilbert space $\mathscr{H}(S^{d-1})$: unitary representation of $\widetilde{SO}(2, d)$	❌