

Space-Time , Quantum Mechanics,

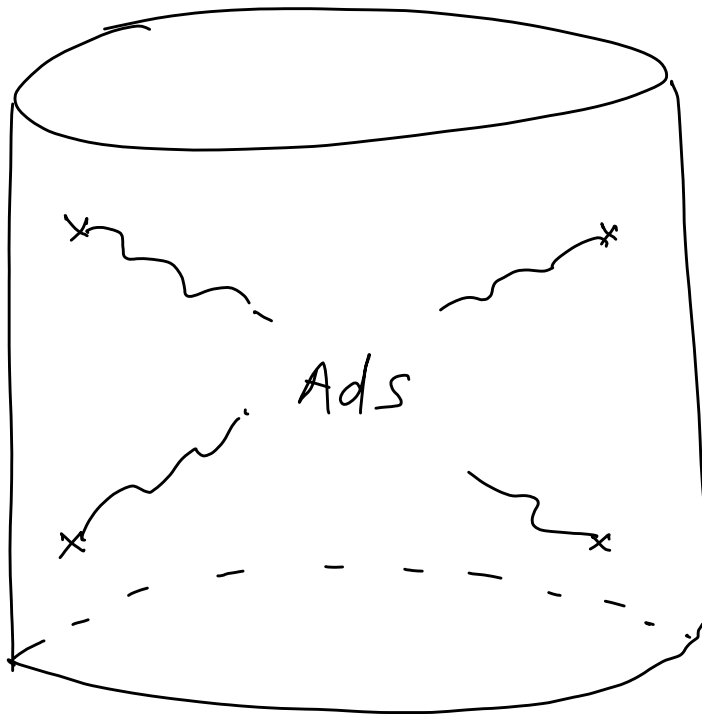
and

Clusterhedra at Infinity

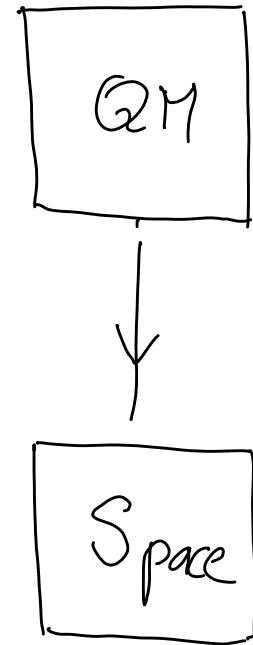
w/ S. He , T. Lam , H. Thomas
M. Spradlin, G. Salvatori
H. Frost, P.-G. Plamondon;
Y. Bai, G. Yan

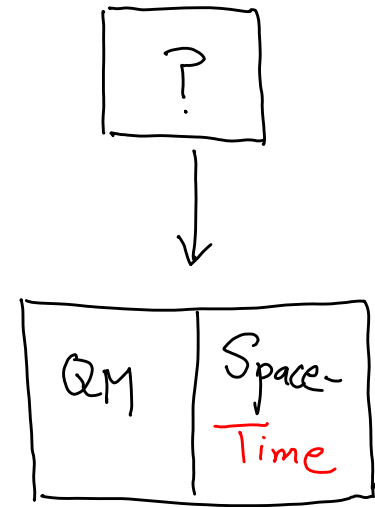
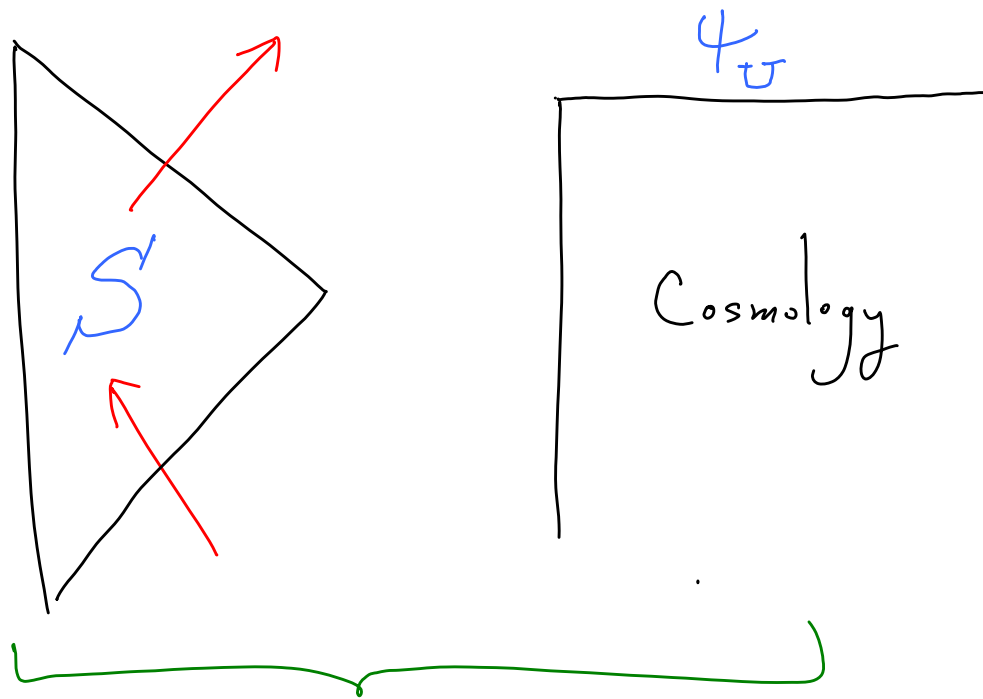
J. Trnka,
J. Bourjaily, ,
P. Benincasa,
S. Carron-Huot
C. Cheung
J. Kaplan

S. Goncharov
A. Postnikov,
F. Cachazo
A. Hodges



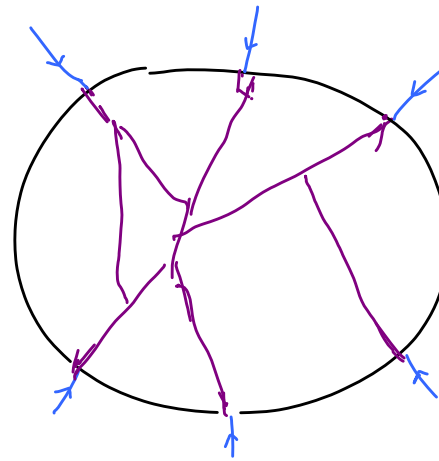
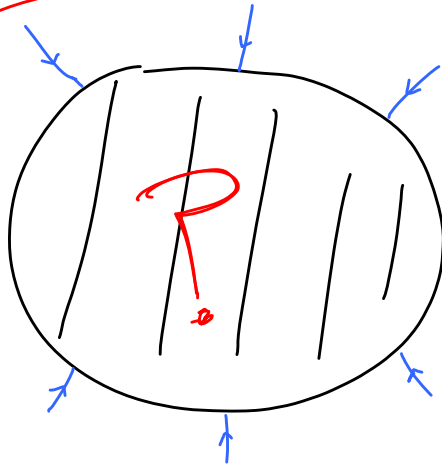
↑ time



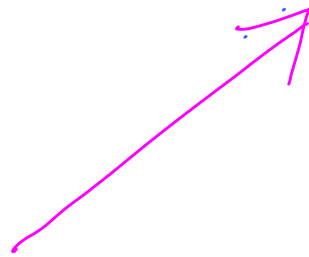


Unlike 2AdS, no
obvious notion of time+locality

What is the Q to which A is the Answer?

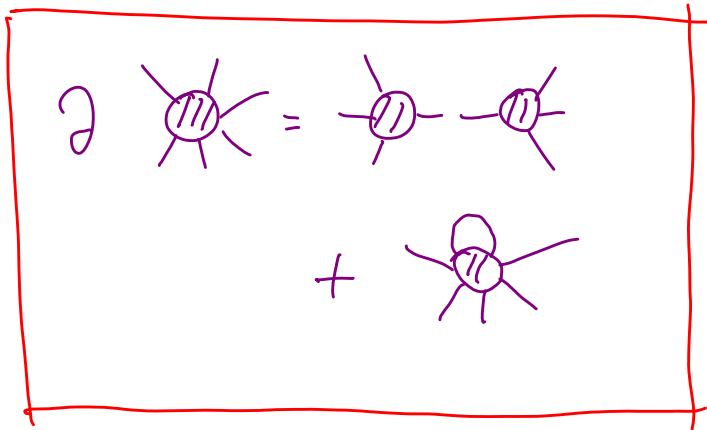
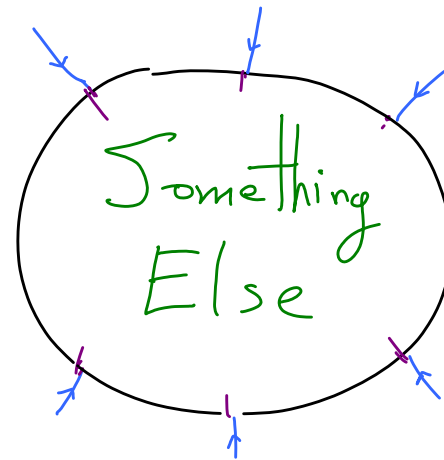
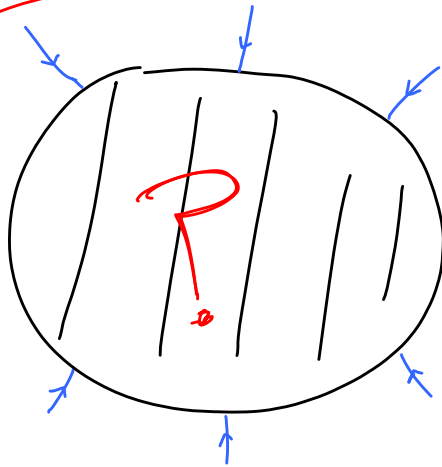


$$2 \text{ (circle with 3 lines)} = \text{(circle with 1 line)} - \text{(circle with 1 line)} + \text{(circle with 1 line)}$$



Local, Unitary evol. in
Spacetime/Hilbert Space.

What is the Q to which A is the Answer?

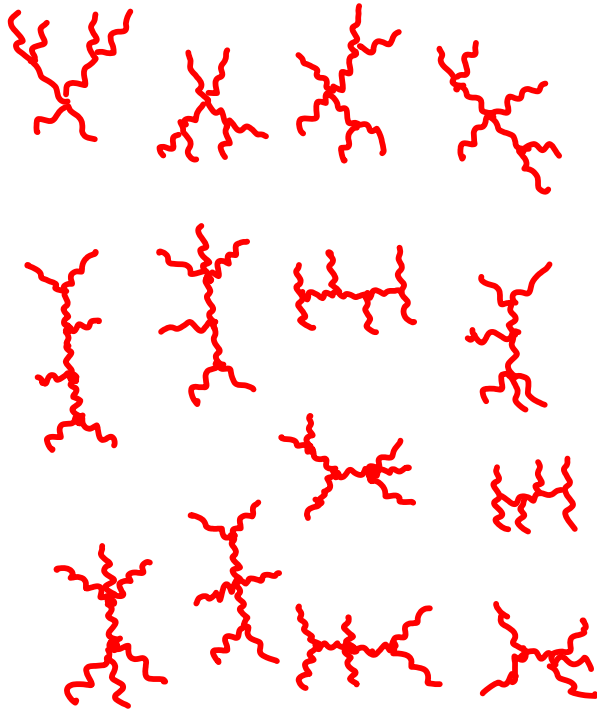


Rules of Spacetime + QM -
Locality + Unitarity -
as derivative outputs

L. H. C.



Feynman Follies



+ ...

220 Diagrams

10's of thousands
of terms ...

Result of a brute force calculation:

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 222. 223. 224. 225. 226. 227. 228. 229. 230. 231. 232. 233. 234. 235. 236. 237. 238. 239. 240. 241. 242. 243. 244. 245. 246. 247. 248. 249. 250. 251. 252. 253. 254. 255. 256. 257. 258. 259. 260. 261. 262. 263. 264. 265. 266. 267. 268. 269. 270. 271. 272. 273. 274. 275. 276. 277. 278. 279. 280. 281. 282. 283. 284. 285. 286. 287. 288. 289. 290. 291. 292. 293. 294. 295. 296. 297. 298. 299. 300. 301. 302. 303. 304. 305. 306. 307. 308. 309. 310. 311. 312. 313. 314. 315. 316. 317. 318. 319. 320. 321. 322. 323. 324. 325. 326. 327. 328. 329. 330. 331. 332. 333. 334. 335. 336. 337. 338. 339. 340. 341. 342. 343. 344. 345. 346. 347. 348. 349. 350. 351. 352. 353. 354. 355. 356. 357. 358. 359. 360. 361. 362. 363. 364. 365. 366. 367. 368. 369. 370. 371. 372. 373. 374. 375. 376. 377. 378. 379. 380. 381. 382. 383. 384. 385. 386. 387. 388. 389. 390. 391. 392. 393. 394. 395. 396. 397. 398. 399. 400. 401. 402. 403. 404. 405. 406. 407. 408. 409. 410. 411. 412. 413. 414. 415. 416. 417. 418. 419. 420. 421. 422. 423. 424. 425. 426. 427. 428. 429. 430. 431. 432. 433. 434. 435. 436. 437. 438. 439. 440. 441. 442. 443. 444. 445. 446. 447. 448. 449. 450. 451. 452. 453. 454. 455. 456. 457. 458. 459. 460. 461. 462. 463. 464. 465. 466. 467. 468. 469. 470. 471. 472. 473. 474. 475. 476. 477. 478. 479. 480. 481. 482. 483. 484. 485. 486. 487. 488. 489. 490. 491. 492. 493. 494. 495. 496. 497. 498. 499. 500. 501. 502. 503. 504. 505. 506. 507. 508. 509. 510. 511. 512. 513. 514. 515. 516. 517. 518. 519. 520. 521. 522. 523. 524. 525. 526. 527. 528. 529. 530. 531. 532. 533. 534. 535. 536. 537. 538. 539. 540. 541. 542. 543. 544. 545. 546. 547. 548. 549. 550. 551. 552. 553. 554. 555. 556. 557. 558. 559. 560. 561. 562. 563. 564. 565. 566. 567. 568. 569. 570. 571. 572. 573. 574. 575. 576. 577. 578. 579. 580. 581. 582. 583. 584. 585. 586. 587. 588. 589. 590. 591. 592. 593. 594. 595. 596. 597. 598. 599. 600. 601. 602. 603. 604. 605. 606. 607. 608. 609. 610. 611. 612. 613. 614. 615. 616. 617. 618. 619. 620. 621. 622. 623. 624. 625. 626. 627. 628. 629. 630. 631. 632. 633. 634. 635. 636. 637. 638. 639. 640. 641. 642. 643. 644. 645. 646. 647. 648. 649. 650. 651. 652. 653. 654. 655. 656. 657. 658. 659. 660. 661. 662. 663. 664. 665. 666. 667. 668. 669. 670. 671. 672. 673. 674. 675. 676. 677. 678. 679. 680. 681. 682. 683. 684. 685. 686. 687. 688. 689. 690. 691. 692. 693. 694. 695. 696. 697. 698. 699. 700. 701. 702. 703. 704. 705. 706. 707. 708. 709. 710. 711. 712. 713. 714. 715. 716. 717. 718. 719. 720. 721. 722. 723. 724. 725. 726. 727. 728. 729. 730. 731. 732. 733. 734. 735. 736. 737. 738. 739. 740. 741. 742. 743. 744. 745. 746. 747. 748. 749. 750. 751. 752. 753. 754. 755. 756. 757. 758. 759. 760. 761. 762. 763. 764. 765. 766. 767. 768. 769. 770. 771. 772. 773. 774. 775. 776. 777. 778. 779. 780. 781. 782. 783. 784. 785. 786. 787. 788. 789. 790. 791. 792. 793. 794. 795. 796. 797. 798. 799. 800. 801. 802. 803. 804. 805. 806. 807. 808. 809. 810. 811. 812. 813. 814. 815. 816. 817. 818. 819. 820. 821. 822. 823. 824. 825. 826. 827. 828. 829. 830. 831. 832. 833. 834. 835. 836. 837. 838. 839. 840.

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4. (10) 2002年10月1日起，凡在我国境内销售货物的单位和个人，其销售额中超过100万元的部分，应按5%的税率缴纳附加税。假设某企业2002年10月1日起，其销售额为100万元，则该企业应缴纳的附加税为多少元？

a) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$; b) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$; c) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$; d) $\lim_{x \rightarrow 0} \frac{e^x + 1}{x} = \infty$; e) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; f) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; g) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; h) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; i) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; j) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; k) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; l) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; m) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; n) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; o) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; p) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; q) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; r) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; s) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; t) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; u) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; v) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; w) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; x) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$; y) $\lim_{x \rightarrow 0} \frac{\ln x}{x} = -\infty$; z) $\lim_{x \rightarrow 0} \frac{\ln x}{\ln x} = 1$;

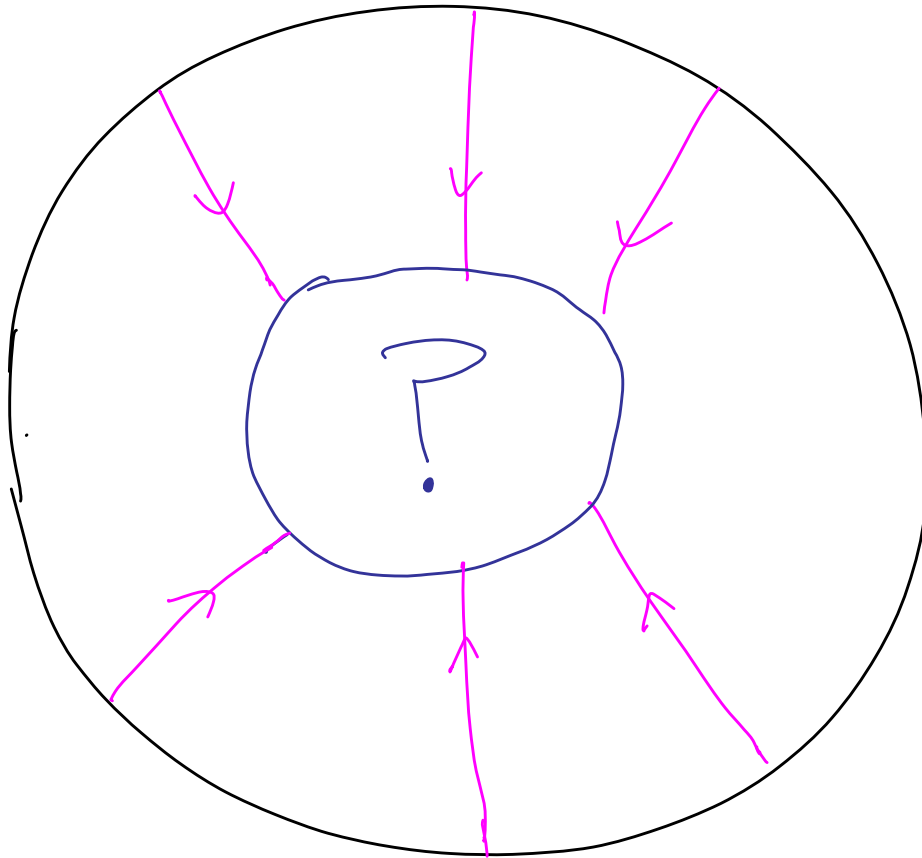
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+ ~ 100 more pages....

$$(1^- 2^+ 3^- 4^+ 5^+ 6^+)$$

$$\frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} (!)$$

Standard way of doing physics makes
usual rules of spacetime + QM
manifest — but is obviously hiding
some extraordinary new structures!



For ~ 10 years
(+ more intensely
in last 2 years)
we are seeing

Positive
Geometries in
Kinematic Space

The Canvas

* Physical momenta

* "Twistor" variables

* "Celestial Sphere"

* Spatial Future

Kin. Space
 $\sim \mathcal{D}$ Minkowski

(Amps)

(ψ_{Univ})

What ideas
breathe
physics-life
into this
space?

The Canvas

| | |
|-----------------------|--|
| * Physical momenta | Kin. Space
$\sim \mathcal{D}$ Minkowski |
| * "Twistor" variables | (Amps) |
| * "Celestial Sphere" | |
| ⋮ | |
| * Spatial Future | (ψ_{Univ}) |

Emerging Picture:

Combinatorics

↕
Positive Geometries

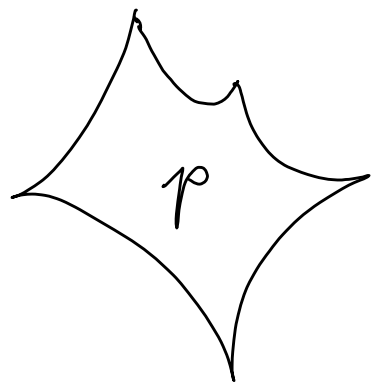
↕
Canonical Forms

* Amplituhedra, Associahedra, Cosmological Polytopes, ...

* Describes REAL WORLD PHYSICS in reasonable approximation, exposes hidden symmetries + new mathematical structures

* Pure magic in physics right under our nose

Positive Geometries



Region $\mathcal{P}$ w/ boundaries
of all Codimension

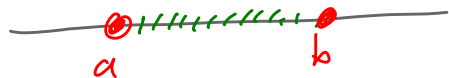
Real Geometry



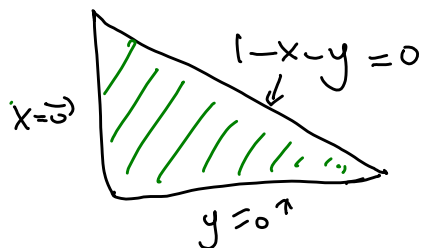
Canonical Forms

$\Omega_{\mathcal{P}}$: unique form with
logarithmic singularities
on (+ only on) boundaries
of $\mathcal{P}$ $\{ \text{locally } \Omega \rightarrow \frac{dx_1}{x_1} \dots \frac{dx_p}{x_p} \}$

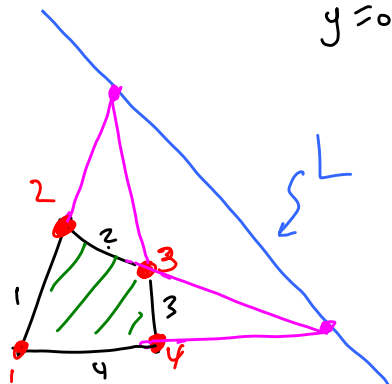
Complex Form



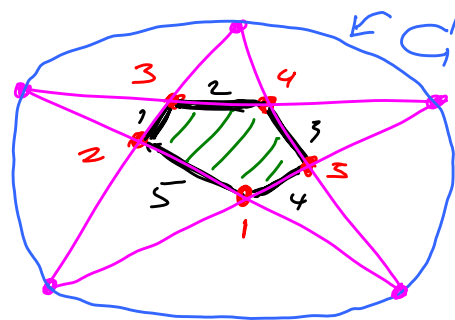
$$\Omega = \frac{dx}{x-a} - \frac{dx}{x-b}$$



$$\Omega = \frac{dx dy}{x y (1-x-y)}$$



$$\Omega = \frac{dx dy}{L_1 L_2 L_3 L_4} \quad \text{L}$$

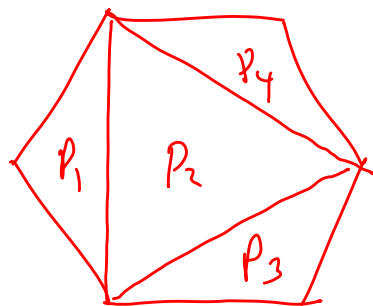


$$\Omega = \frac{dx dy}{L_1 L_2 L_3 L_4 L_5} \quad \text{C}$$

Singularities of $\Omega \longleftrightarrow$

Boundaries/Faces
of P

Triangulation



P tiled by P_i



$$\Omega_P = \sum_i \Omega_{P_i}$$

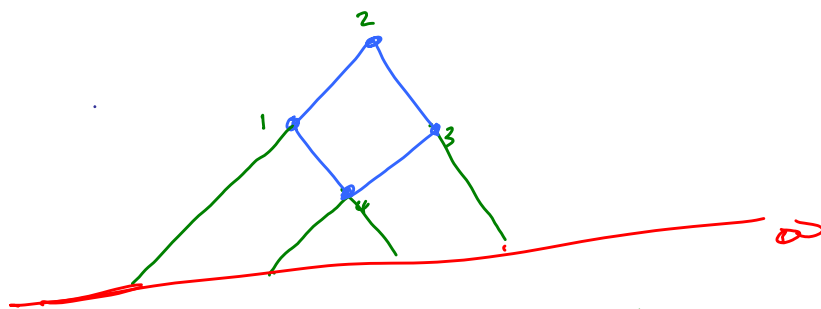
Spurious poles
cancel in sum

Ex: "Simple Polytopes"

$$\Omega$$

$\sum_{\text{vertices}}$

$$\text{sgn}(X_{a_1} \cdots X_{a_n}) d \log X_{a_1} \cdots d \log X_{a_n}$$



Spurious Poles @ ∞ cancel in sum.

Bi-adjoint ϕ^3 theory

* Interaction $f_{abc} f^{ABC} \phi_A^a \phi_B^b \phi_C^c$

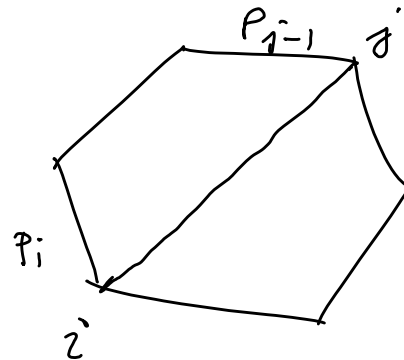
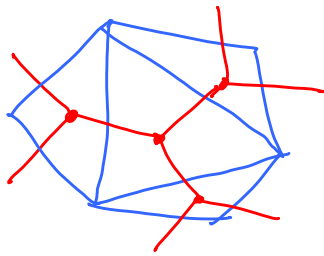
* "Trace decomposition":

$$\text{Tr}(T_{a_1} \dots T_{a_n}) \text{Tr}(T_{A_1} \dots T_{A_n})$$

$$\text{Tr}(\dots T_{a\dots}) \text{Tr}(\dots T_{a\dots}) \text{Tr}(\dots T_{A\dots}) \text{Tr}(\dots T_{A\dots})$$

* Usual $1/N \leftrightarrow$ genus expansion, where diagrams must be consistent with orderings for both "a" and "A". Focus on case where "a", "A" orderings are same, diagrammatics identical to YM theory

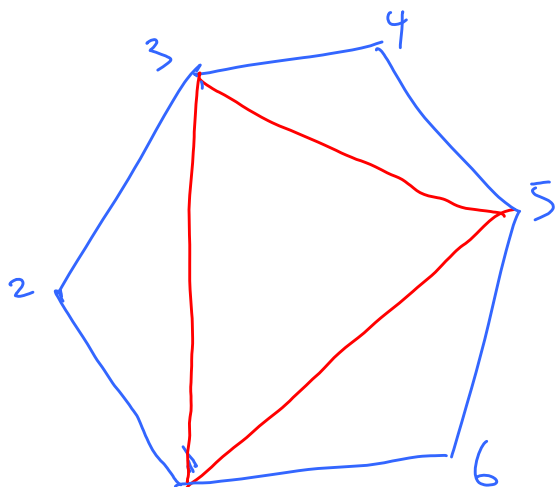
ϕ^3_{B-A} at Tree Level



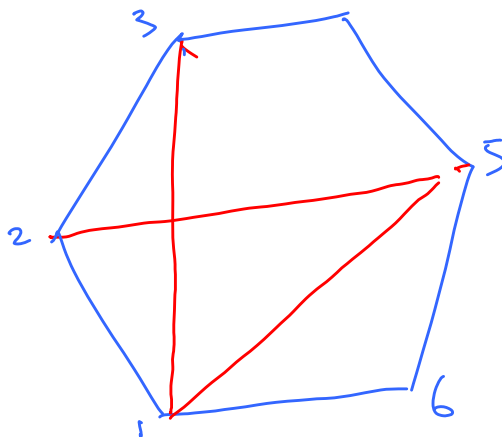
$$X_{ij} = (p_i + \dots + p_{j-1})^2 = X_{ji}$$

$$[X_{ii+1} = 0]$$

$$A^{\text{tree}} = \sum_{\text{Triang. } T} \prod_{ij \in T} \frac{1}{X_{ij}}$$

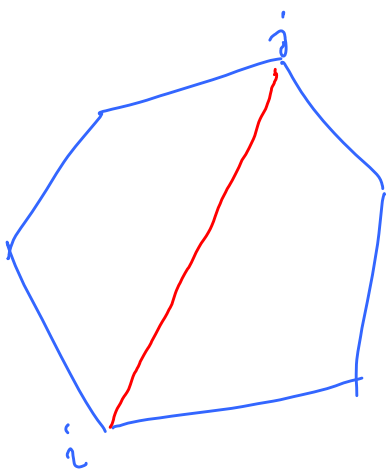


Allowed

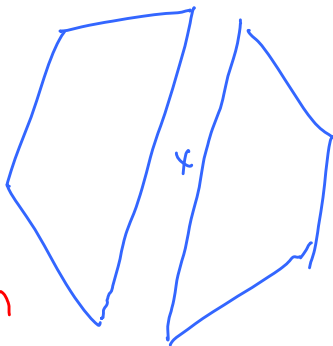


Crossing Chords not allowed.

Locality



$x_{ij} \rightarrow 0$
Factorization



Unitarity

Big Clue Seen By $\mathcal{N}=4$ Theorist Experimentalist

Amplitude has poles when $(\sum p)^2 \rightarrow 0$, + Factorizes!

$$\partial \quad \text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3}$$

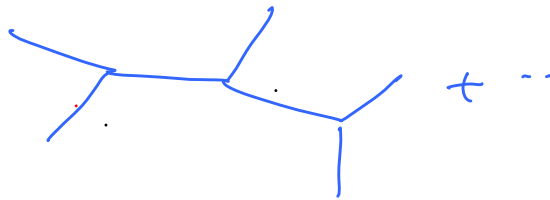
Diagram 1: A circle with three internal lines and four external lines.

Diagram 2: A circle with two internal lines and four external lines.

Diagram 3: A circle with one internal line and four external lines.

F

"It's



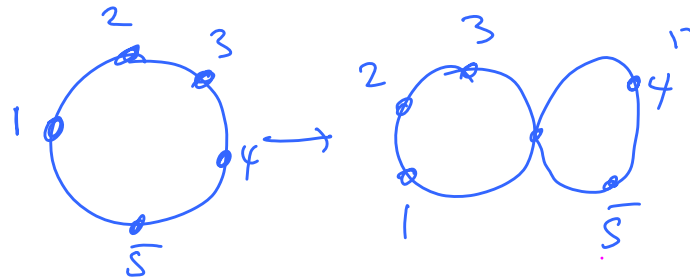
"

Particles in
Spacetime

D

"

"It's



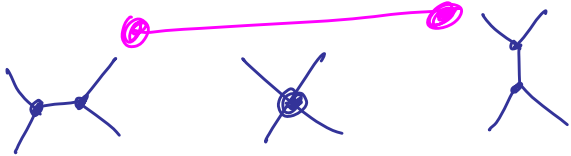
String Worksheet

Bigger Clue Seen By Lazier but Smarter Experm.

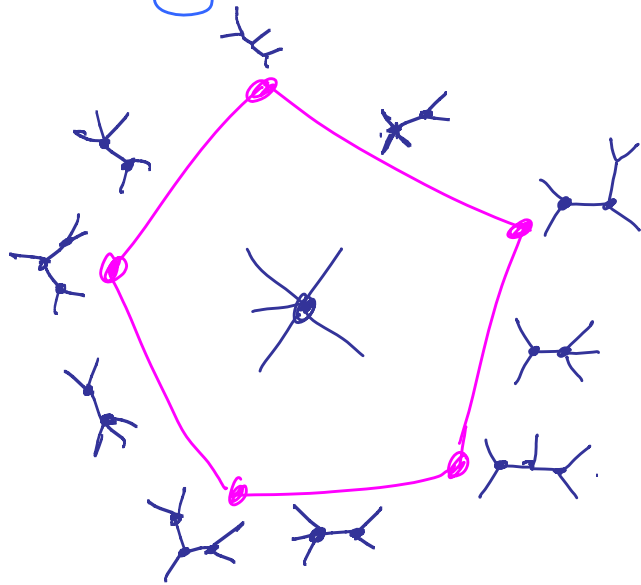
"Only certain patterns of poles come together!"

$n=4$ s, t but not s, t

$n=5$ s_{12}, s_{123} + cyclic but not other comb.



$n=4$

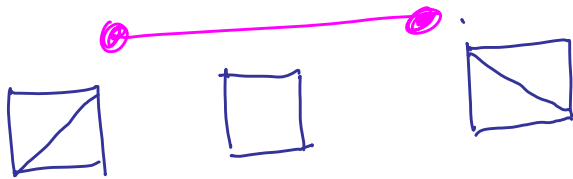


Bigger Clue Seen By Lazier but Smarter Experm.

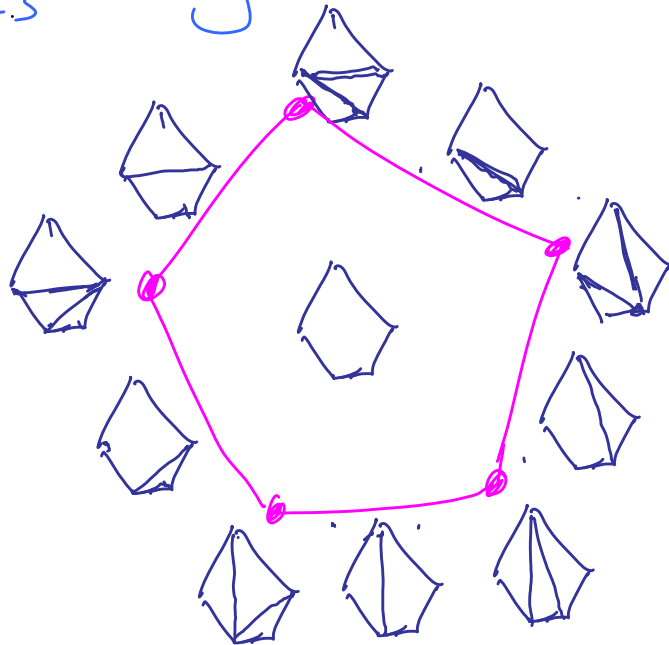
"Only certain patterns of poles come together!"

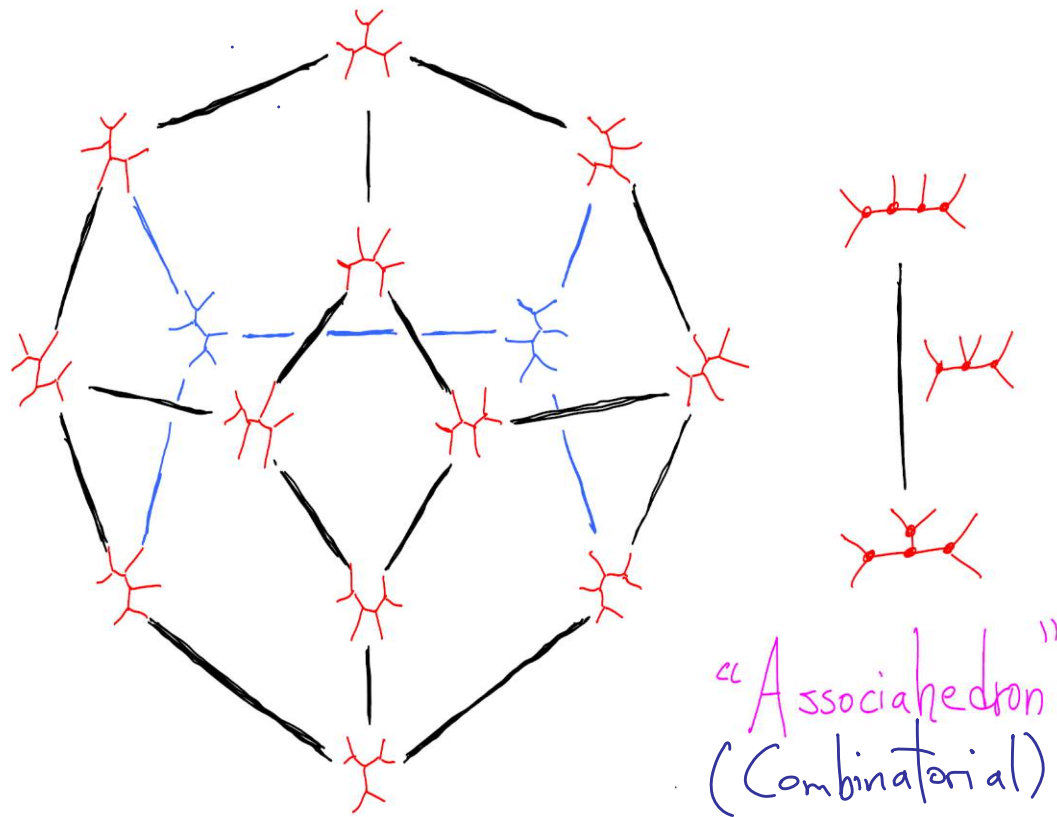
$n=4$ s, t but not s, t

$n=5$ s_{12}, s_{123} + cyclic but not other comb.



$n=4$





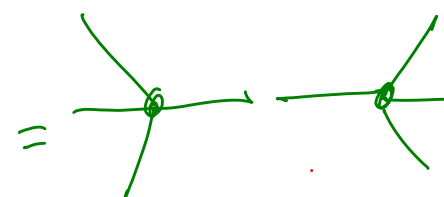
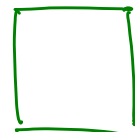
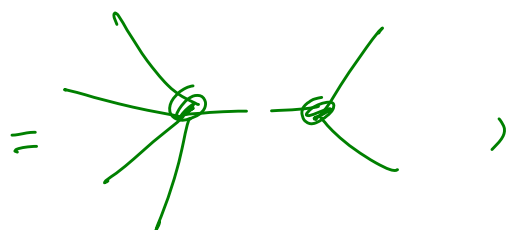
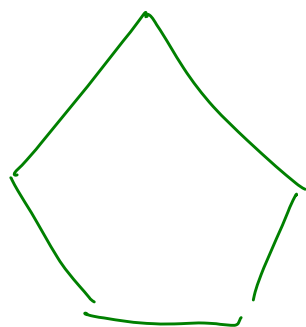
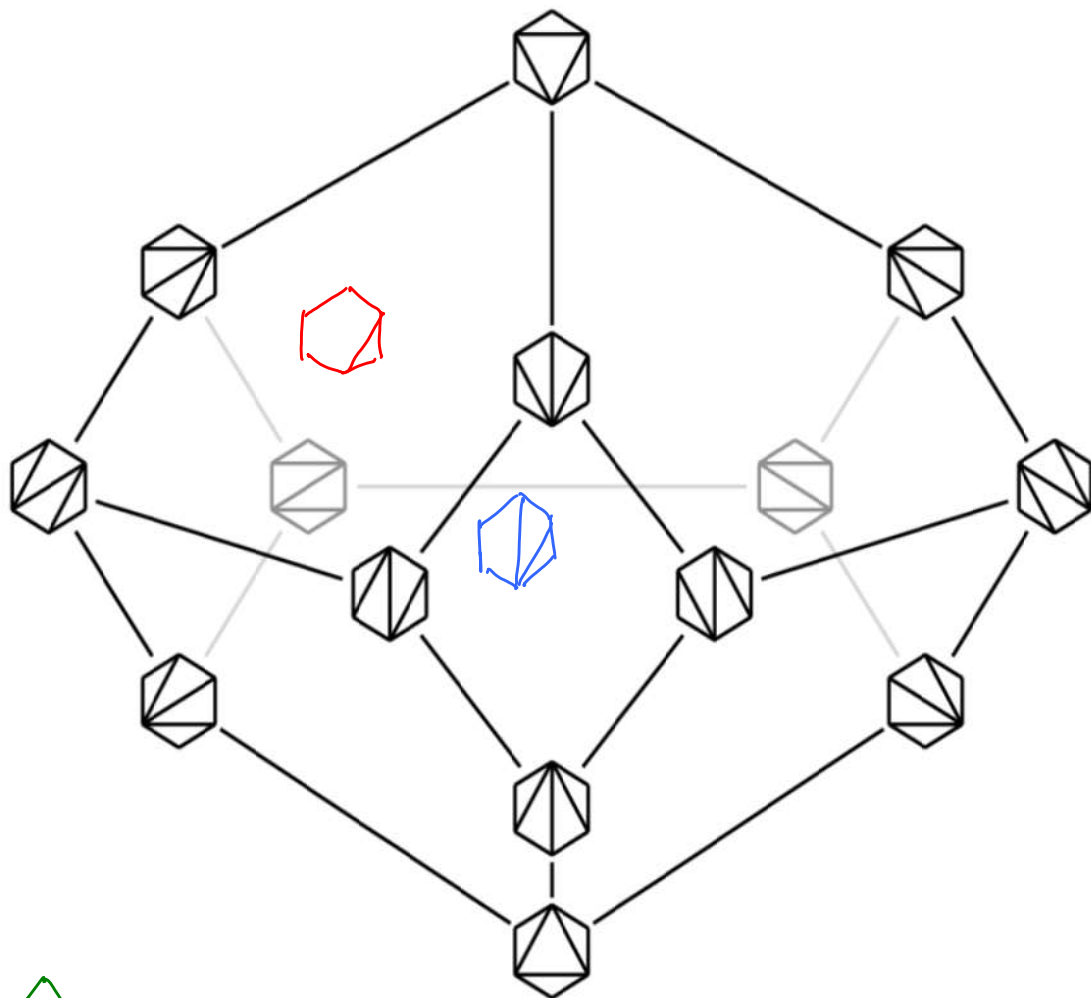
Particle + String
picture don't
make this
obvious!

But Assoc.
does see
everything
since

$$\partial A_n = A_{n_1} \times A_{n_2}$$

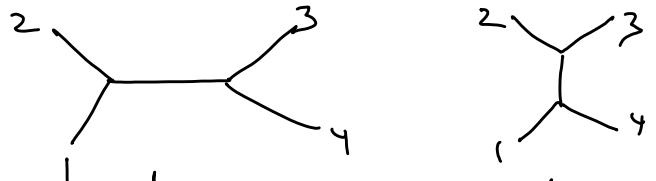
HOW CAN WE MAKE THIS OBVIOUS?

A Qualitative Clue to the "theory"
in Kinematic Space



Hidden Symmetry: Projective Invariance

$n=4$



$$\frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}} = d \log X_{13}/X_{24}.$$

General n : invariance of $\Omega_{(n-3)}$ under $X_{ij} \rightarrow \alpha(x) X_{ij}$

$\Rightarrow d \log X_{ij} \rightarrow d \log X_{ij} + d \log \alpha \rightarrow$ for each set

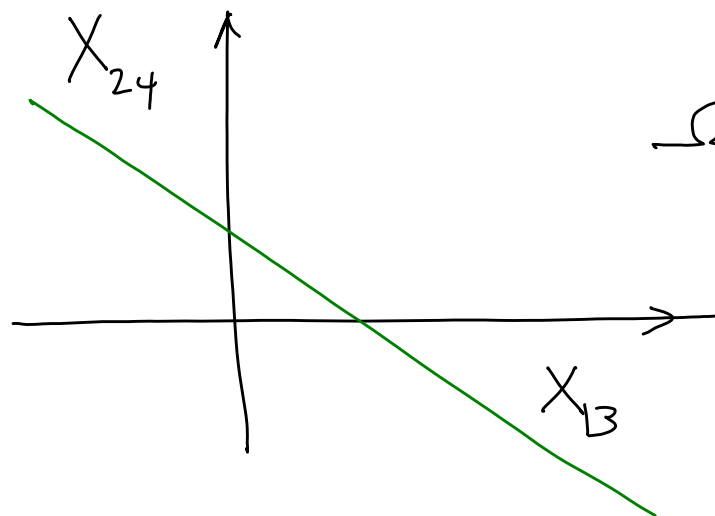
of $(n-4)$ comp. propagators, coeff. of $d \log \alpha$ vanishes.

Many more eqns than unknown signs.

Remarkably: signs can be chosen so $\Omega_{(n-3)}$ is projective!

[Analog of Dual Conf Inv. in $N=4$ SYM].

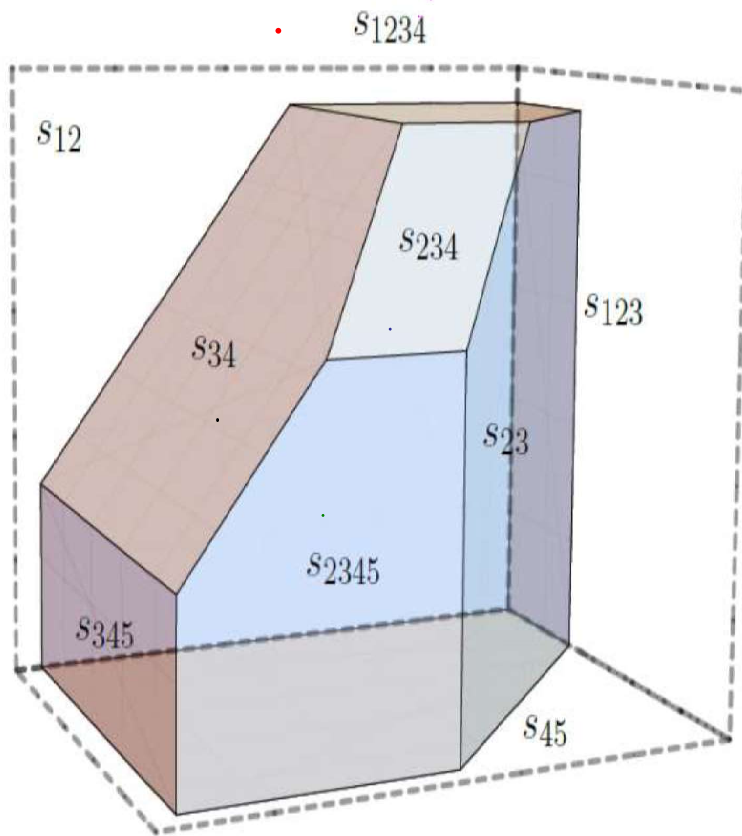
Pull back to Amplitude



$$|\Omega| = dX_{13} \left(\frac{1}{X_{13}} + \frac{1}{X_{24}} \right)$$

$$X_{13} + X_{24} = c$$

General n : $|\Omega|_{ABHY} = d^{(n-3)} X \cdot (\text{Amplitude})$



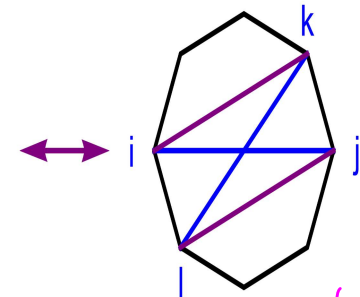
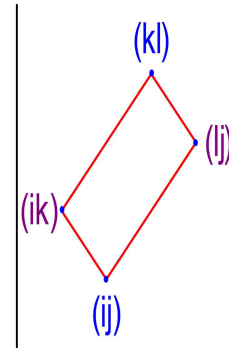
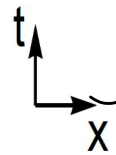
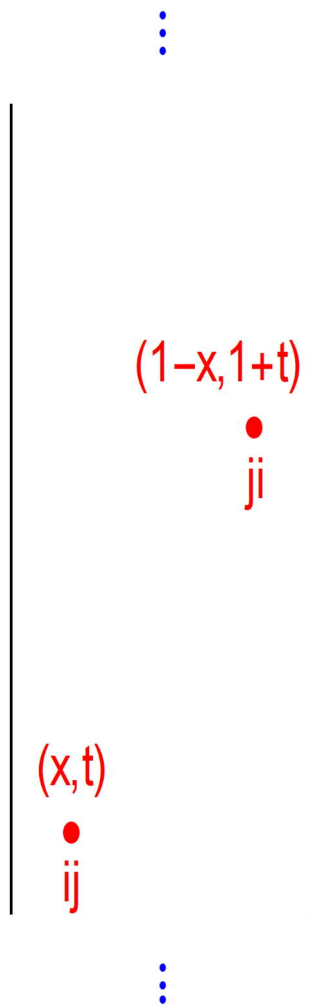
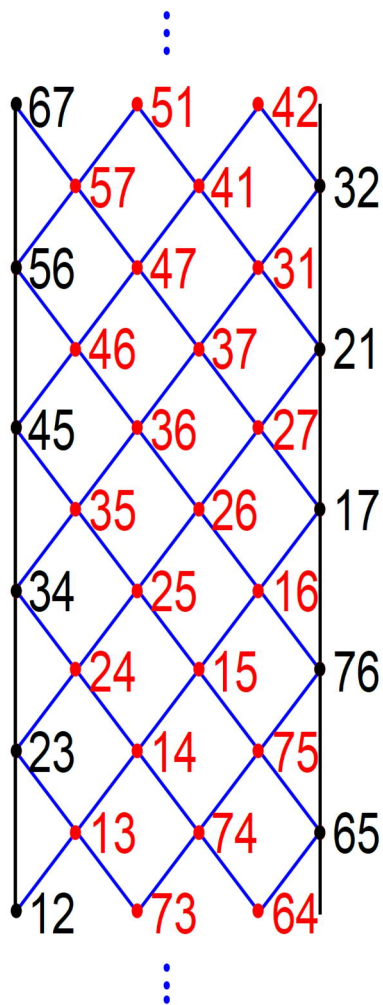
"ABHY"
 Associahedron
 in Kinematic
 Space

↓
 "Cluster Algebra"

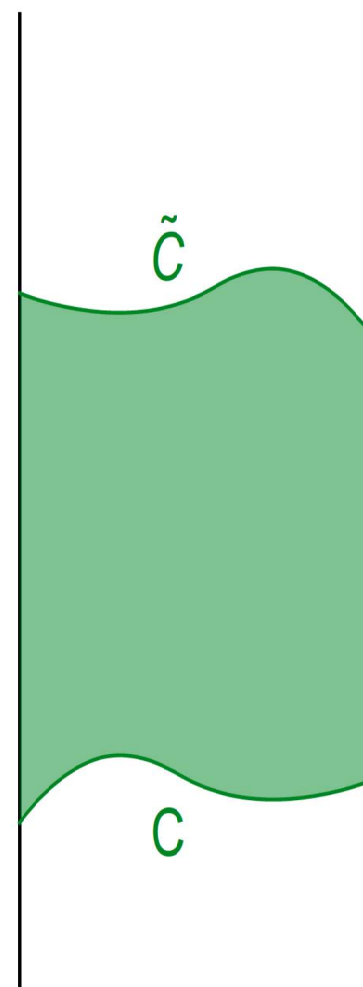
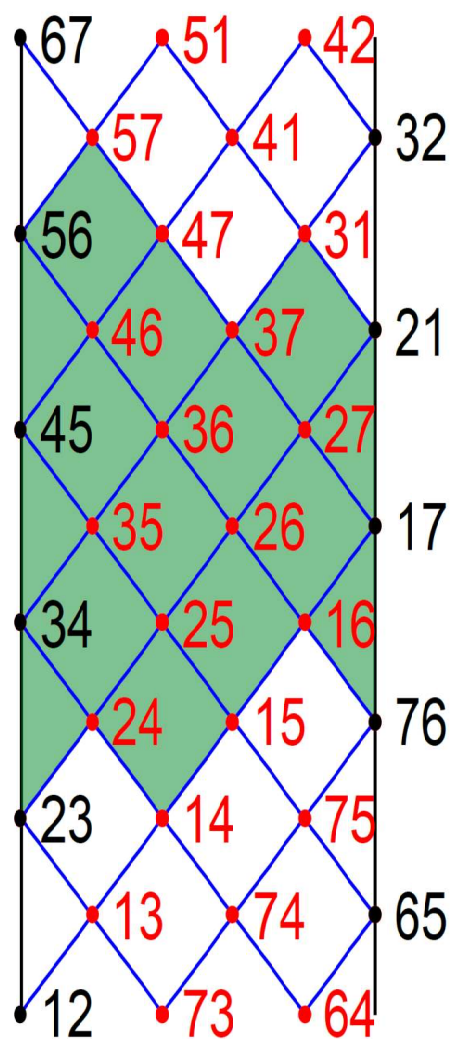
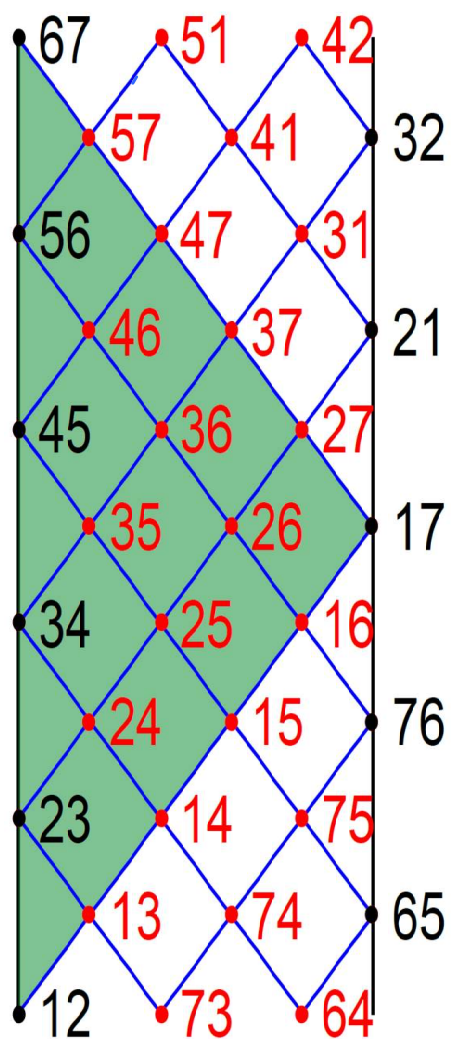
Polytopes
 Associated with
 Dynkin Diagrams
 (H. Thomas et al.)

HOW CAN WE MAKE THIS OBVIOUS?
 A Qualitative Clue to the "theory"
 in Kinematic Space

Kinematic "Spacetime"



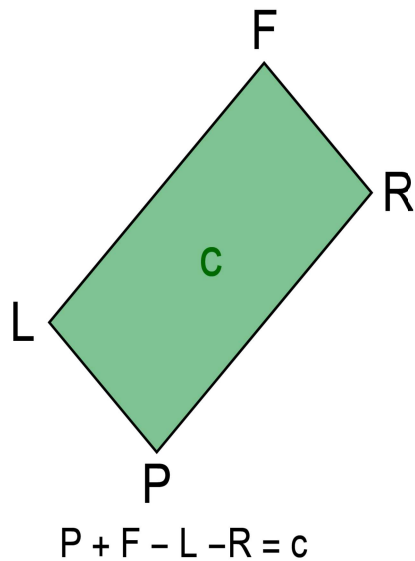
P, F corners $\longleftrightarrow$ incompatible chords
of Causal Diamond



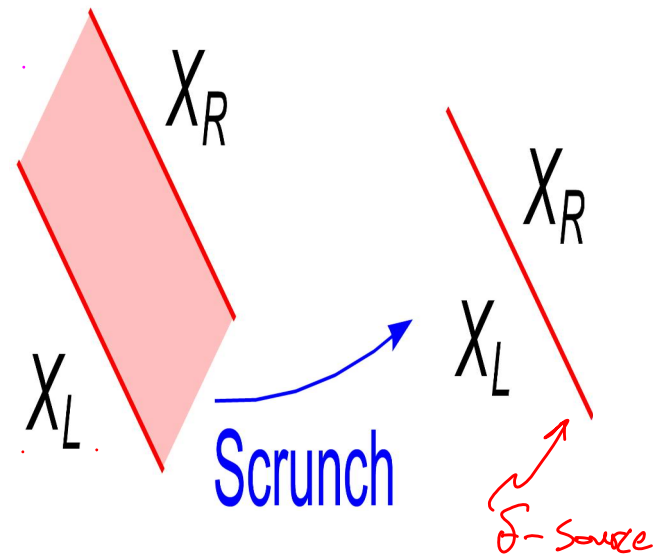
Wave Equation in Kin. "Spacetime"

$$\partial_u \partial_v X = c$$

"Gauss Law"

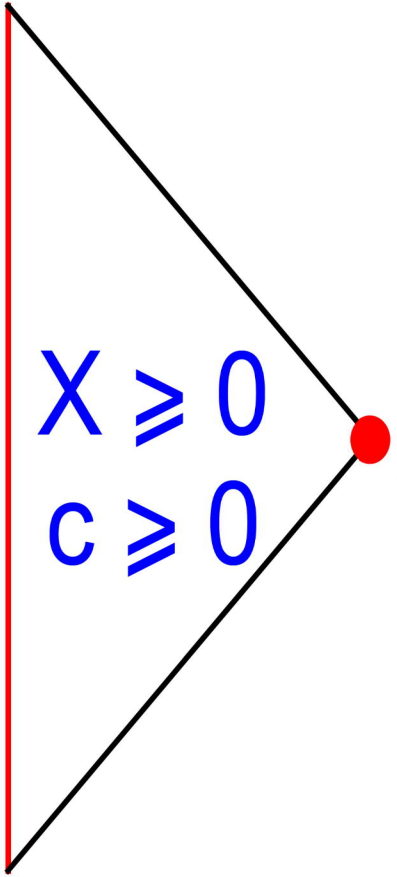


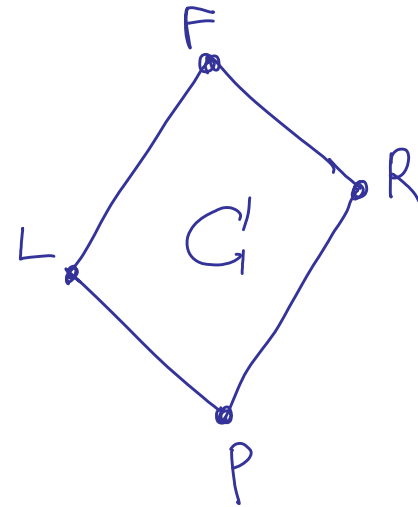
"Scrunch rule"



Positive Wave Equation

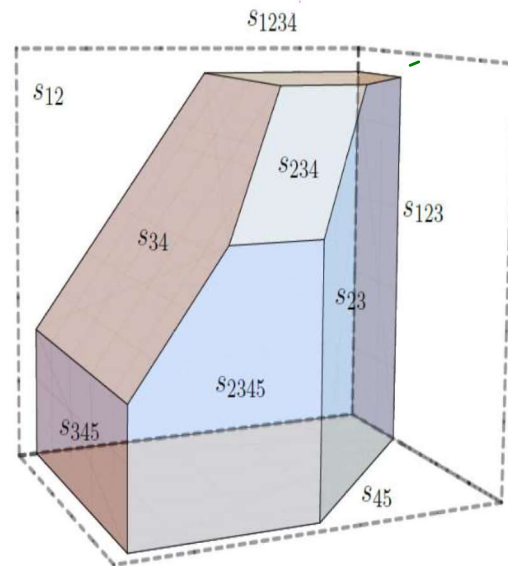
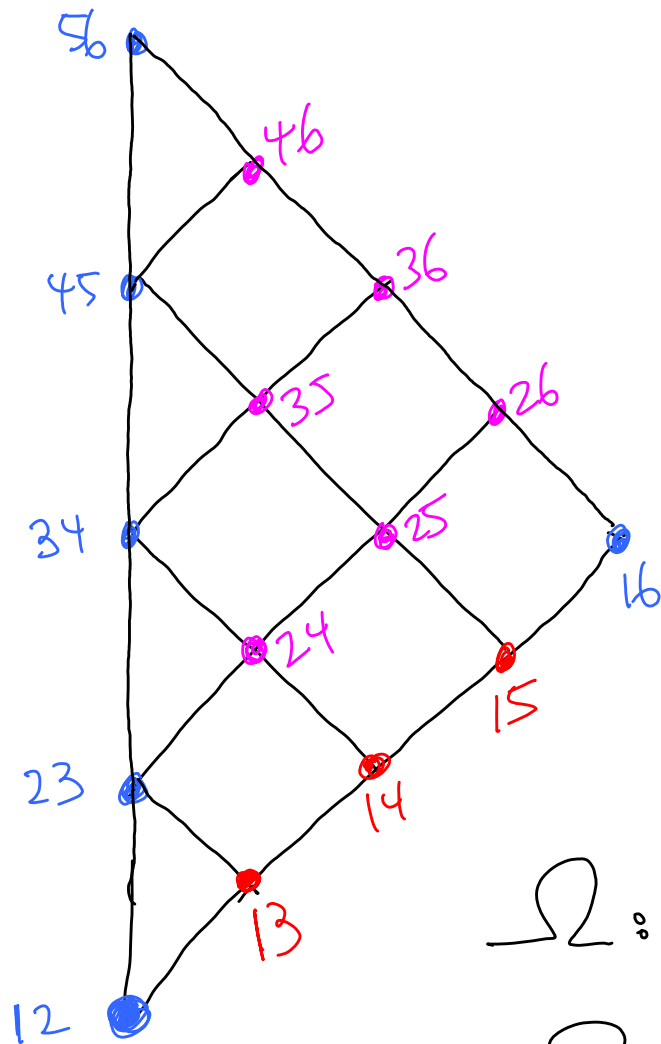
$$\partial_u \partial_v X = c$$


$$X = 0 \quad \begin{matrix} X \geq 0 \\ c \geq 0 \end{matrix} \quad X = 0$$



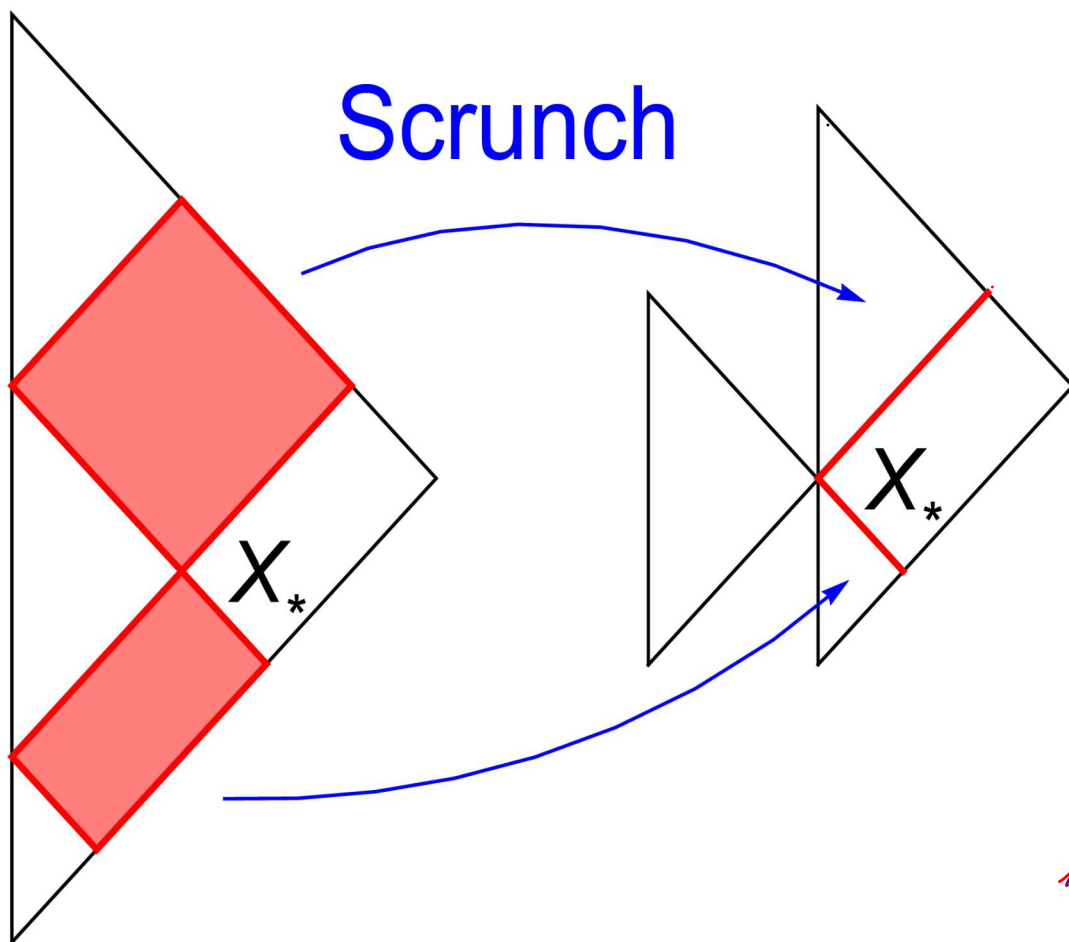
$$X_P + X_F - X_L - X_R = C_1$$

Positive WE $\longrightarrow$ ABHY Associahedron
in Kinematic Space.

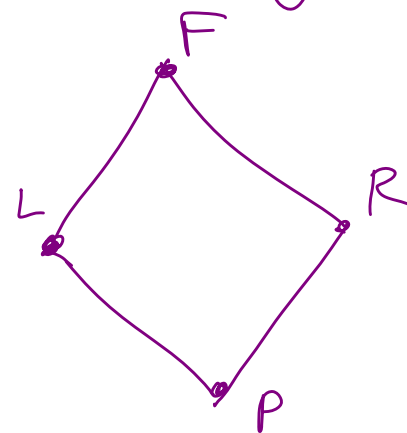


Ω : Fully determined by
 $\Omega|_{WE} = \omega^{can} [Assoc.]$

Factorization!



Crucial point:

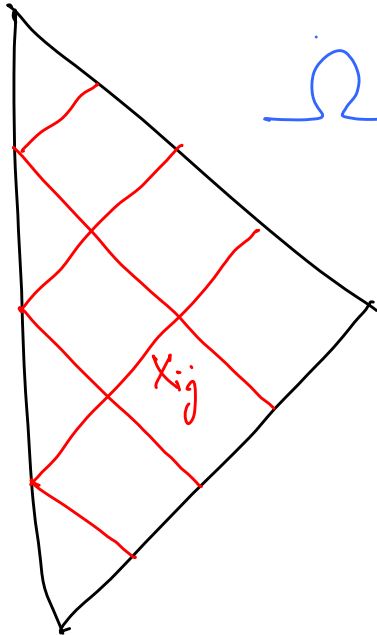


can't put $P=0, F=0$.

$$\overrightarrow{P+F} - L - R = c$$

neg = pos X

Scattering Form as Canonical Form



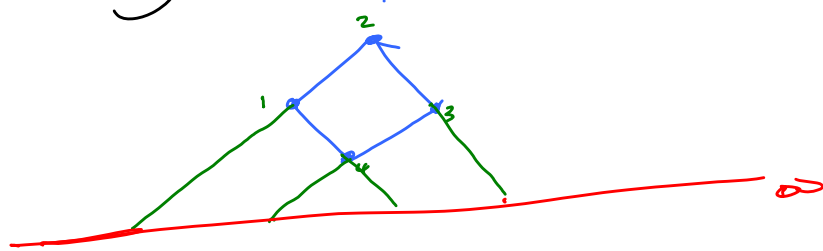
Ω fixed by : $\Omega \Big|_{\substack{\text{Wave} \\ \text{Eqn}}} = \omega^{\text{can}} [\text{Assoc}]$

* Trivially Projective Invariant

* Since $\det[g_1, \dots, g_{n-3}]$ of g -vectors
in a cluster cone $= \pm 1$,

$$\Omega \Big|_{WE} = \omega^{\text{can}} [\text{Assoc}] = (dVol) \times \text{Amplitude!}$$

* Feynman diagrams are one particular triangulation :

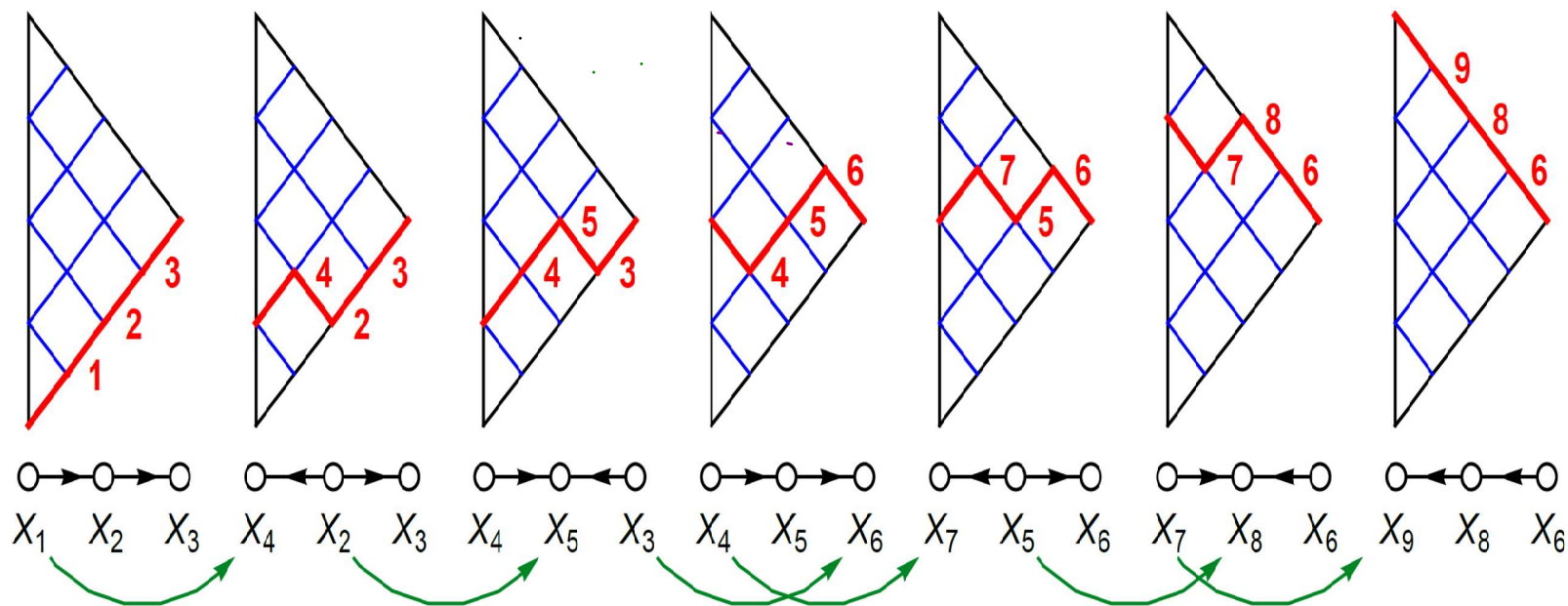


Spurious poles @
 ∞ break proj.
 invariance term-
 by-term

* Many other triangulations! Manifest proj.
 invariance term-by-term, vastly more efficient
 than Feynman Diagrams

* Another general method for computing
 can. forms, applied here, yields string amplitudes

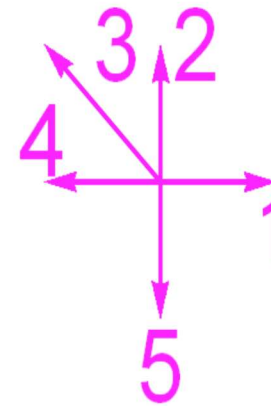
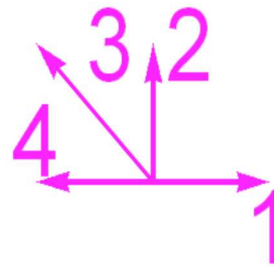
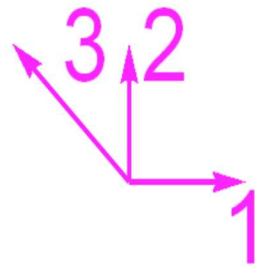
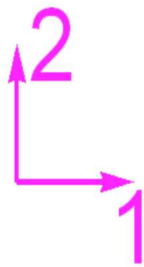
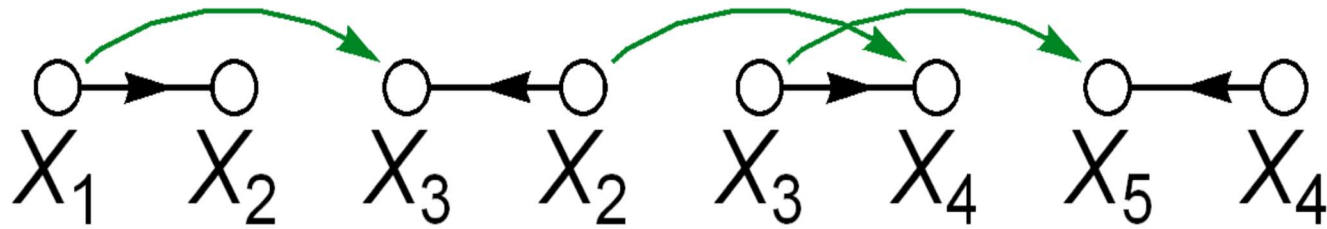
"Time Evolution" in Kinematic Space



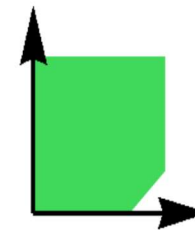
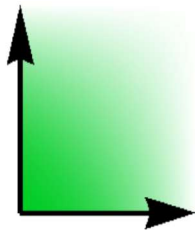
"Walk" by mutating only on sources

$$X_v + X'_v - \sum_{v \rightarrow w} X_w = c_v$$

(~ "tropicalized mutation")

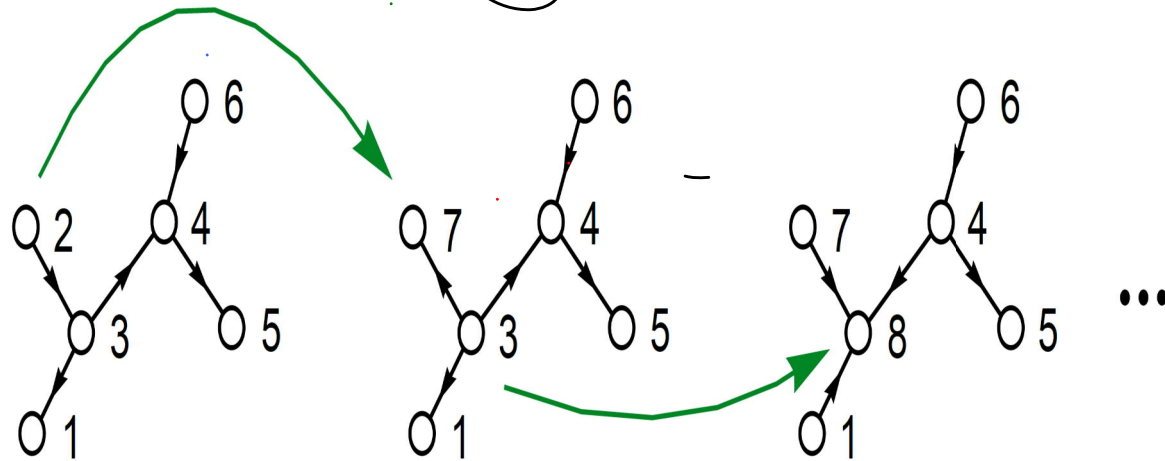


STOP



STOP

Abstract Away "Time Evolution"



$$X_v + X'_v - \sum_{v \rightarrow w} X_w = c_v$$

(Auslander-Reiten Walk)

Can Define For Any Acyclic Quiver

$$Q \xrightarrow[\text{@ } i]{\text{mutate}} Q' \implies X' = A_i X$$

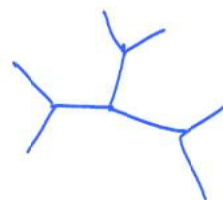
A_i generators of Coxeter Group of Q

ABHY Polytope Finite $\longrightarrow$ Coxeter Group Finite

$\longrightarrow$ Dynkin Classification $\longrightarrow$ ABHY Real.

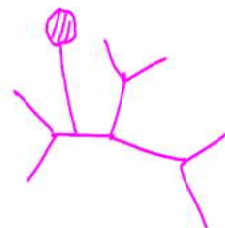
of all Finite Cluster Algebra Polytopes

A_n



"tree"

B_n

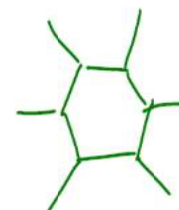
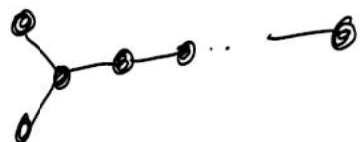


"tadpole"

C_n



D_n

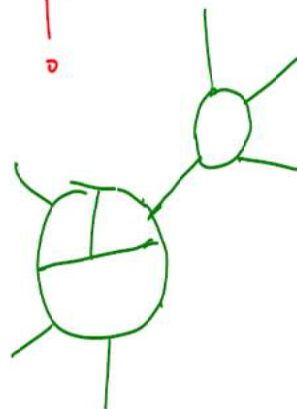


"loop"

Exceptional Finite



?

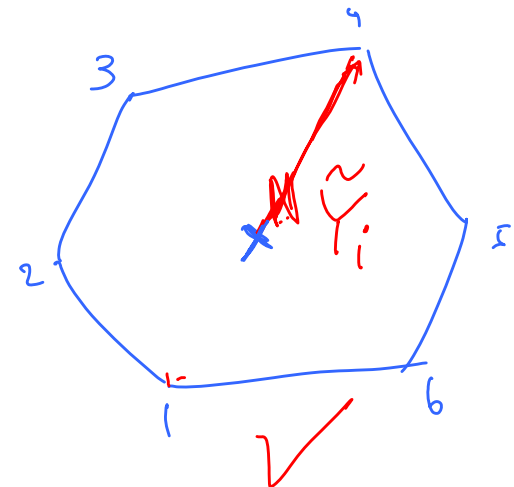
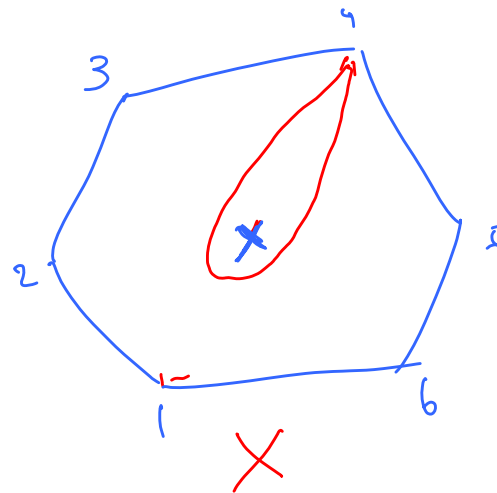
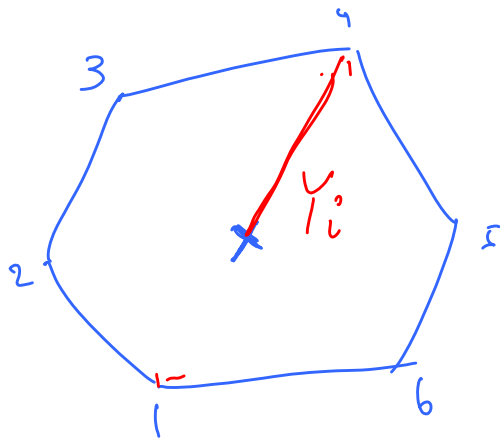
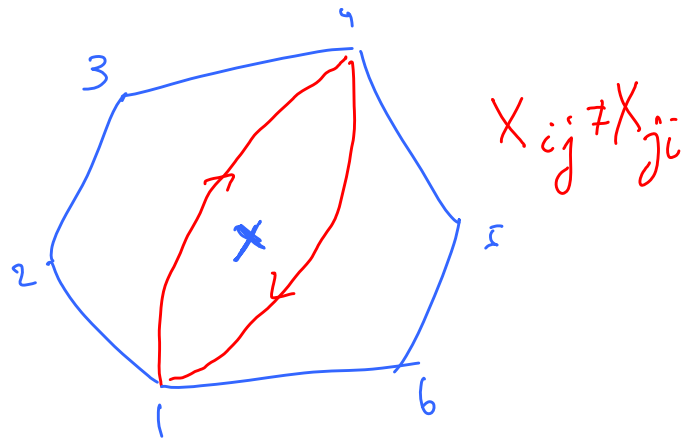
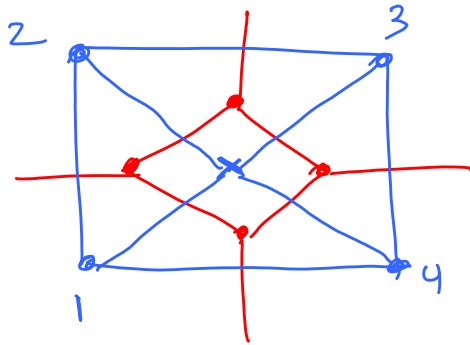


Infinite type

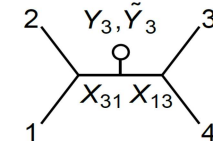
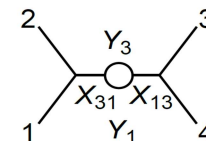
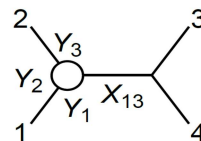
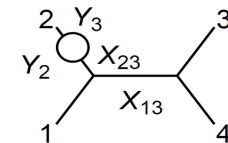
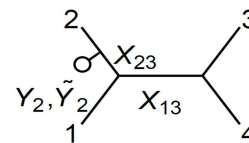
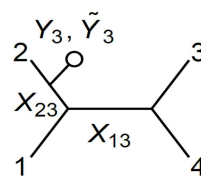
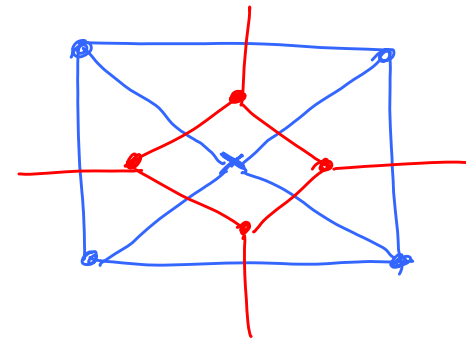
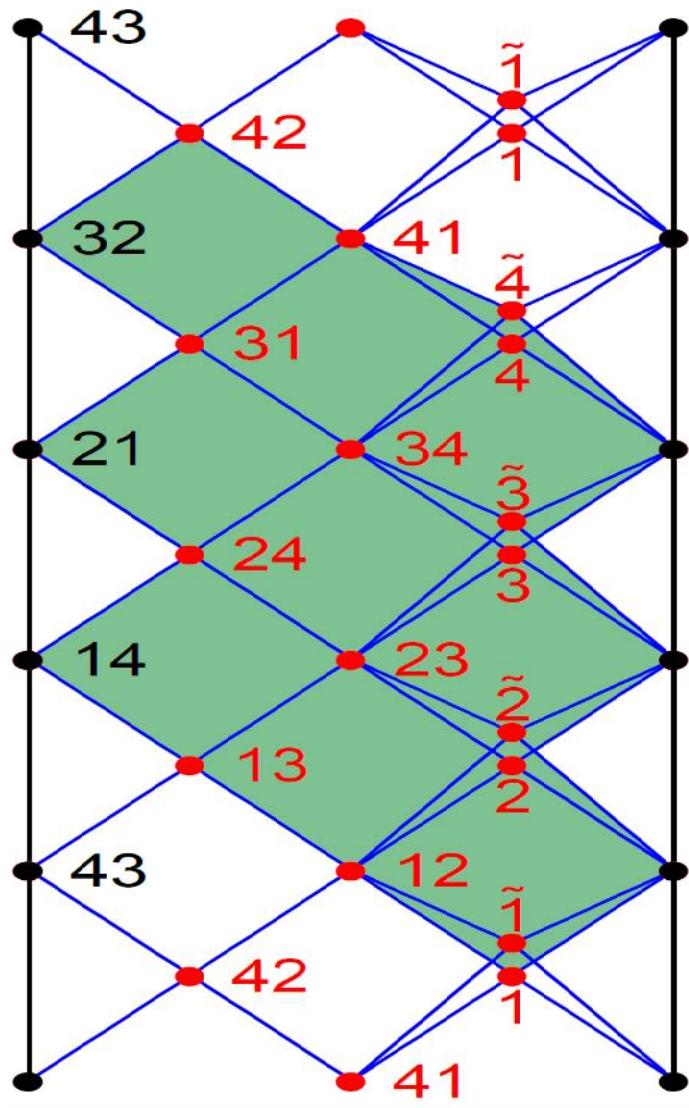


(Cluster Algebra of Punctured Surfaces)

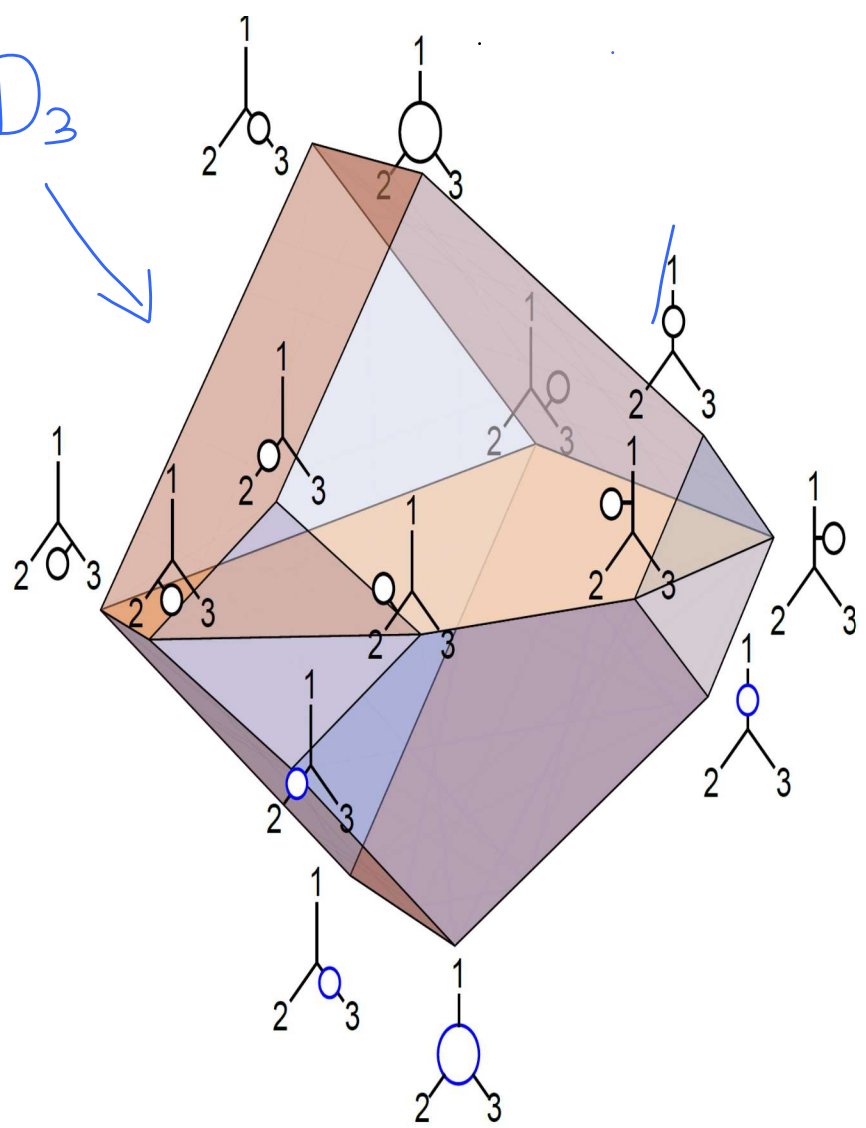
Variables for 1-loop Integrand



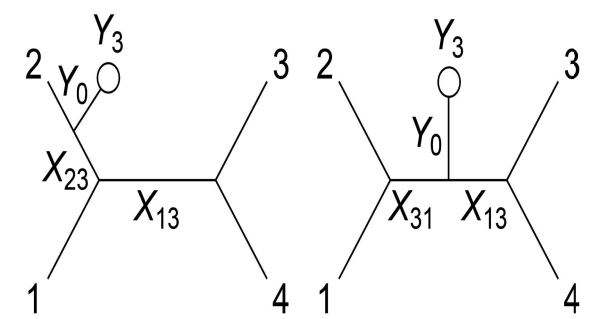
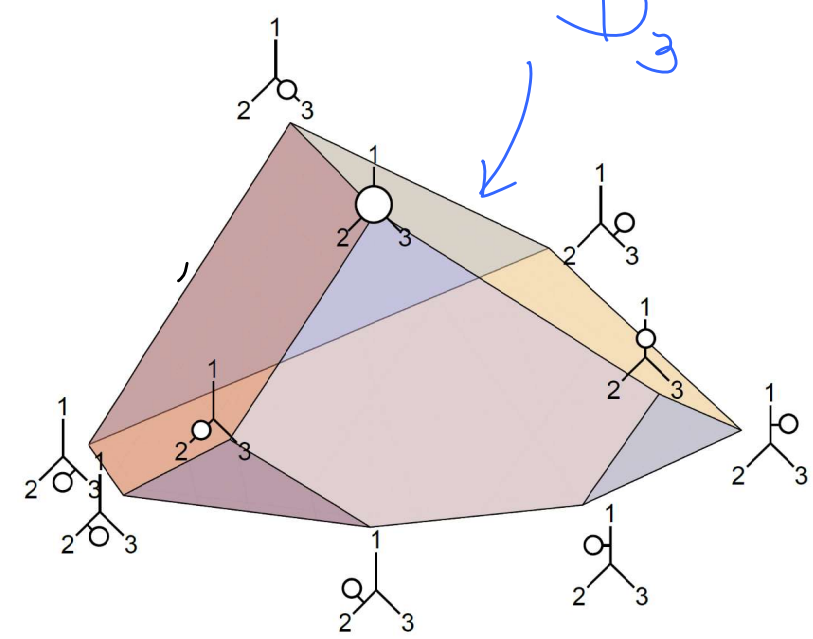
Kinematic "Spacetime" @ $l-\text{loop}$



D_3



D_3

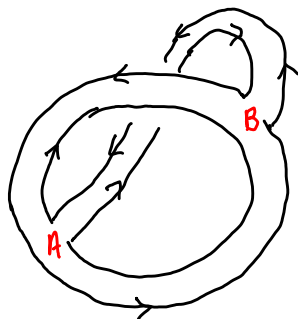


h -trace Feynman Diagrams, genus g , L -loops

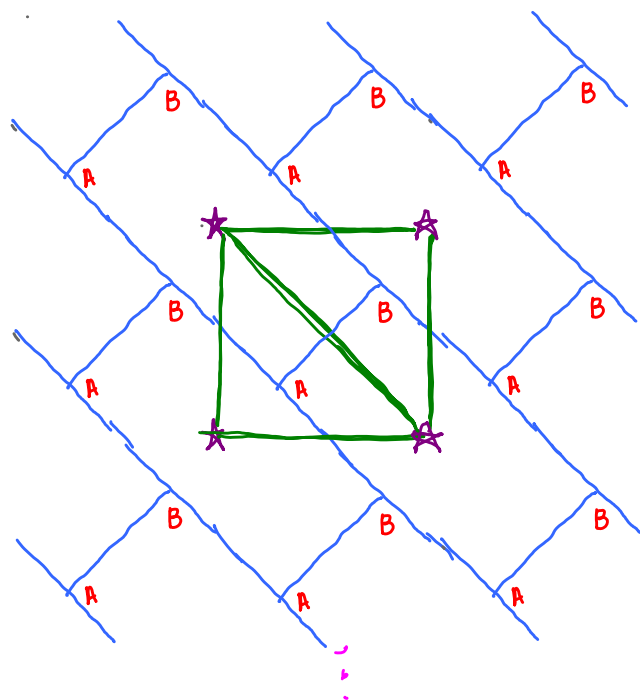
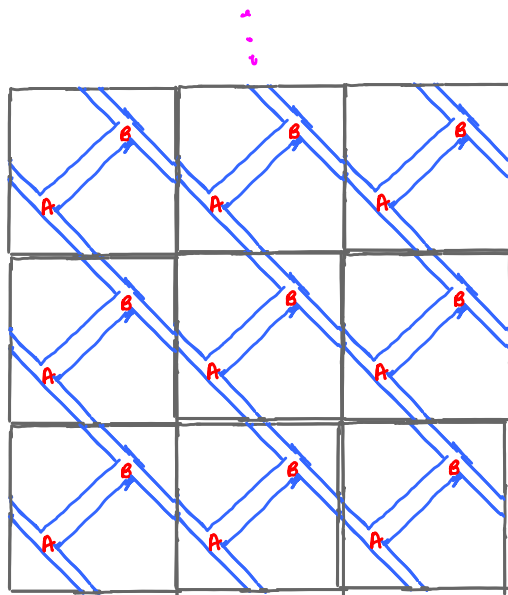
Dual Diagram



Triangulation of surface with
 h boundaries, genus g , and $p = L + 1 - 2g - h$
punctures

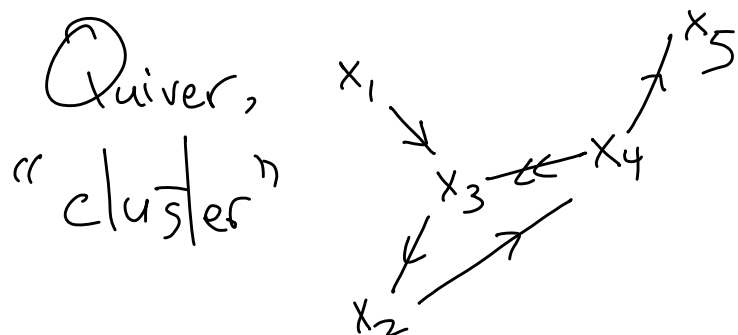


$$L=2, h=0, \\ g=1$$



$p=1,$
Triangulation of
once-punctured
torus

Some Cluster Algebra Basics



• "Mutation" @ v

1) Reverse all arrow

2) If $a \rightarrow v \rightarrow b$ add $a \rightarrow b$.

3) Delete 

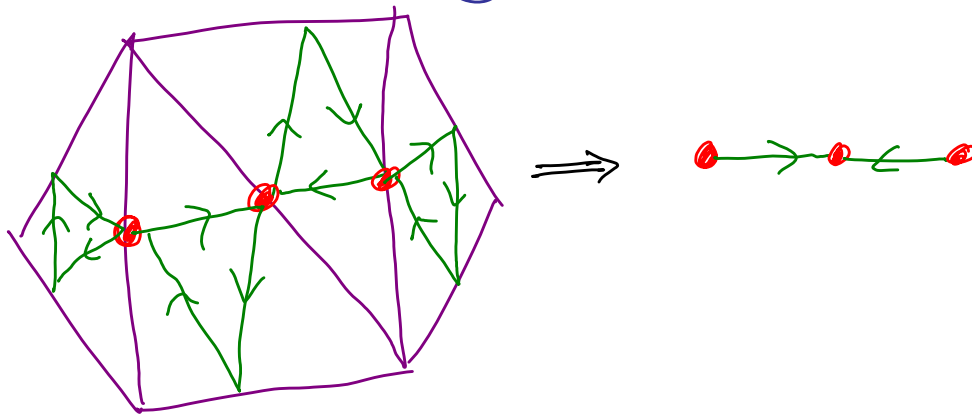
$$\bullet x_v x_{v'} = \prod_{w \rightarrow v} x_w + \prod_{w \leftarrow v} x_w$$

• Also "tropicalized", pair:

$$\left\{ \begin{aligned} &X_v + X_{v'} - \sum_{w \leftarrow v} X_w, \\ &X_v + X_{v'} - \sum_{w \rightarrow v} X_w \end{aligned} \right\}$$

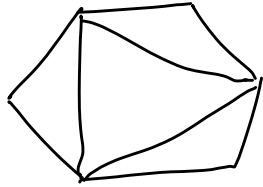
Surface Cluster Algebras

* Quivers from triangulations of surface

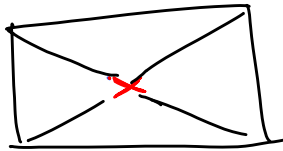


* Mutation $\longleftrightarrow$ Skein relations: $X = \left(\begin{array}{c} \text{crossing} \\ \text{cup} \end{array} \right) \left(\begin{array}{c} \text{cup} \\ \text{crossing} \end{array} \right) + \left(\begin{array}{c} \text{cup} \\ \text{cup} \end{array} \right)$

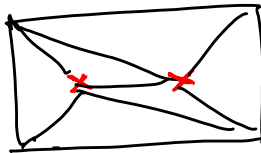
All-loop, all orders in $1/N$ Φ_{BA}^3



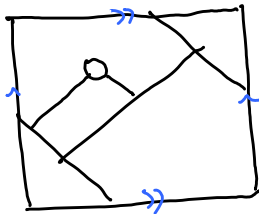
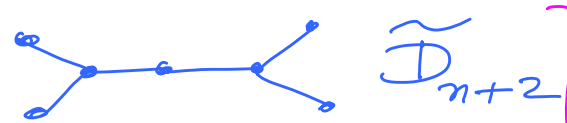
trees $\longrightarrow$



1-loop $\longrightarrow$

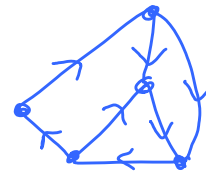


2-loop $\longrightarrow$



2-loop, leading $1/N$ [corrections]

$\longrightarrow$



$\left. \begin{matrix} \tilde{D}_{n+2} \\ \text{"Infinite"} \\ \text{Type"} \end{matrix} \right\}$

Seeking "Clusterhedra"

g-vectors

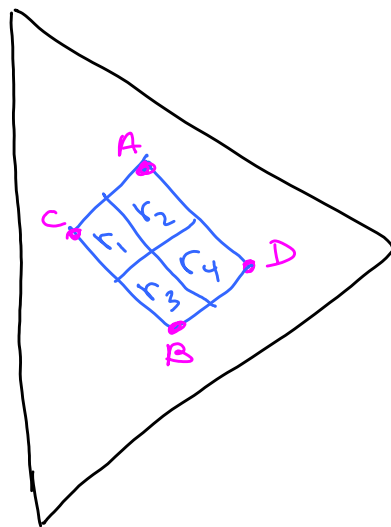
- Canonical way of labelling cluster variables, given choice of initial cluster $(x_1, \dots, x_n)$.
- For each mutation μ , there is one of $\left\{ X_u + X_{v'} - \sum_{w \rightarrow v} X_w, X_v + X_{v'} - \sum_{w \leftarrow v} X_w \right\}$ that is satisfied by (+ determines) g-vectors.
- We'll call this set of relations [again, dep. on choice of initial cluster], $\{R_\mu\}$.

Emergence of "Kinematic Spacetime" from Cluster Algebra.

Consider all " $X + X - X - \dots$ " relations $\{R_\mu\}$.

There is a tiny subset of them R_{tex} that generate all the rest by positive lin combination:

$R_\mu = \sum c_{\mu, \text{tex}} R_{\text{tex}}$. R_{μ^*} are "pixels"
or "meshes" in "Kin Spacetime".



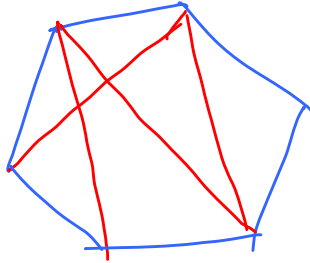
Mat. rel. $A+B-C-D$

"

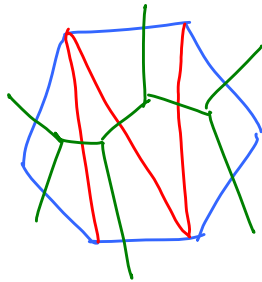
$$r_1 + r_2 + r_3 + r_4$$

Understood for all finite-type cluster algebras

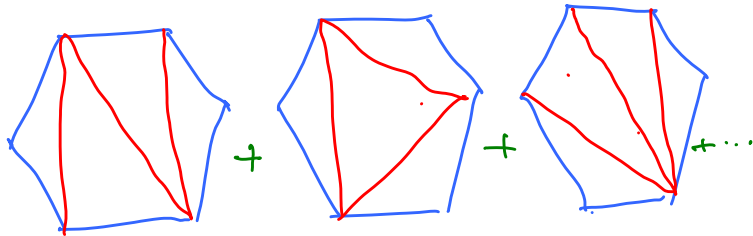
Spacetime + QM $\leftrightarrow$ Cluster q -vector Cones + Fan



Propagator X_{ij}

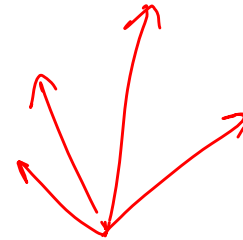
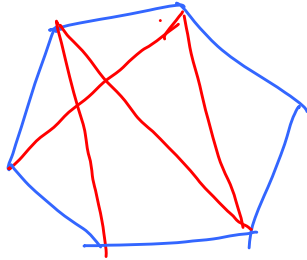


Spacetime process

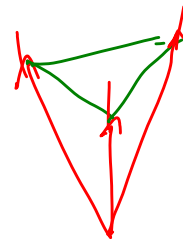
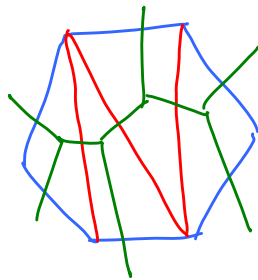


Add an amplitude
for all processes with
"+1", because of QM

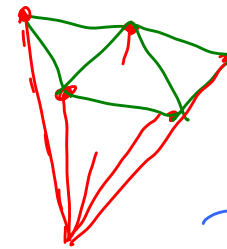
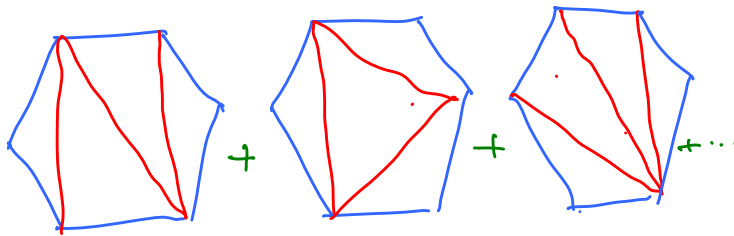
Spacetime + QM $\leftrightarrow$ Cluster g -vector Cones + Fan



g -vectors.

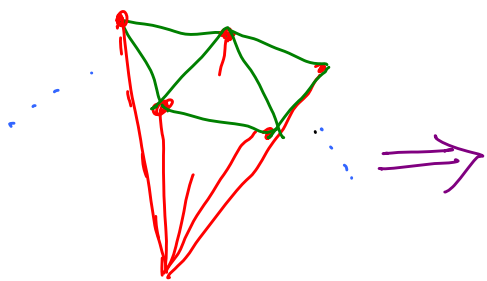


g -vector cone



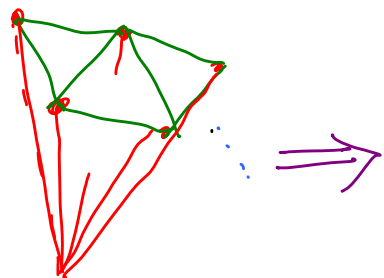
g -vector cones
tile a fan

Amazing +
Deep Fact!



$$\Omega = \sum \frac{\text{sgn} |g_{a_i} g_{a_n}|}{d \log X_{a_1} \dots d \log X_{a_n}}$$

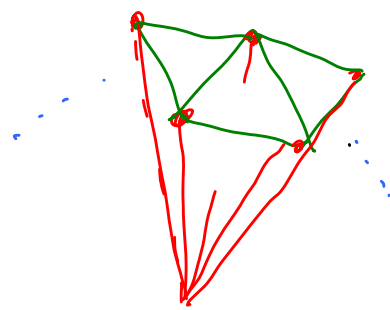
is projective invariant



$$\Omega \Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i \right)$$

$$\times \sum_{\text{diag}} (+1) \pi \frac{1}{X'_s}$$

Amplitude

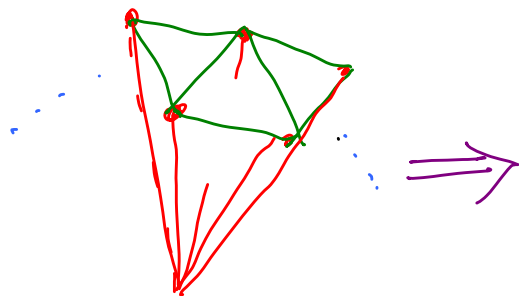


$$\Rightarrow \Omega \Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i \right)$$

$$\times \sum_{\text{diag}} +1 \cdot \pi \frac{1}{X'_s}$$

Because

$$|g_1 \dots g_m| = \pm 1!$$



$$\Omega \Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i \right)$$

[g-vectors are normal
to facets of polytope]

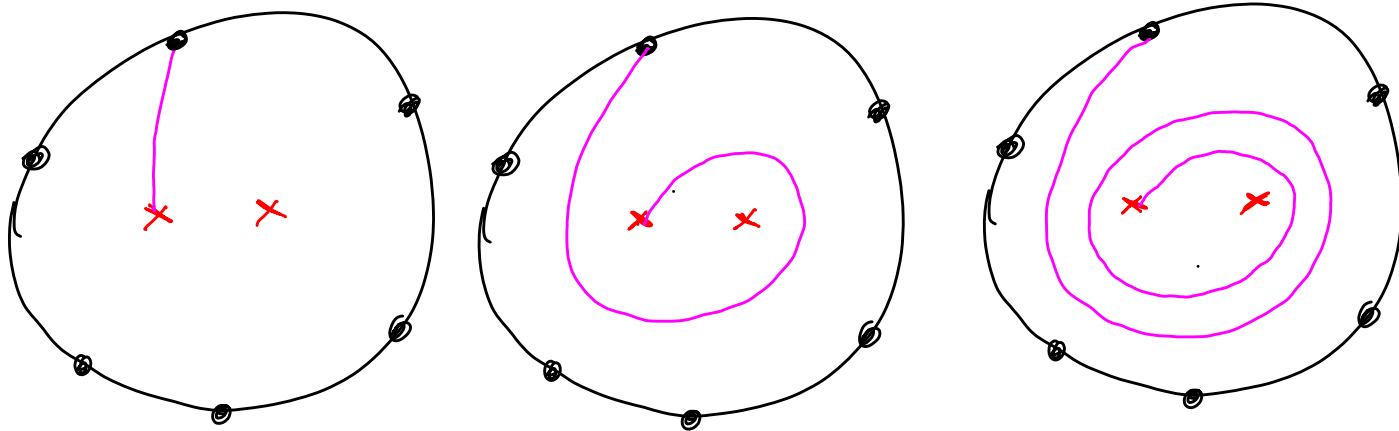
$$\times \sum_{\text{diag}} + 1 \cdot \prod \frac{1}{X'_s}$$

ABHY construction automatically orders k_a such that
Polytope given by $X_A \geq 0$ has correct shape!

* Much of this carries over to
infinite-type cluster algebras —

except that g -vector fan is infinite
but doesn't cover all of space

Infinite Winding



- * Infinite action of Mapping class groups, Braid_L
- * Also, g-vector cones still fit snugly together, but leave lower-codim. holes in the space
- [$\Rightarrow$ Projective invariance works as "telescopic cancellation" but we have to "compactify" the infinite space, somehow.]

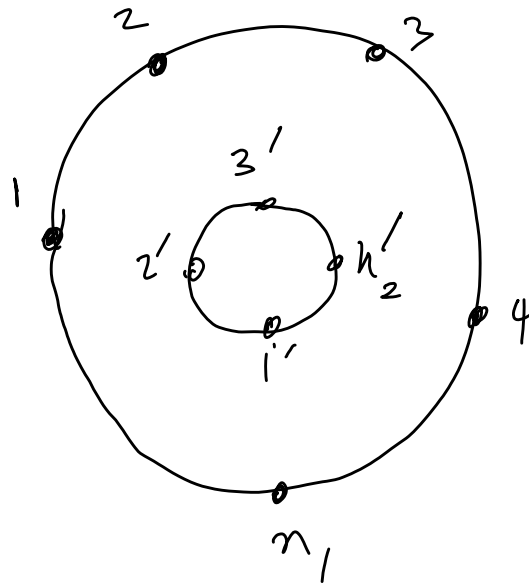
"Clusterhedra"

Are the polytopes where:

- * Vertices unambiguously associated w/ diagrams/clusters.
- * g -vectors are (among) facet normals.
- * Projective [trivial]
- * Pull back to amplitude with all $+$'s
(= all facet normal dets. $= \pm 1$, non-trivial!)

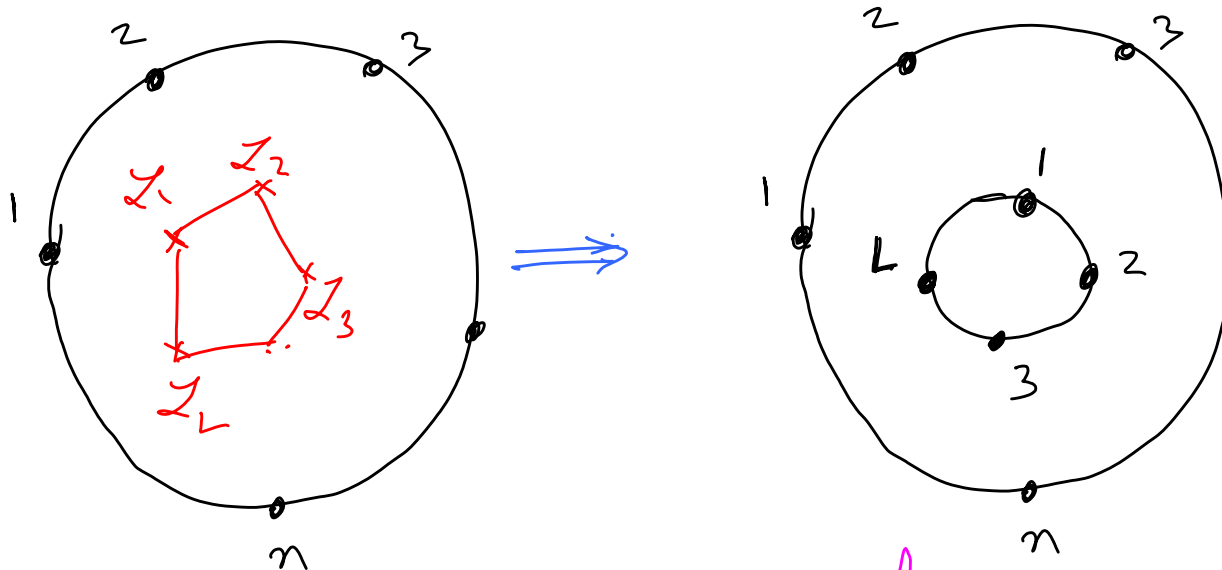
"Kinematical Spacetime" richer @ multiloop level +
we will find Clusterhedra naturally living there

Double-trace, 1-loop



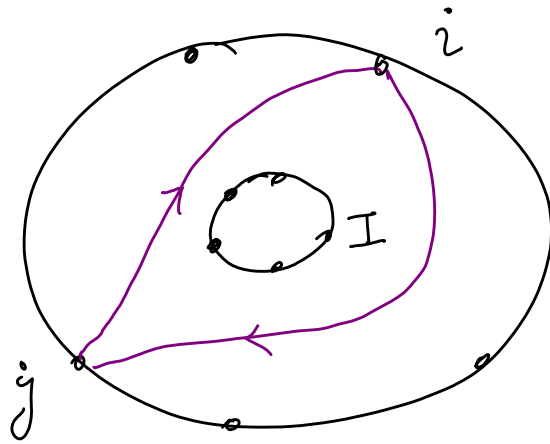
A_{n_1, n_2}

Class of All-loop Cuts

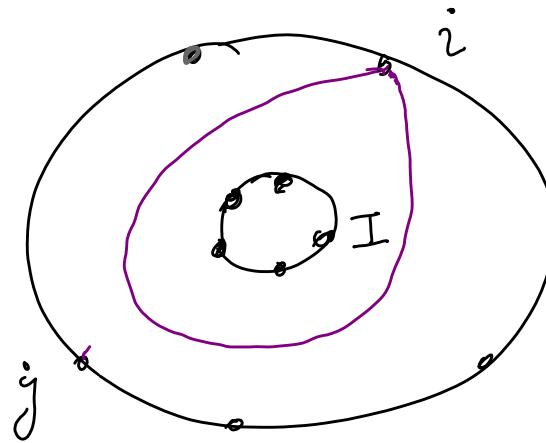


$A_{n,L}$

Variables for $A_{n,L}$



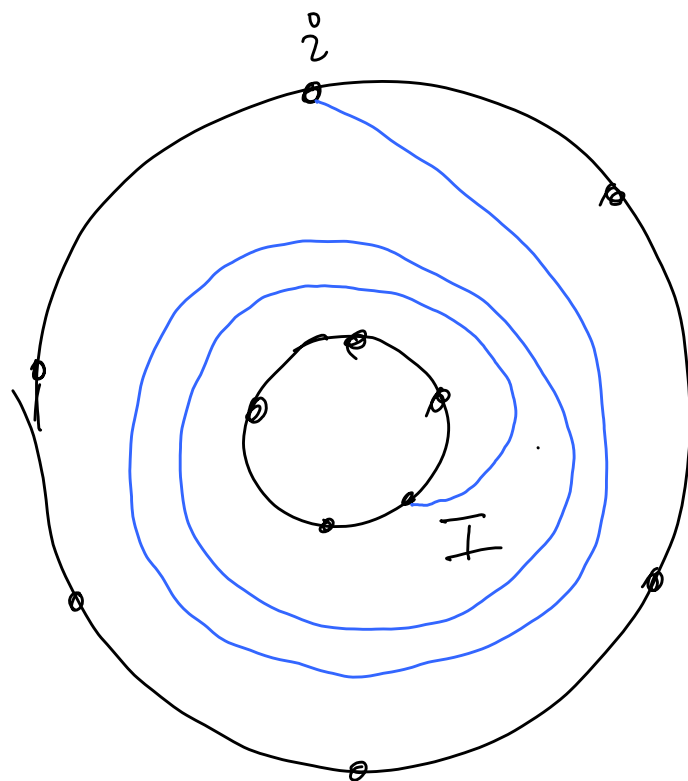
$$X_{ij} \neq X_{ji}$$



$$X_{i+n}$$

"Out/Out" or " $\mathbb{I}_n/\mathbb{I}_n$ "

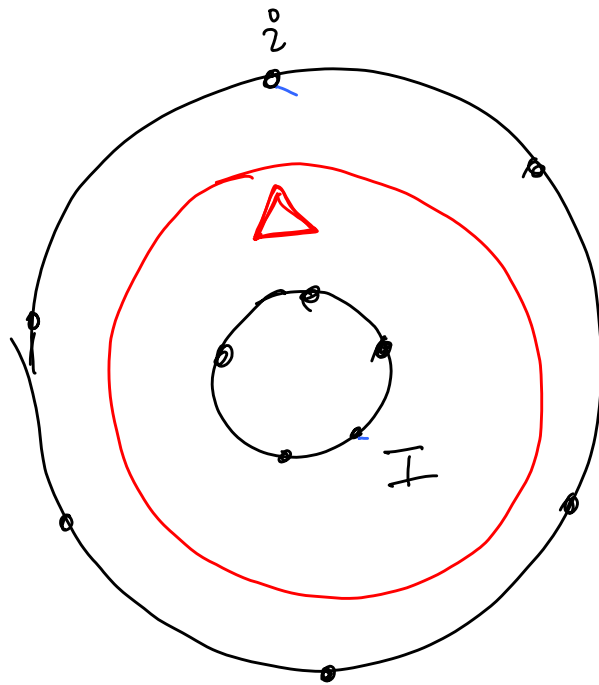
Variables for $A_{n,L}$



$$X^{\omega} \quad iI$$

" $I_n - Out$ "

Variables for $A_{n,L}$

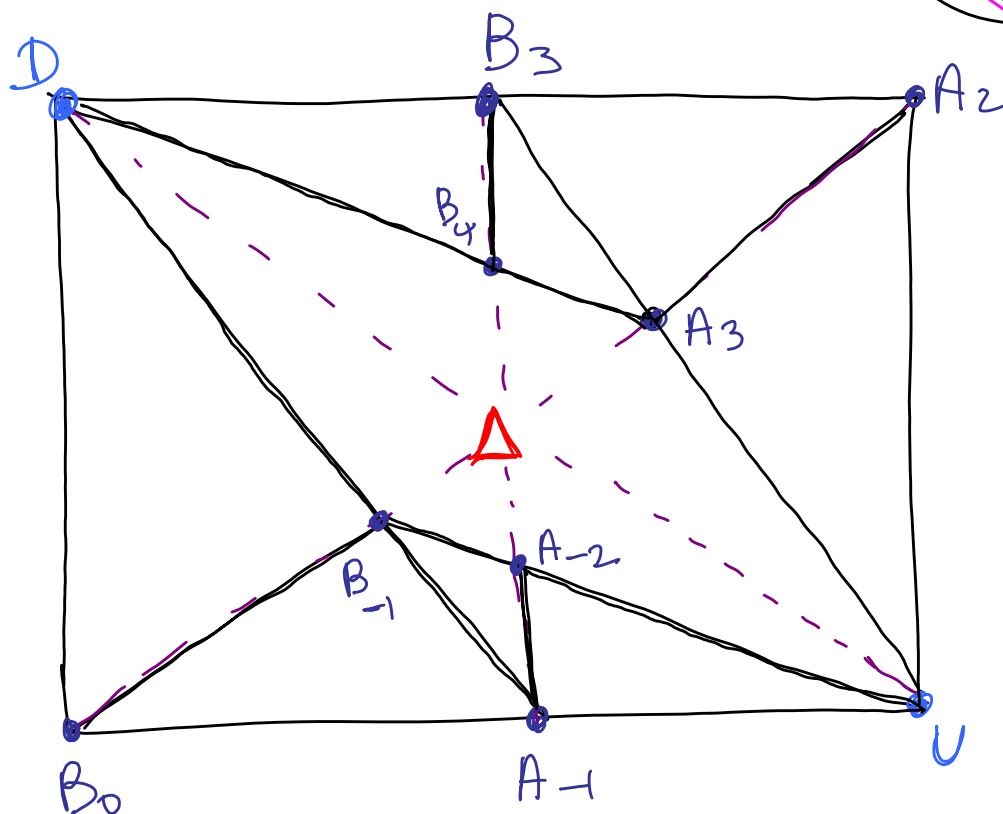
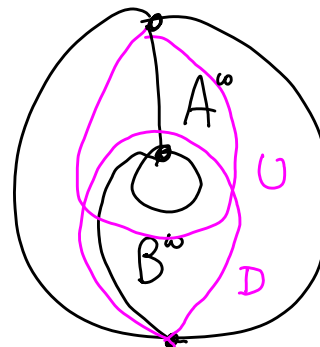


$$\left\{ \begin{array}{l} X^{+w} + \Delta = X^{+w+1} \\ X^{-w} + \Delta = X^{-w-1} \end{array} \right.$$

Δ : Not Cluster Variable,
but crucial part of story.

The g-vectors asymptote to Δ

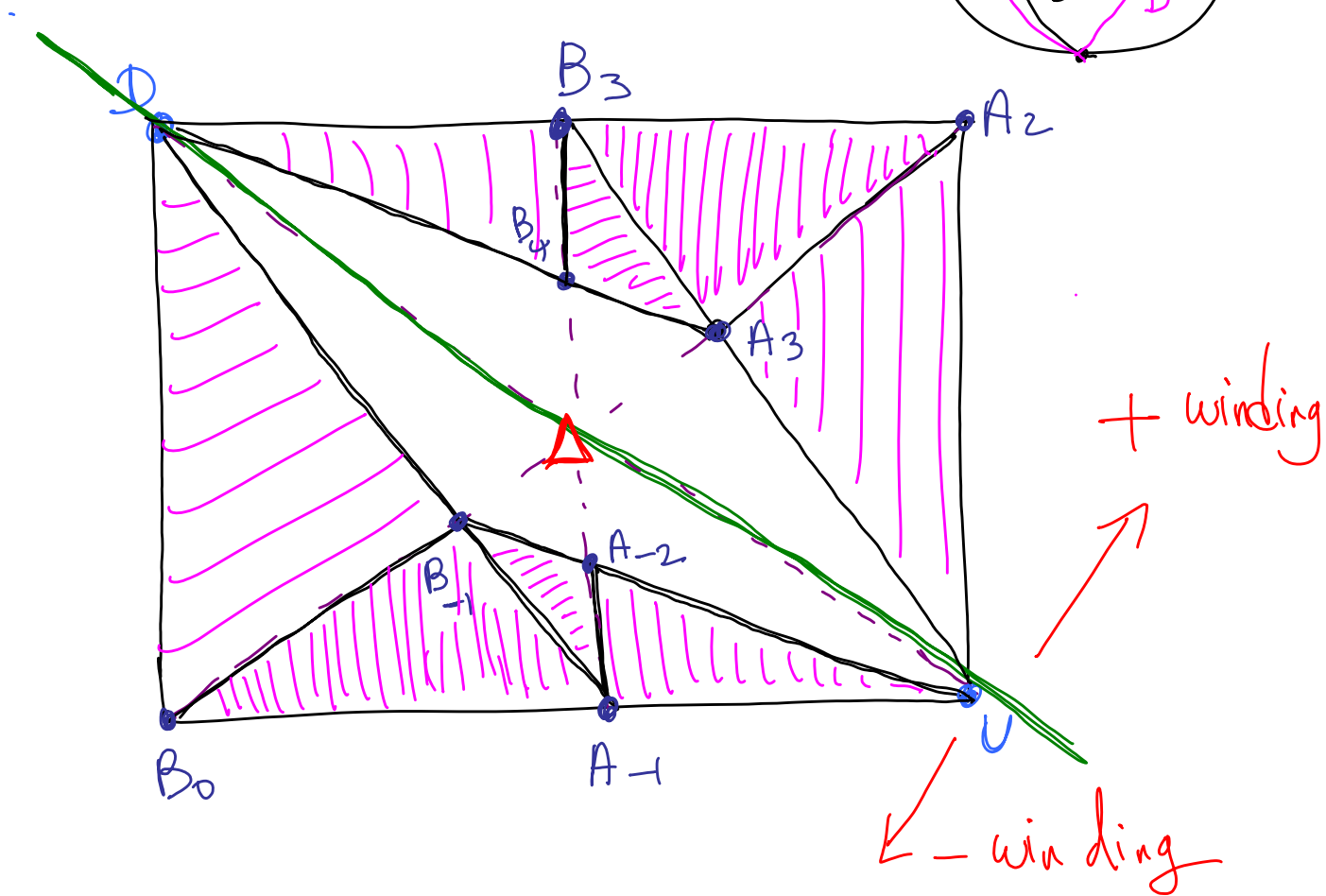
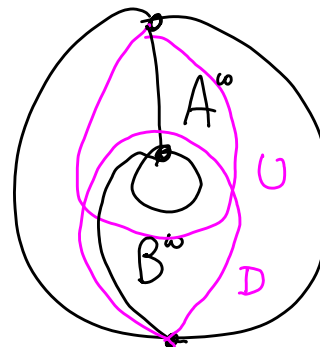
For example in $A_{2,1}$:



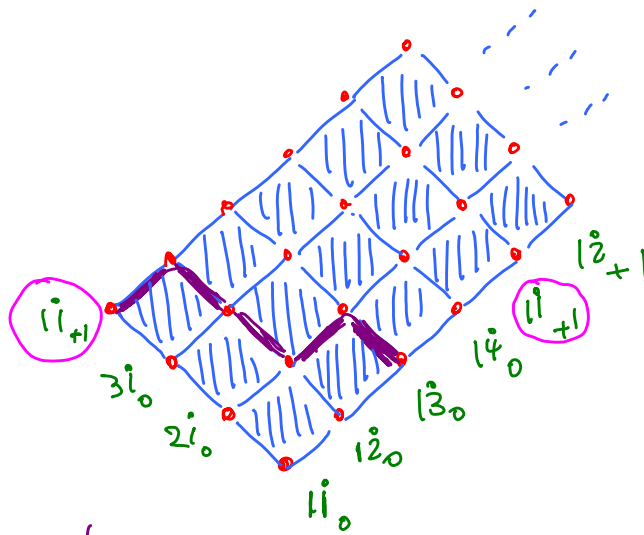
(2-dim
projection
of 3-d
g-vectors)

The g-vectors asymptote to Δ

For example in $A_{2,1}$:

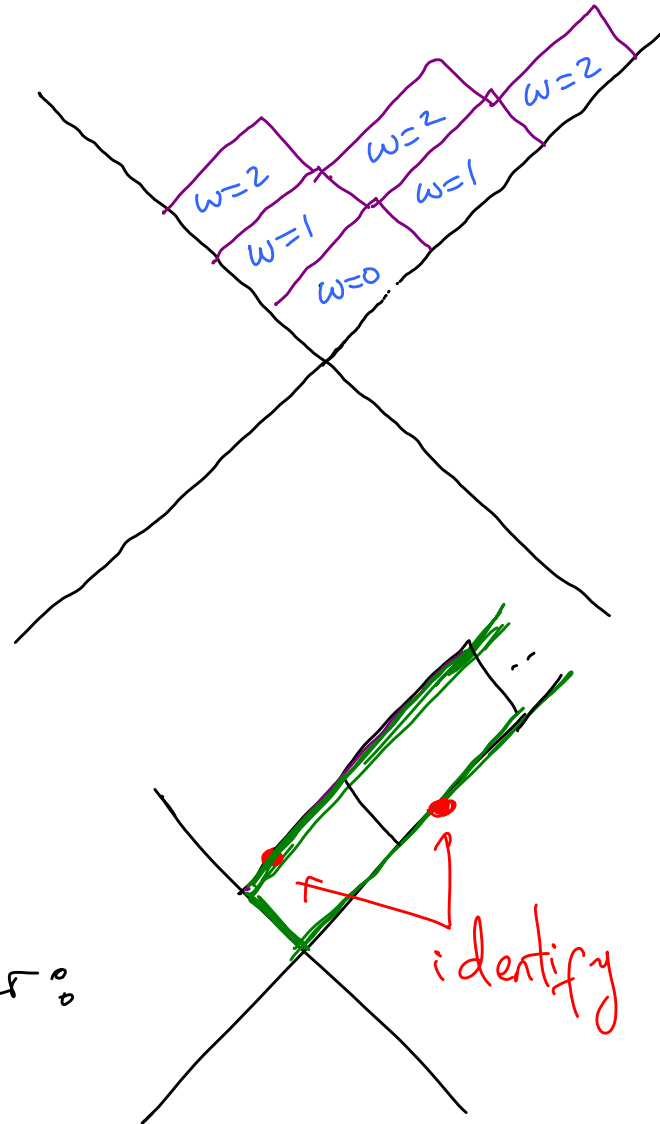


In-Out "Kinematical Spacetime"

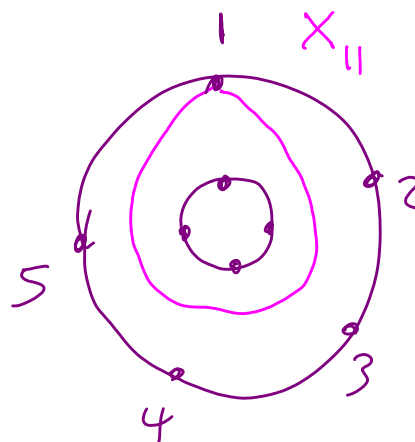
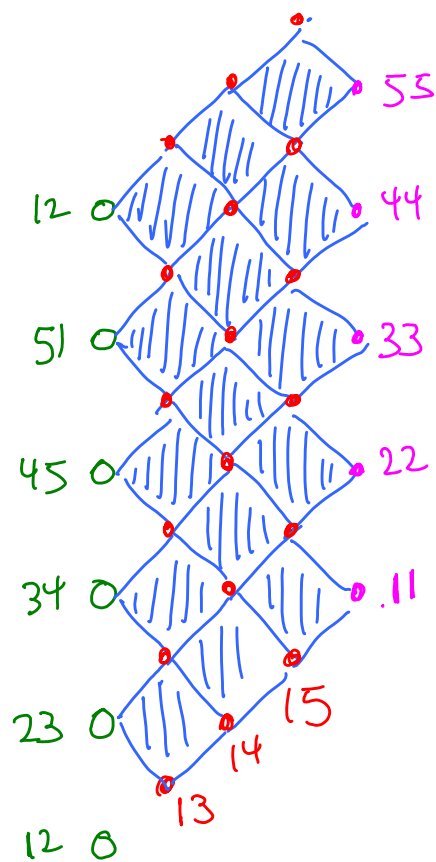


Initial "Cauchy Surface"
set by initial cluster

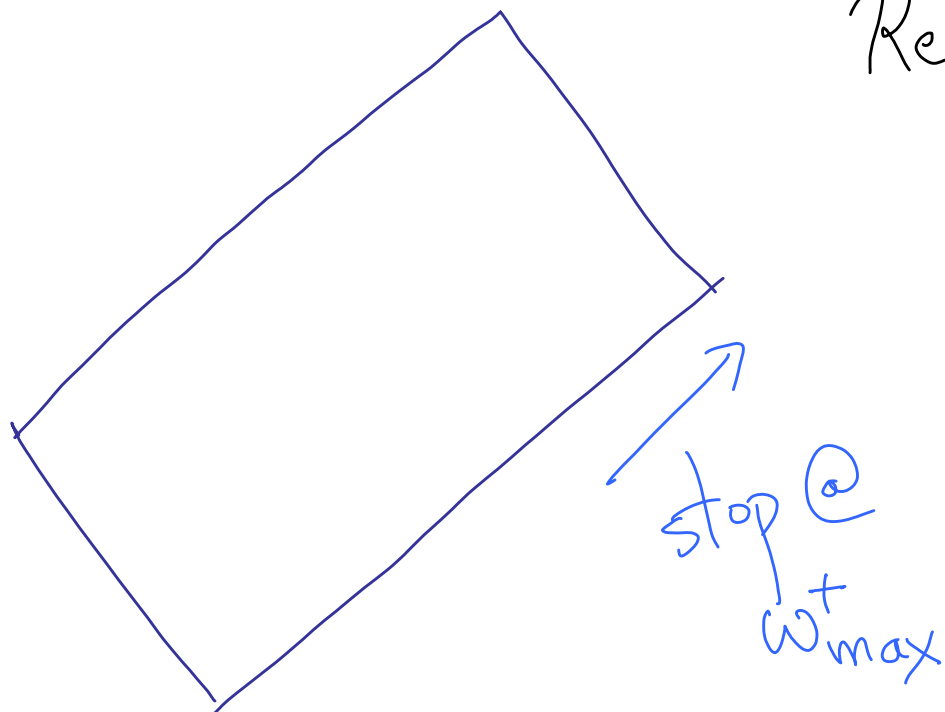
* So, we have an infinite cylinder:



In-In / Out-Out "Kinematic Space"



"Compactifying" winding with Δ



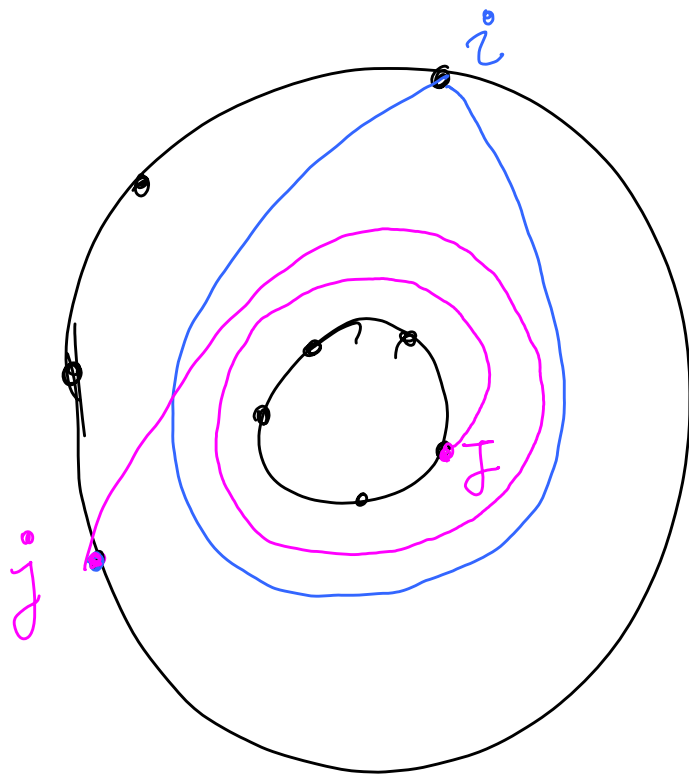
Relation $X_w + X_{w+1} - X_{\dots}$
 $\underbrace{\quad}_{\text{missing!}}$

$\Downarrow$ Replace with.

$$X_w + \Delta - X_{w+1}$$

Gluing "In-In/Out-Out" to "In-Out"

The "mesh" relations are $\left\{ \begin{array}{l} X_{ii} \\ X_{II} \end{array} + X_{jI}^{\omega_{\max}^+, \omega_{\min}^-} - \dots \right\}$



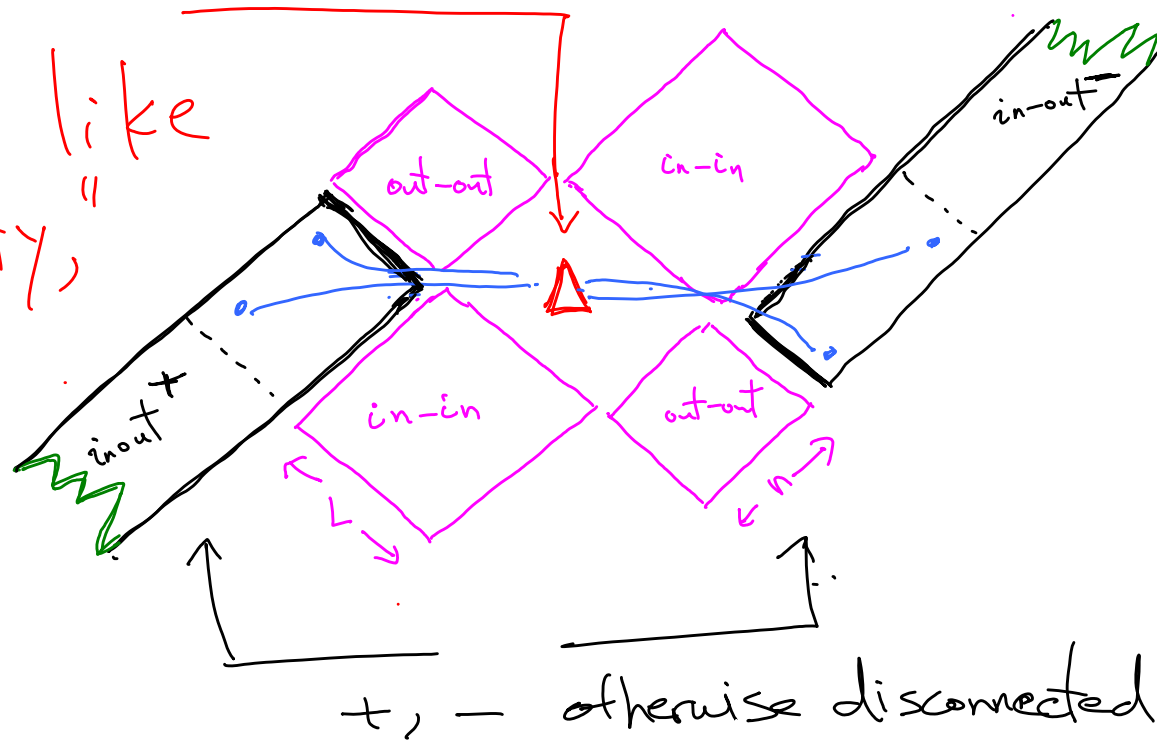
Gluing $+$ to $-$ winding

$$\begin{array}{c} \times \\ \times \\ \times \end{array} \begin{array}{c} + \omega_{\max} \\ iI \end{array} + \begin{array}{c} \times \\ \times \\ \times \end{array} \begin{array}{c} \rightarrow \omega_{\max} \\ iI \end{array} - \dots - \# \triangle$$

* Existence of these relations
is non-trivial $\rightarrow$ surface cluster
structure telling us to use " Δ " for
"compactification" of cluster complex

The Final "Kinematical Spacetime" for A_n, L

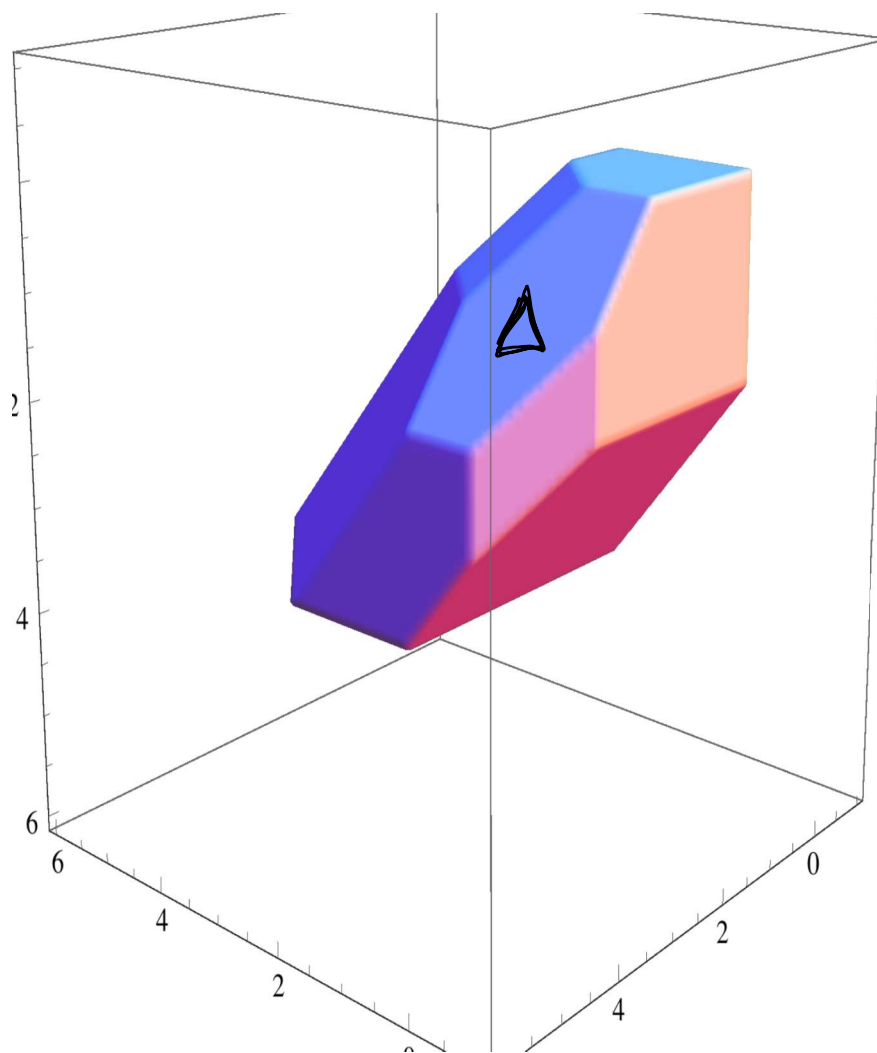
Δ is like
"gravity,"
coupling
all
sectors



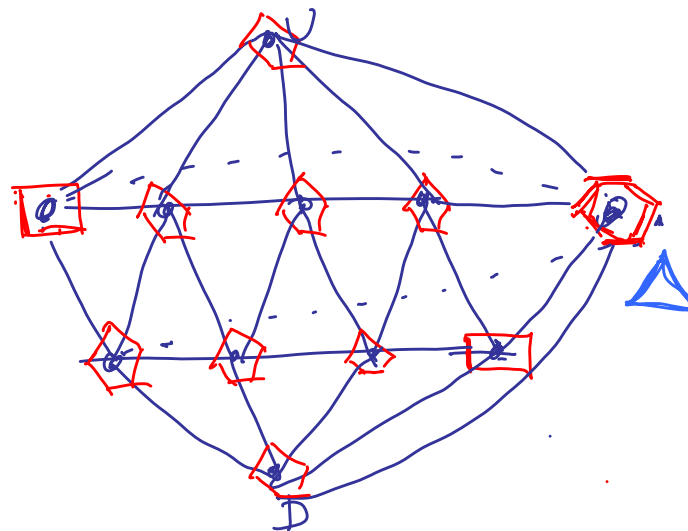
The relations $\{R_\mu\}$ of this "kin. Spacetime"
positively generate all the rest!

Affine Clusterhedra.

- * Exactly the same ideas work for $\tilde{D}_{n+2}$
= 2-loop. So, we have clusterhedra through
2-loops and all L cut of $A_{n,L}$.
- * Δ is "mutation partner" for every missing var;
all vertices $\implies$ all vertices unamb. assoc. w/ diagonals.
- * All non- Δ vertices counted same # of
times
- * All normal facet det's involving Δ are
 ± 1 , get all $+1$'s for amplitude
upon pullback



$A_{2,1}$ Clusterhedron



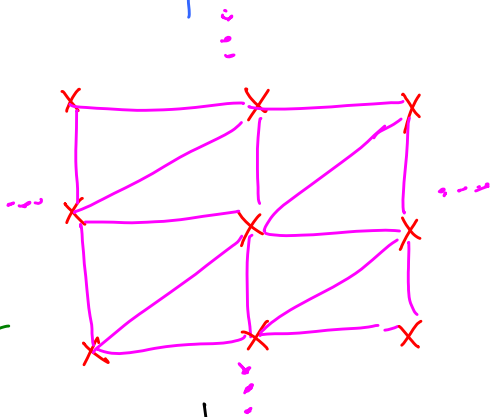
So we are done through 2-loops +
some all-loop "cuts". On our way
to all loops / all orders in $1/N$.

Understand decoupled "Topological Sectors",
closed loop " Δ "s should "gravitationally"
couple all sectors.

Novelty beyond Affine-Type: Kinematic Space is "Cosmological"

e.g. once punctured torus

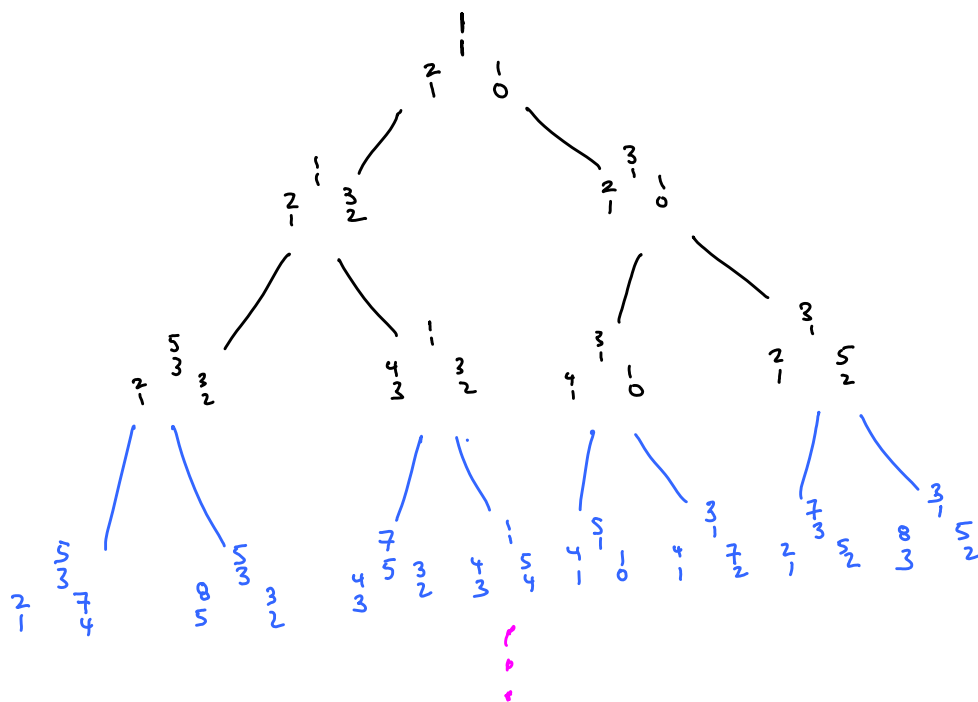
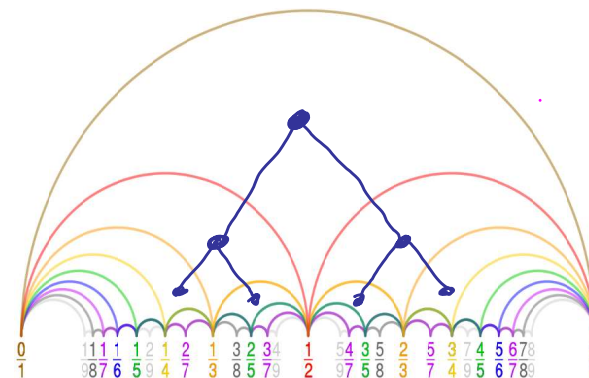
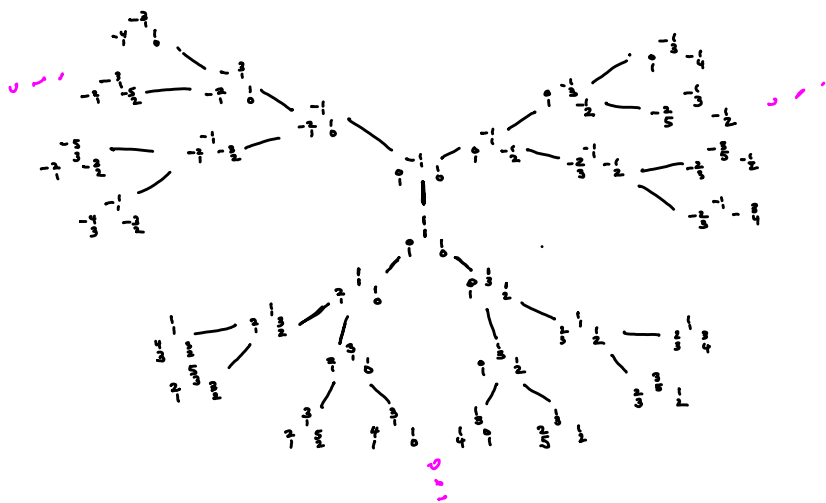
+ $SL(2, \mathbb{Z})$
for gen
triang-



Markov
Quiver

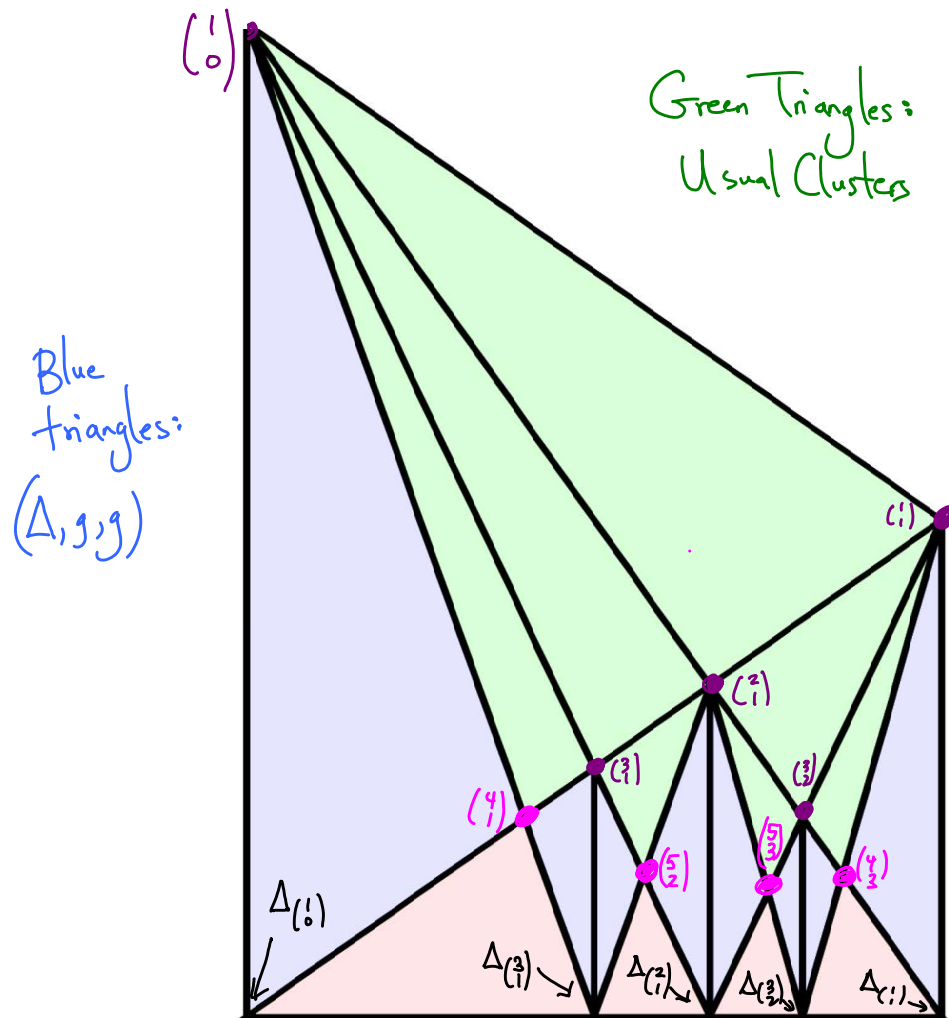


- Cluster variables:
lines of slope (p/q)
connecting punctures
- Δ 's : lines of slope
 (p/q) missing punctures



"dS"
in Kin. Space

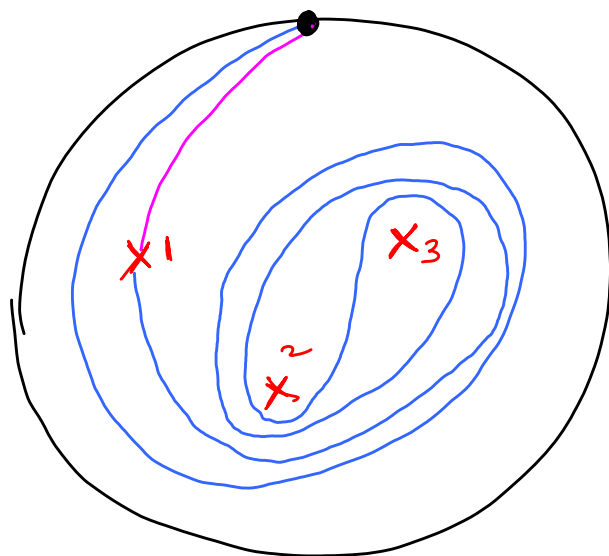
Markov Clusterhedron (Fan)



Red triangles:
 (Δ, Δ, g)

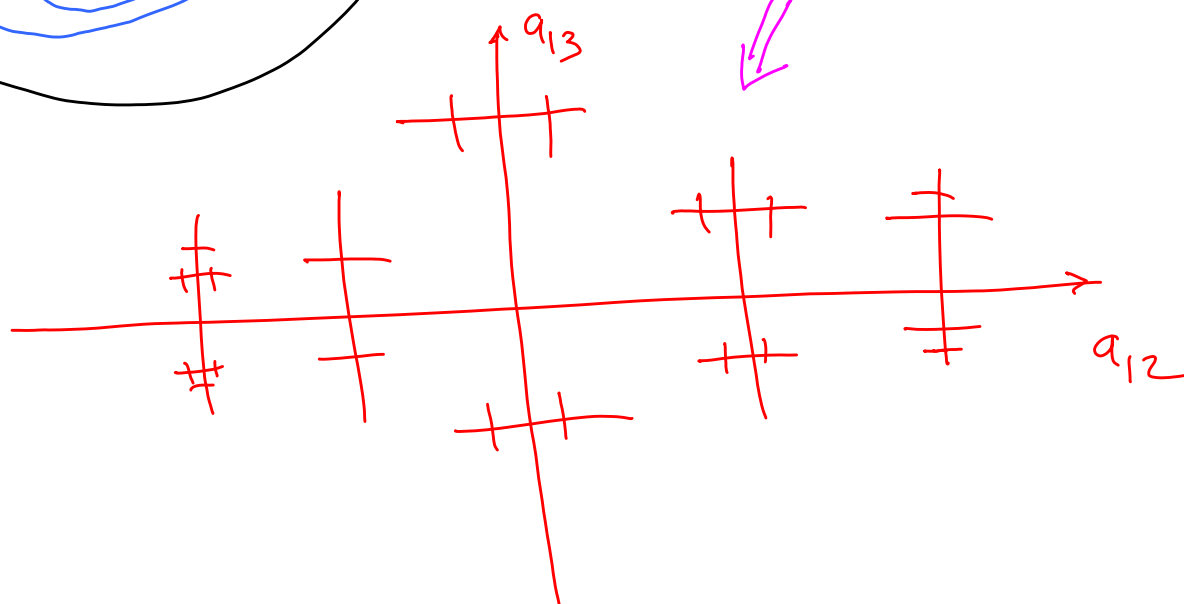
Again: Δ'
"compactify",
giving
clusterhedron

$L \geq 3$ punctures $\rightarrow$ "Eternal Inflation"



Arcs labelled by
Braid₃/Braid₂

Free₂ (a_{12}, a_{13})



Outlook

- * Over the past few years, we have been seeing concrete examples where rules of spacetime + quantum mechanics — locality + unitarity — are seen to emerge from simple new structures in kinematical space @ infinity
- * All-loop planar $\mathcal{N}=4$ SYM $\rightarrow$ Amplituhedra
- * Non-SUSY ϕ_{BA}^3 , heading towards all-loop, all orders in $1/N \rightarrow$ Clusterhedra
- * These ideas are directly related to real-world physics in reasonable approximations, expose hidden symmetries, and describe universal features of wide class of theories
- * How are they related? Relation to String theory?
 $\Delta \leftrightarrow$ gravity more than analogy? More general theories?

