Space-Time, Quantum Mechanics, and Cluster Hedra at Infinity

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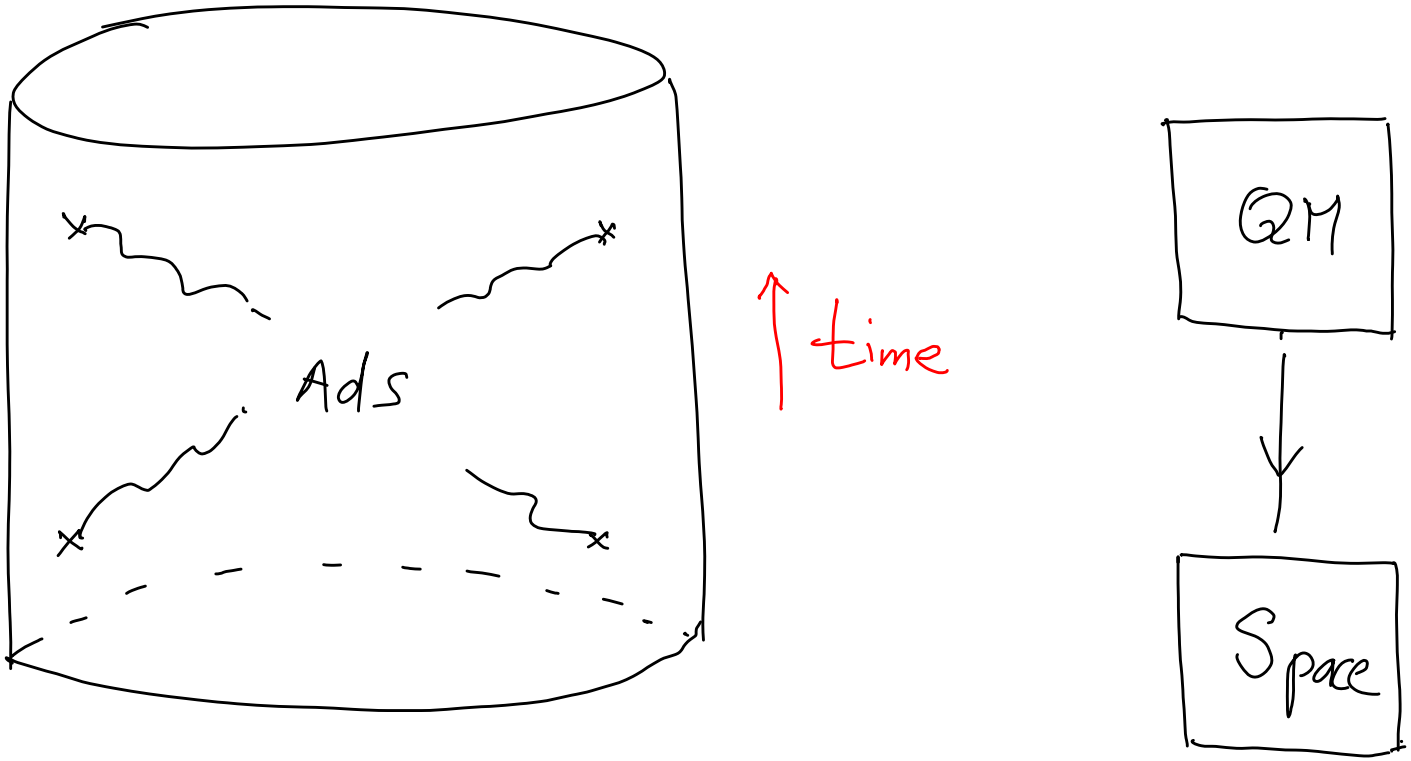
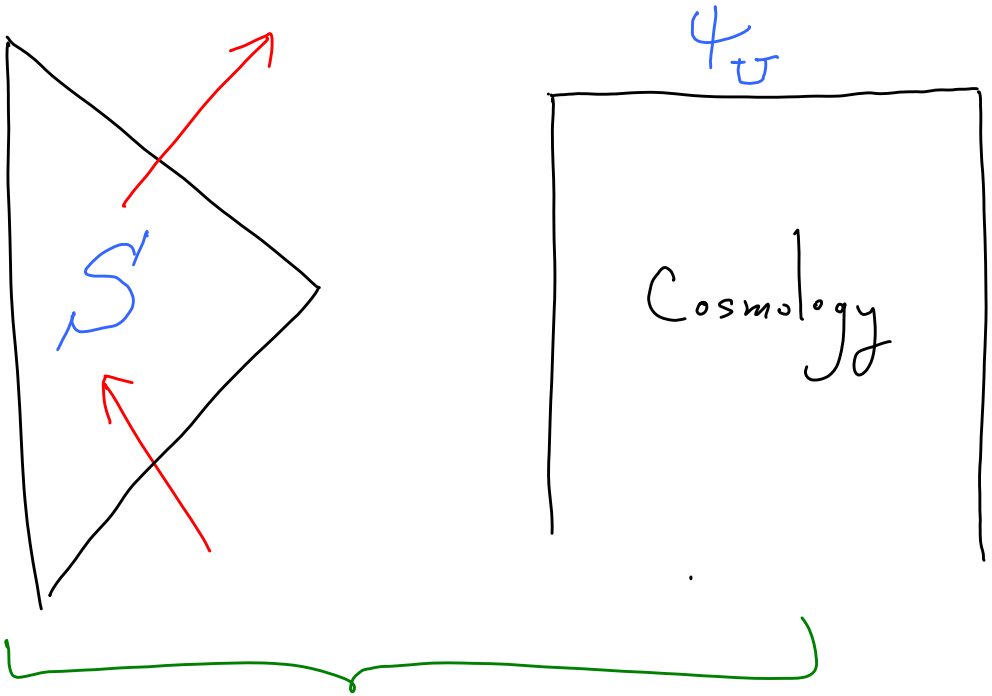
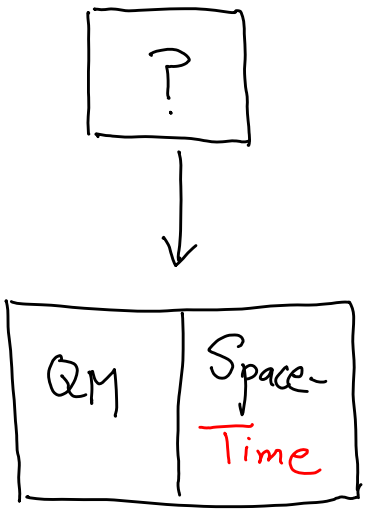
S. Goncharov, A. Postnikov, F. Cachazo, A. Hodges

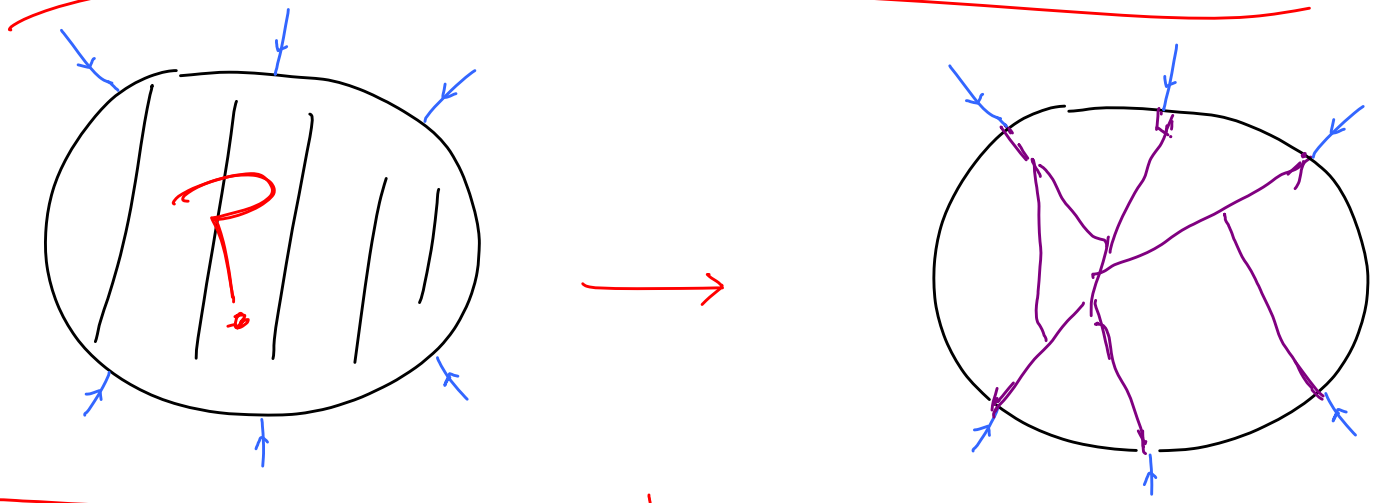

Ads
time
QM
Space

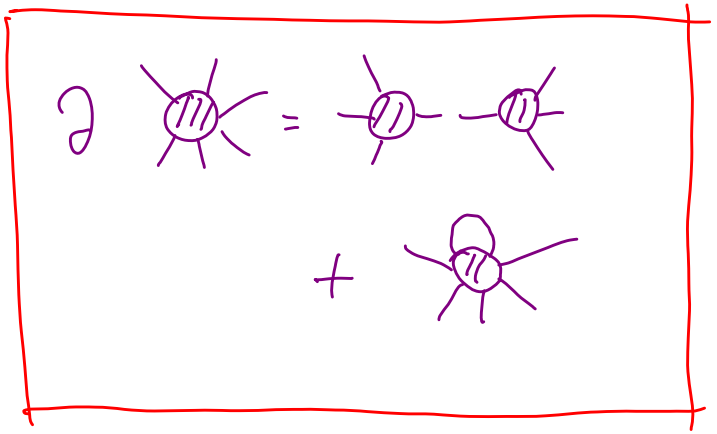




Unlike ∂AdS, no
obvious notion of time + locality

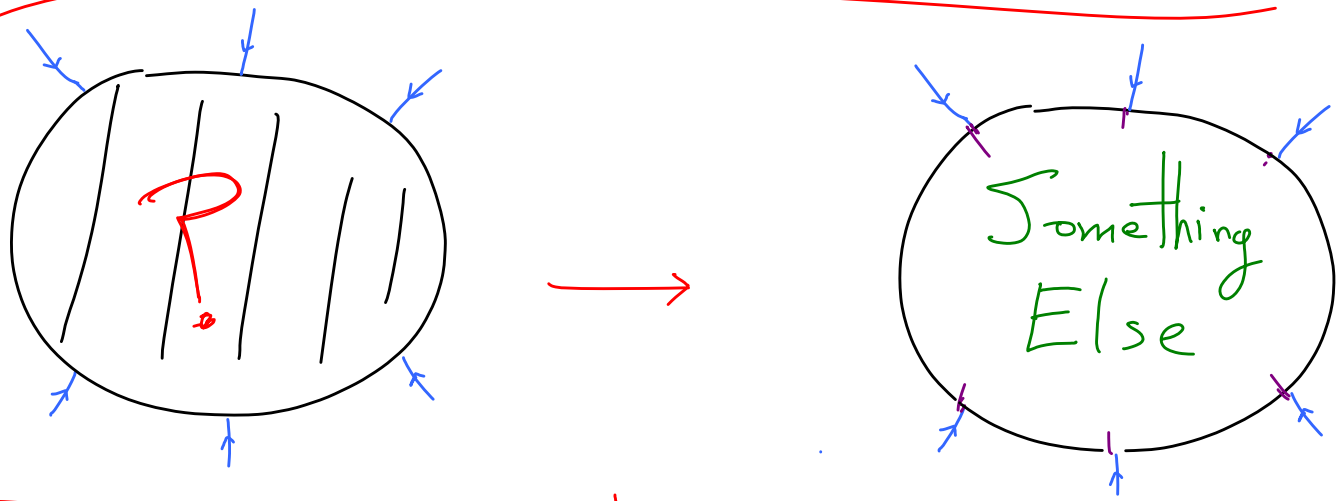
What is the Q to which A is the Answer?

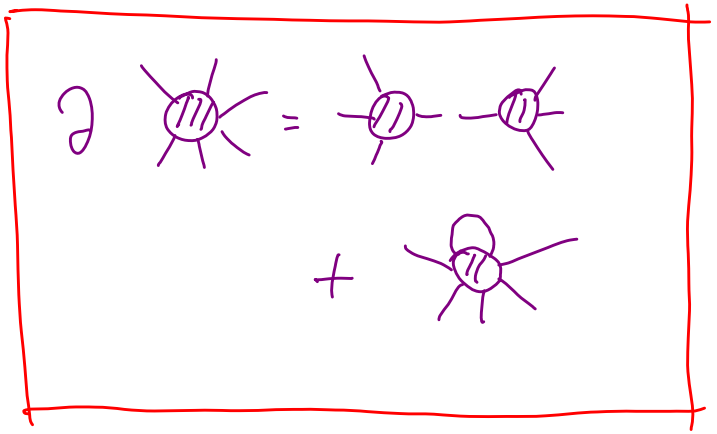




Local, Unitary evol. in Spacetime / Hilbert Space.

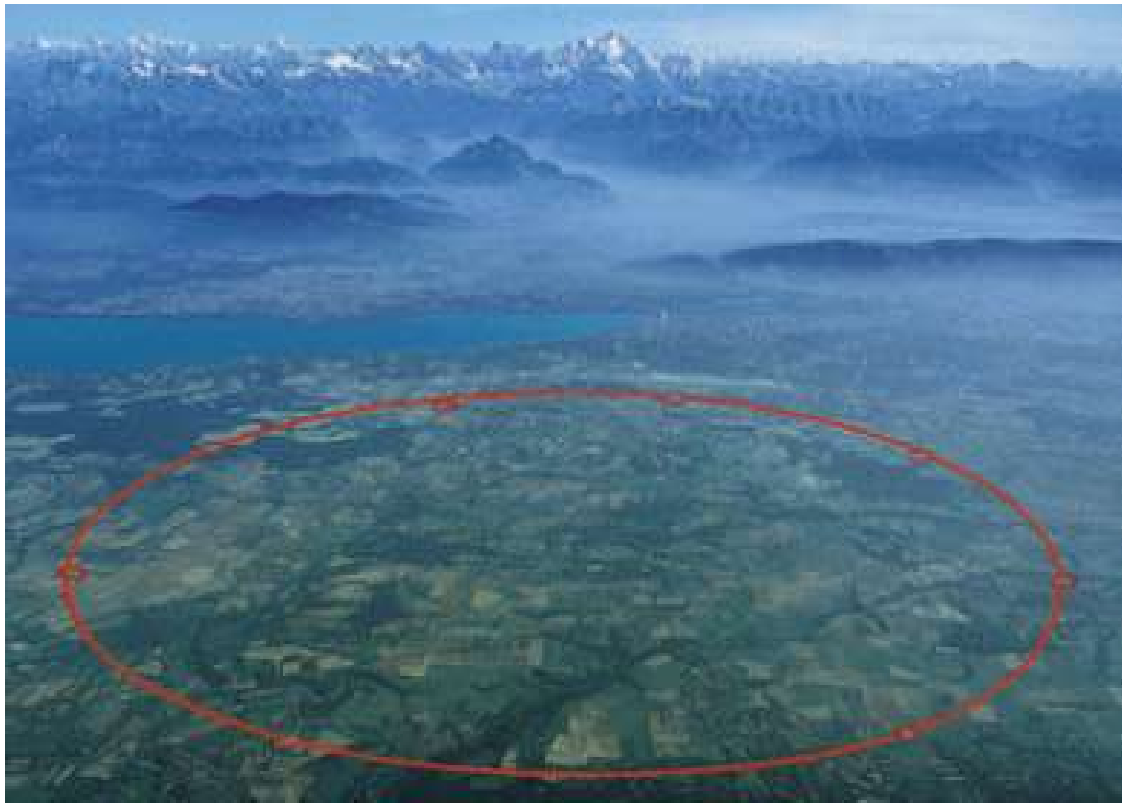
What is the Q to which A is the Answer?



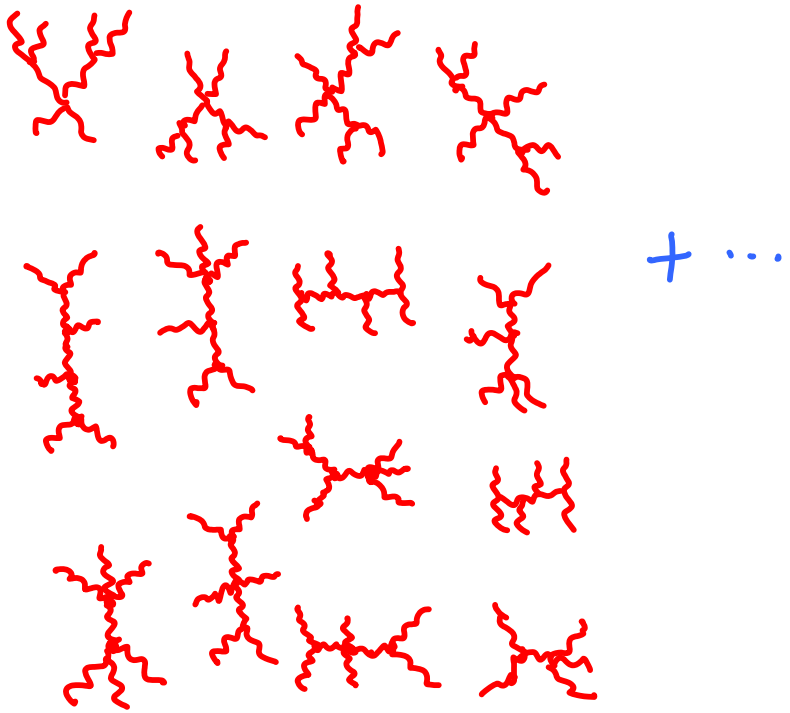


Rules of Spacetime + QM-
Locality + Unitarity-
as derivative outputs

L . H . C .



Feynman Follies



+ ...

220 Diagrams

10's of thousands of terms ...

Result of a brute force calculation:

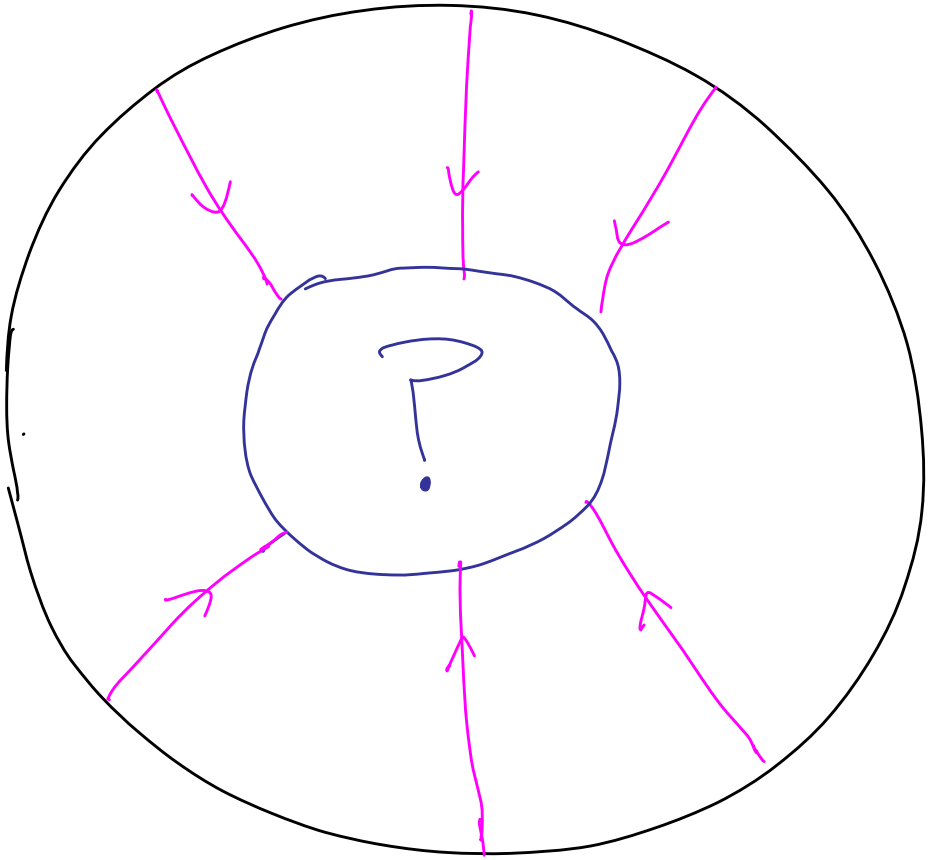
+ ~ 100 more pages ....

$k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5$

$$(1^-2^+3^-4^+5^+6^+)$$

$$\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} \quad (!)$$

Standard way of doing physics makes usual rules of spacetime + QM manifest — but is obviously hiding some extraordinary new structures!



For ~ 10 years
(+ more intensely
in last 2 years)
we are seeing

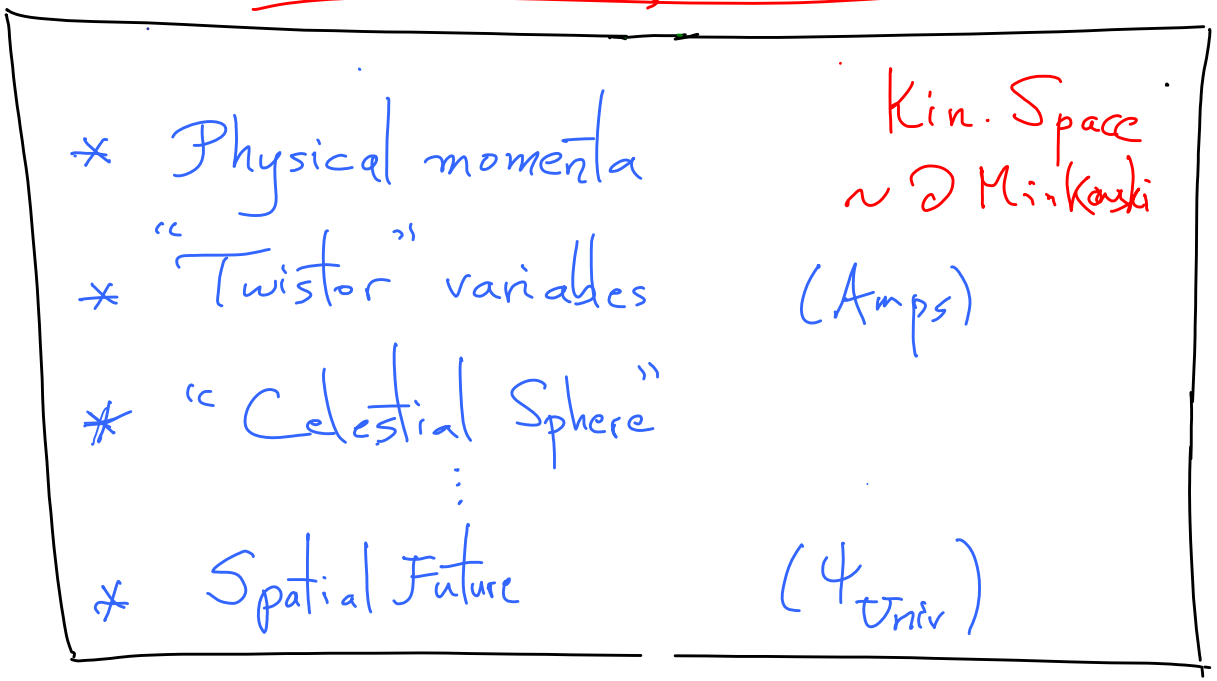
Positive
Geometries in
Kinematic Space

The Canvas

* Physical momenta — Kin. Space ~ ∂ Minkowski
* "Twistor" variables (Amps)
* "Celestial Sphere"
⋮
* Spatial Future (Ψ_{Univ})

What ideas breathe physics-life into this space?

The Canvas



Emerging Picture:

Combinatorics

↕

Positive Geometries

↕

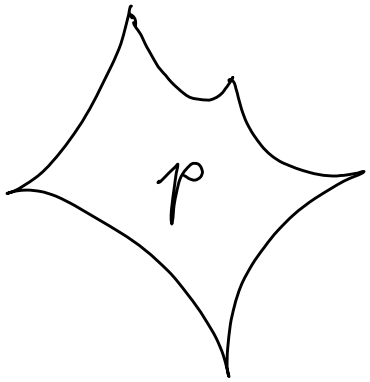
Canonical Forms

⋮

* Amplituhedra, Associahedra, Cosmological Polytopes, ...
* Describes REAL WORLD PHYSICS in reasonable approximation, exposes hidden symmetries + new mathematical structures
* Pure magic in physics right under our nose

Positive Geometries

Canonical Forms

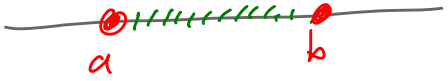


Region $\mathcal{P}$ w/ boundaries of all Codimension

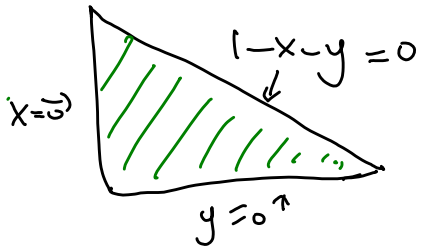
Real Geometry

$\Omega_{\mathcal{P}}$: unique form with logarithmic singularities on (+ only on) boundaries of $\mathcal{P}$ $\{\text{locally } \Omega \to \frac{dx_1}{x_1} \cdots \frac{dx_D}{x_D}\}$

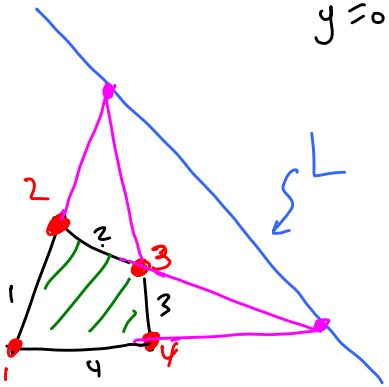
Complex Form

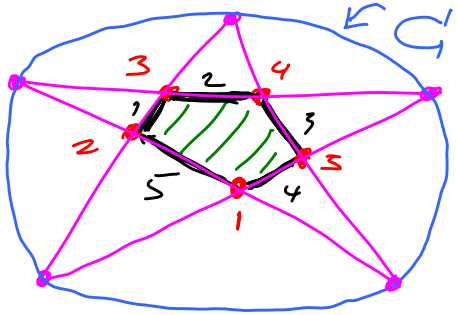


$$\Omega = \frac{dx}{x-a} - \frac{dx}{x-b}$$



$$\Omega = \frac{dx\,dy}{xy(1-x-y)}$$



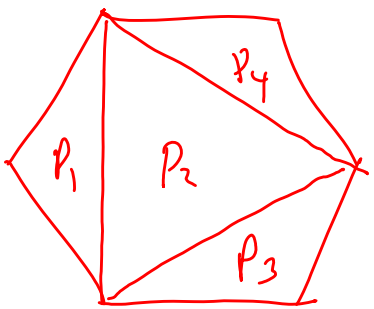


$$\Omega = \frac{dx\,dy}{L_1 L_2 L_3 L_4} \quad L$$

$$\Omega = \frac{dx\,dy}{l_1 l_2 l_3 l_4 l_5} \quad C$$

Singularities of Ω $\longleftrightarrow$ Boundaries/Faces of P

Triangulation



$\mathcal{P}$ tiled by P_i

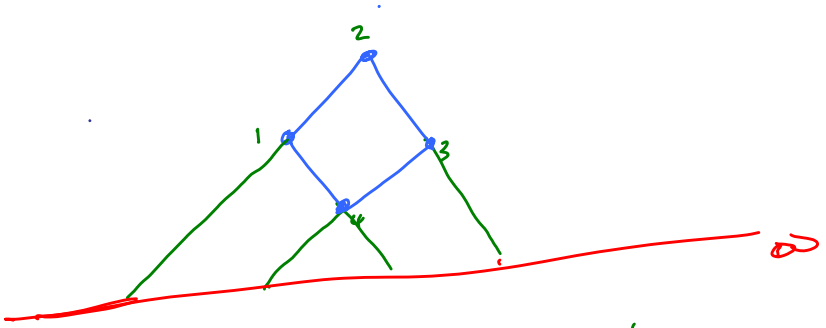
$\Downarrow$

$$\Omega_{\mathcal{P}} = \sum_i \Omega_{P_i}$$

Spurious poles cancel in sum

Ex: "Simple Polytopes"

$$\Omega = \sum_{\text{vertices}} \operatorname{sgn}|X_{a_1} \cdots X_{a_n}| \; d\log X_{a_1} \cdots d\log X_{a_n}$$



Spurious Poles @ ∞ cancel in sum.

Bi-adjoint ϕ^3 theory

* Interaction $f_{abc} f^{ABC} \phi_A^a \phi_B^b \phi_C^c$

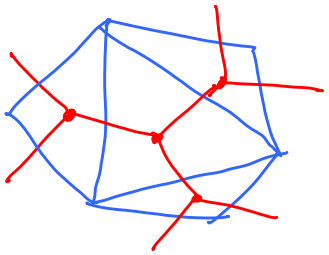
* "Trace decomposition":

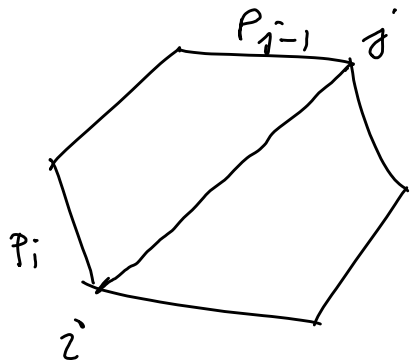
$$Tr(T_{a_1} .. T_{a_n}) Tr(T_{A_1} .. T_{A_N})$$

$$Tr(.. T_a ..) Tr(.. T_a ..) Tr(.. T_A ..) Tr(.. T_A ..)$$

* Usual $1/N \leftrightarrow$ genus expansion, where diagrams must be consistent with orderings for both "a" and "A". Focus on case where "a", "A" orderings are same, diagrammatics identical to YM theory

$\phi^3_{B\text{-}A}$ at Tree Level

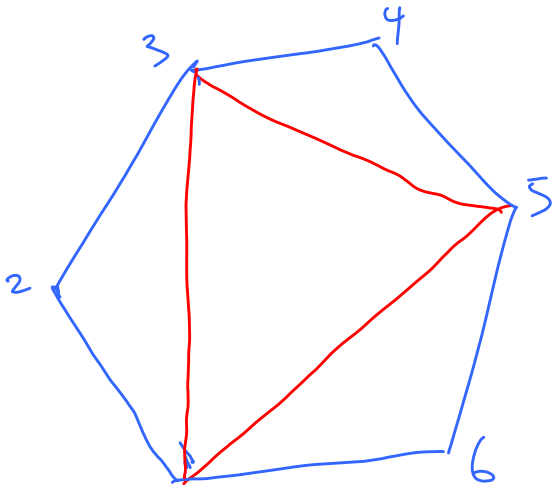




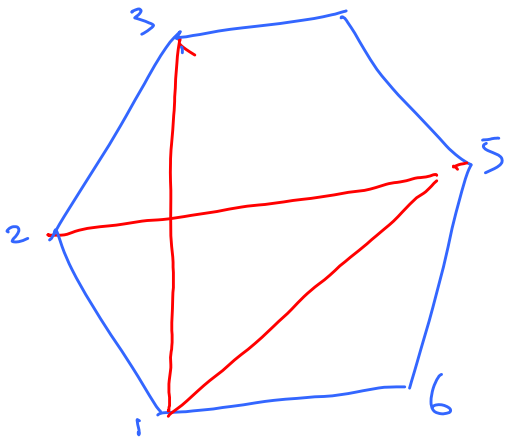
$$X_{ij} = (p_i + \cdots + p_{j-1})^2 = X_{ji}$$

$$[X_{i,i+1} = 0]$$

$$A^{tree} = \sum_{Triang.\,T} \prod_{ij \in T} \frac{1}{X_{ij}}$$

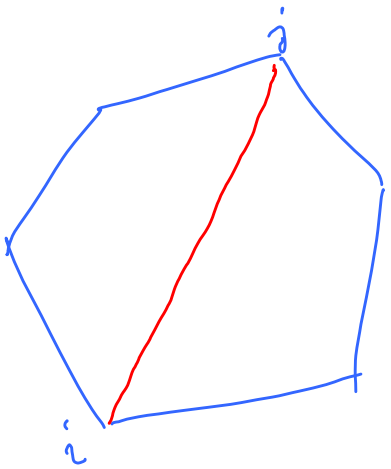


Allowed



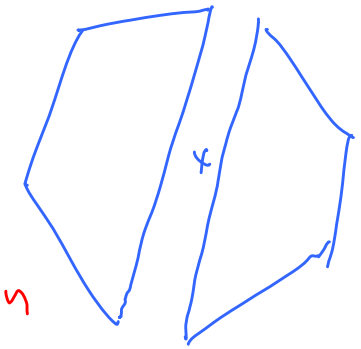
Crossing Chords not allowed.

Locality



$X_{ij} \to 0$

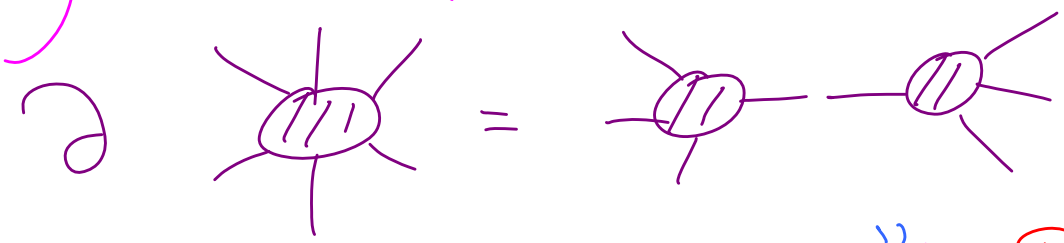
Factorization

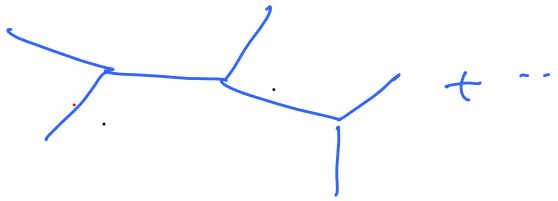


Unitarity

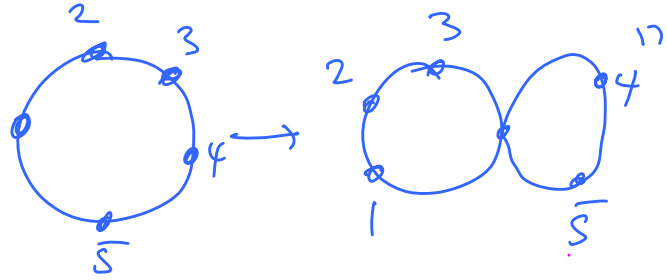
Big Clue Seen By Closet-Theorist Experimentalist

Amp has poles when $(\Sigma p)^2 \to 0$, + Factorizes!



F: "It's  + ..."

Particles in Spacetime

D: "It's 

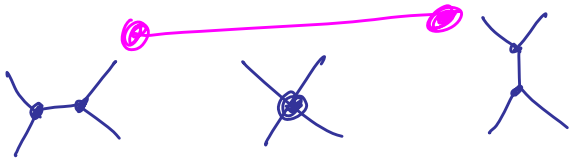
String Worldsheet

Bigger Clue Seen By Lazier but Smarter Experim.

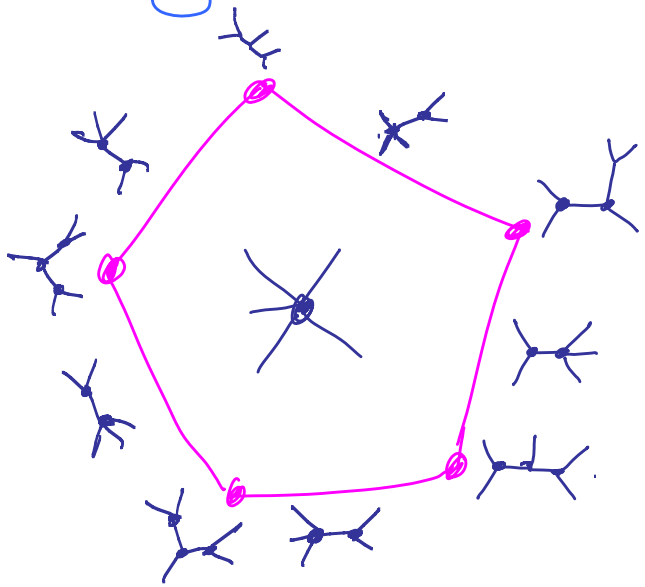
"Only certain patterns of poles come together!"

$n = 4$ s, t but not s,t

$n = 5$ s_{12}, s_{123} + cyclic but not other comb.



$n = 4$



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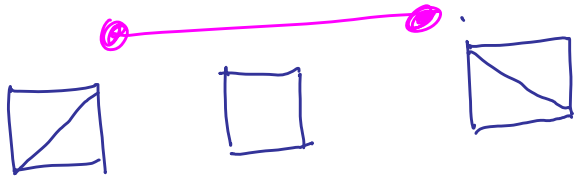
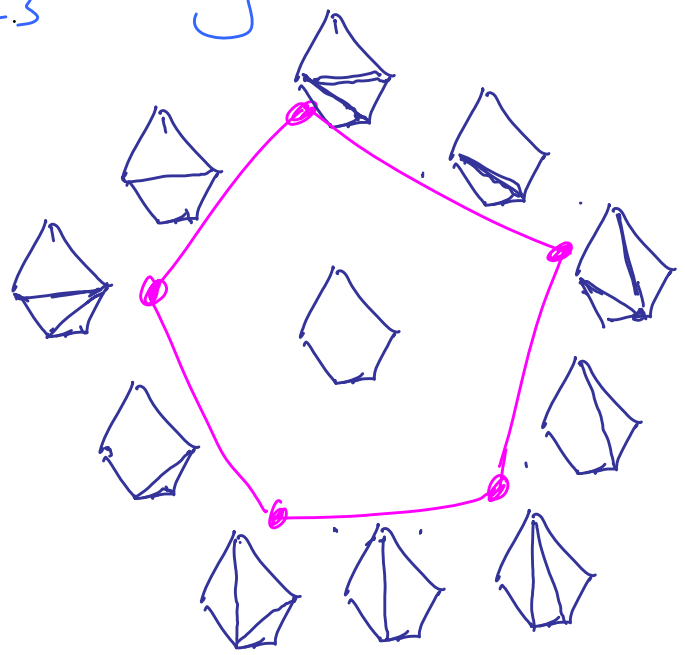
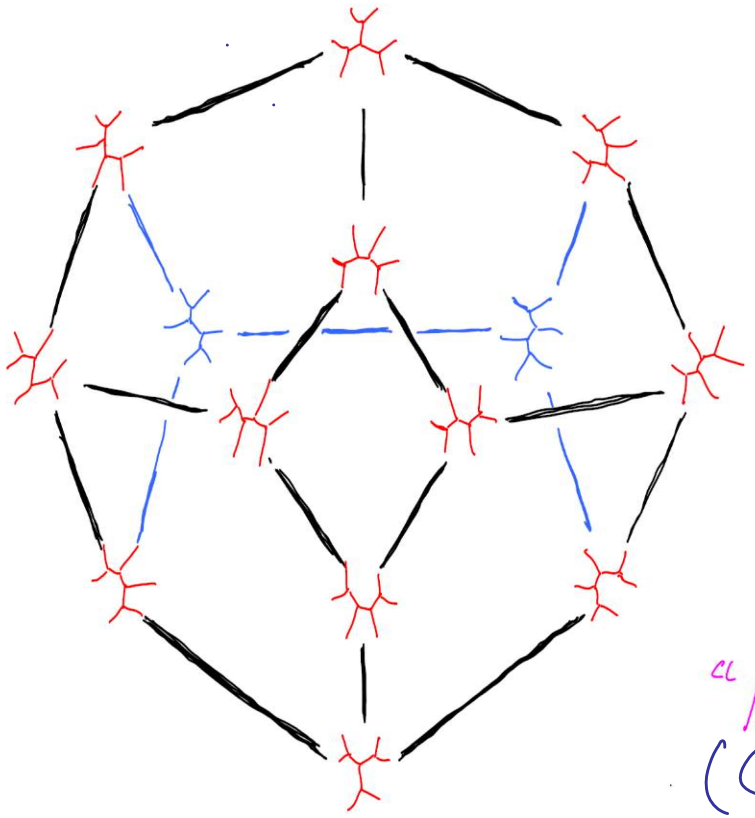
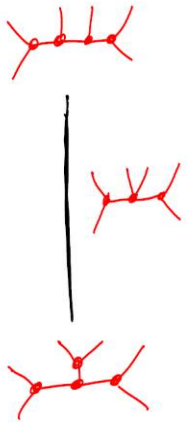


$n = 4$







"Associahedron"
(Combinatorial)

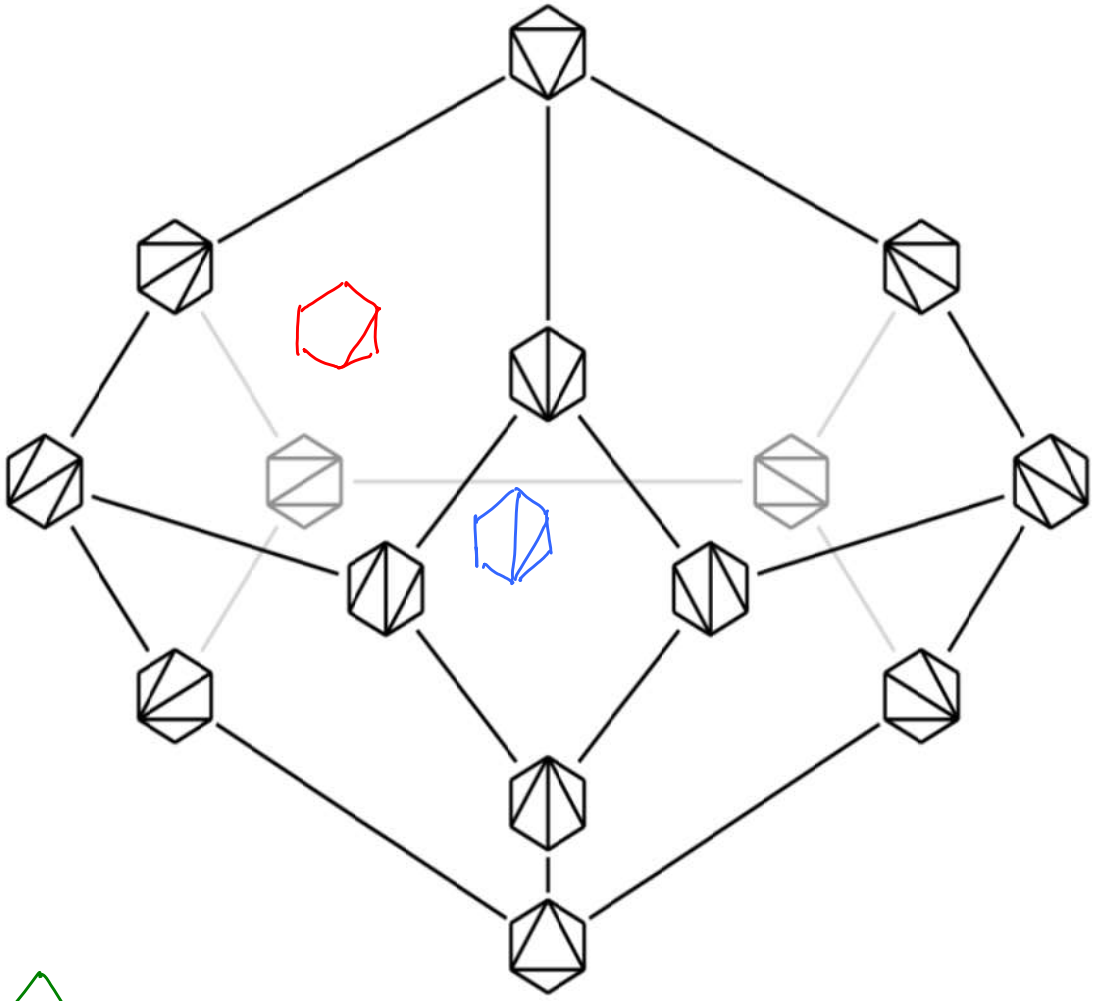
Particle + String picture don't make this obvious!

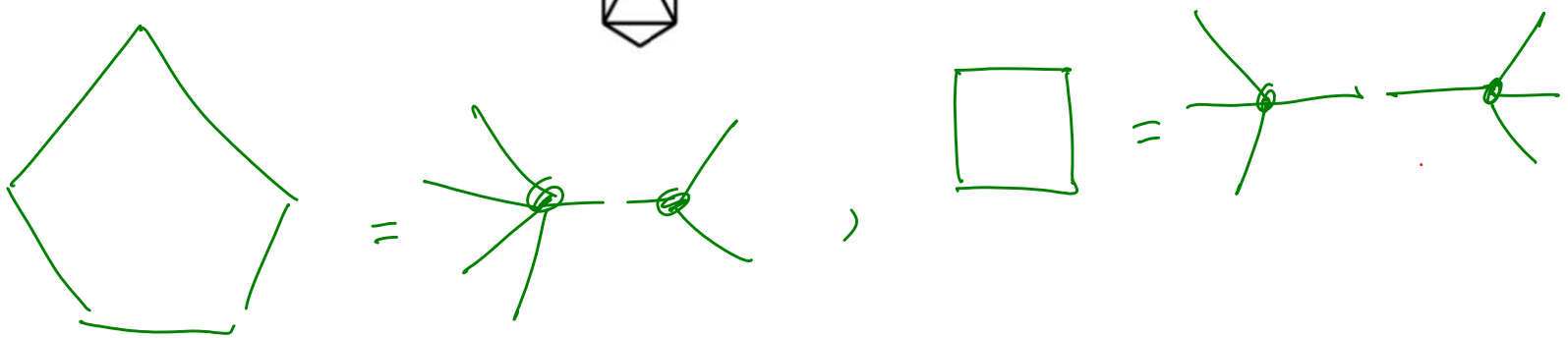
But Assoc. does see everything since

$\partial A_n = A_{n_1} \times A_{n_2}$

HOW CAN WE MAKE THIS OBVIOUS?

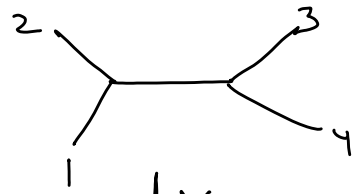
A Qualitative Clue to the "Theory" in Kinematic Space

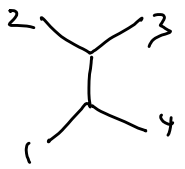




Hidden Symmetry: Projective Invariance

$n=4$





$$\frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}} = d\log X_{13}/X_{24}.$$

General n: invariance of $\Omega_{(n-3)}$ under $X_{ij} \to \alpha(X) X_{ij}$

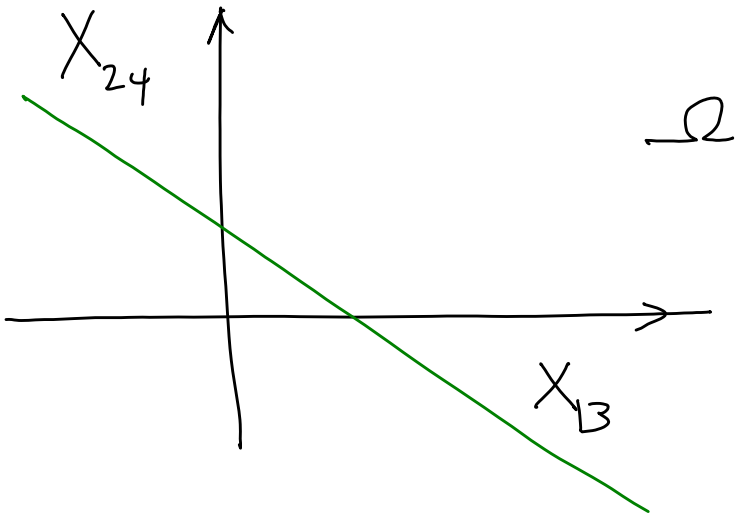
$\Longrightarrow$ $d\log X_{ij} \to d\log X_{ij} + d\log\alpha$ $\longrightarrow$ for each set of $(n-4)$ comp. propagators, coeff. of $d\log\alpha$.  vanishes.

Many more eqns than unknown signs.

Remarkably: signs can be chosen so $\Omega_{(n-3)}$ is projective!

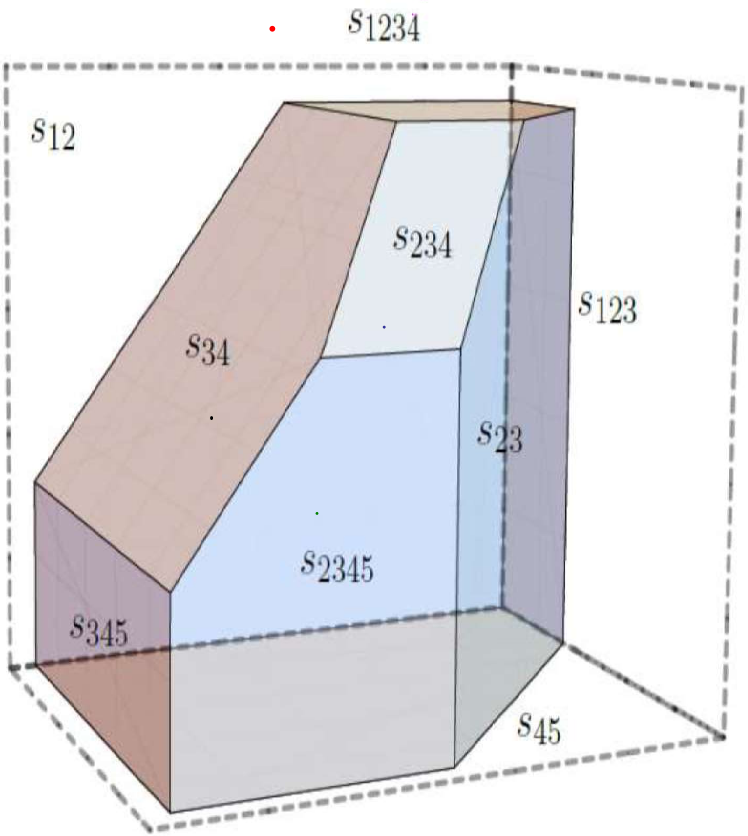
[Analog of Dual Conf Inv. in $\mathcal{N}=4$ SYM].

Pull back to Amplitude



$$\Omega\Big| = dX_{13}\left(\frac{1}{X_{13}}+\frac{1}{X_{24}}\right)$$

General n: $\Omega\Big|_{ABHY} = d^{(n-3)}X \cdot (\text{Amplitude})$



"ABHY" Associahedron in Kinematic Space

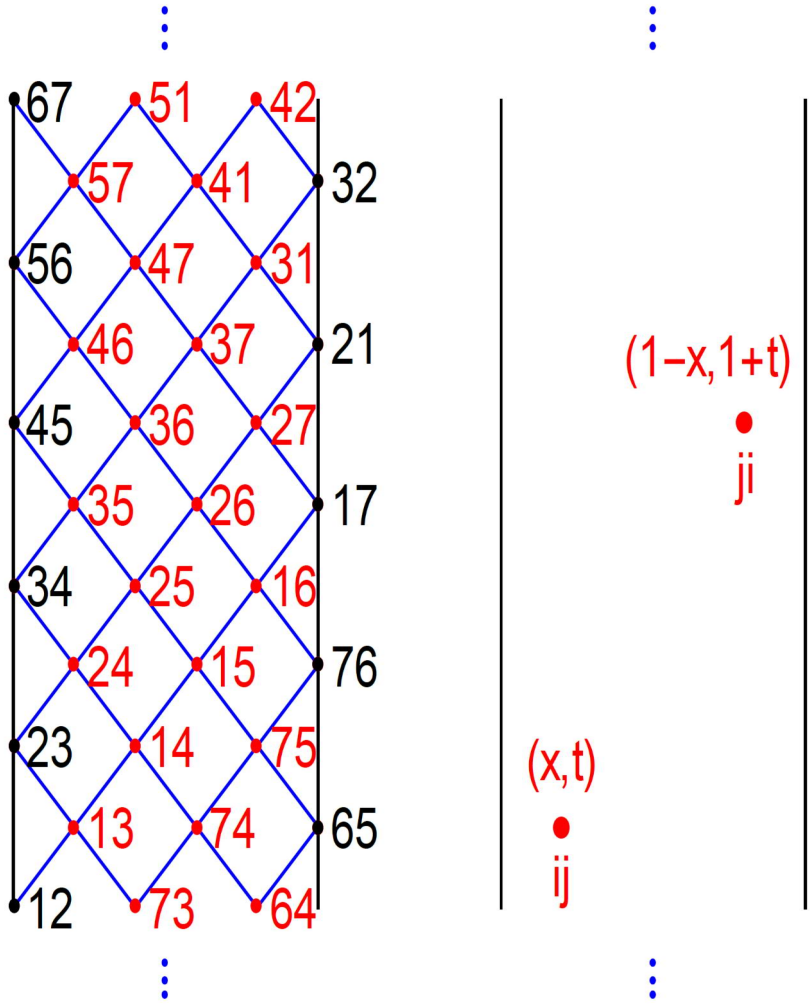
↓

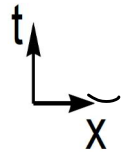
"Cluster Algebra" Polytopes Associated with Dynkin Diagrams (H. Thomas et. al.)

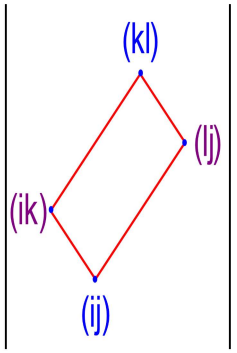
HOW CAN WE MAKE THIS OBVIOUS?

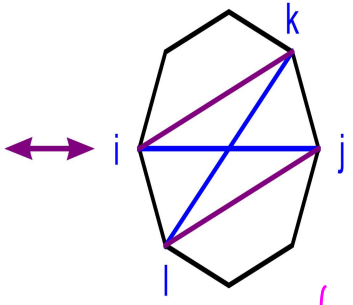
A Qualitative Clue to the "theory" in Kinematic Space

Kinematic "Spacetime"

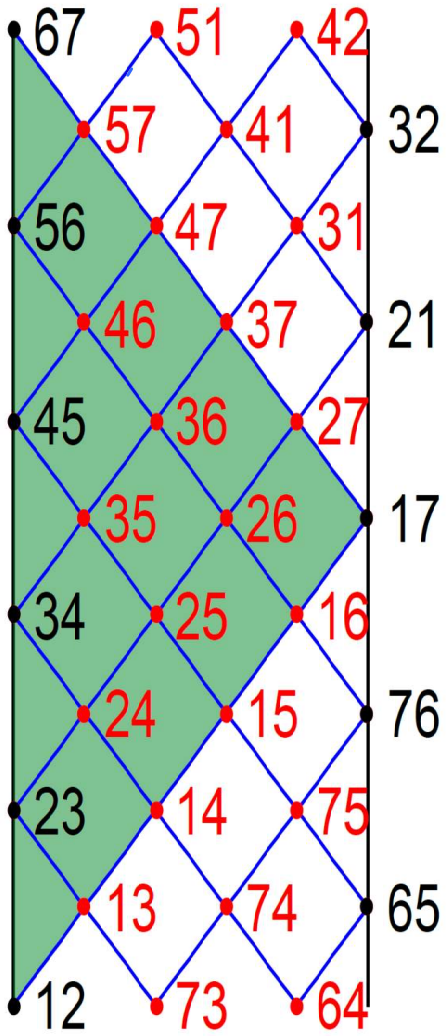








P, F corners ⟷ incompatible chords of Causal Diamond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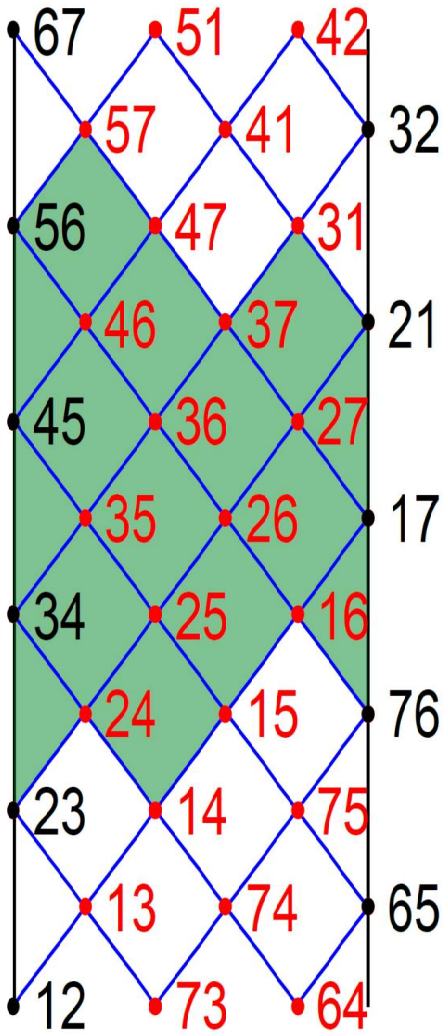
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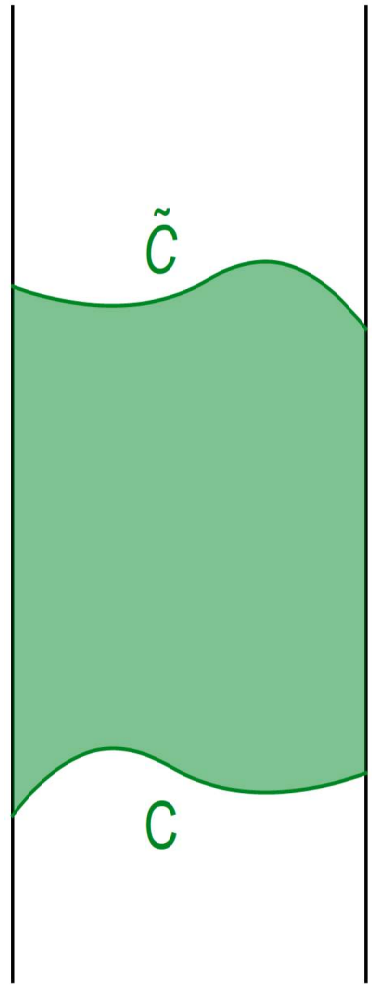
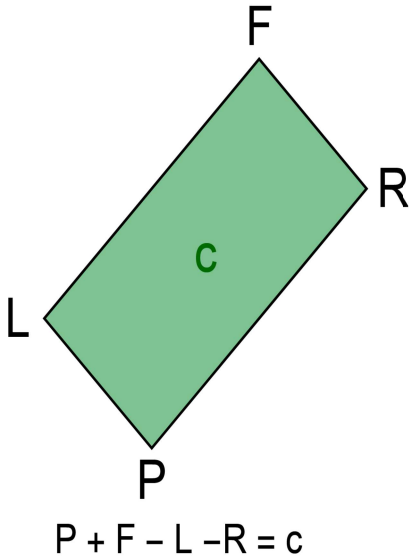

C̃
C

Wave Equation in Kin. "Spacetime"

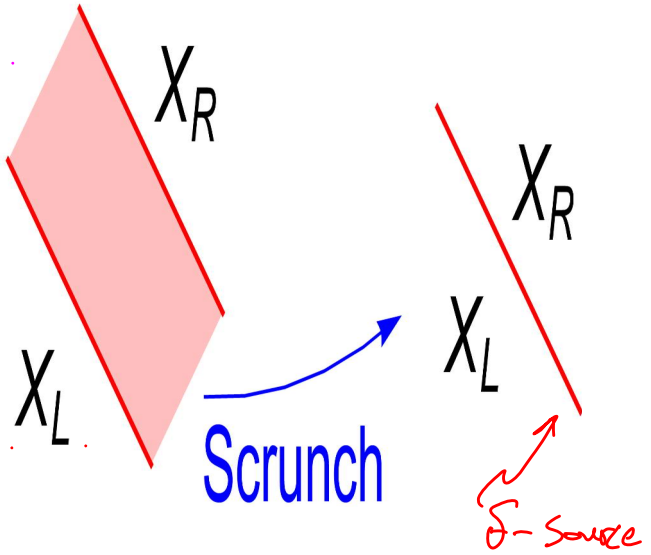
$$\partial_u \partial_v X = c$$

"GaussLaw"

"Scrunch rule"

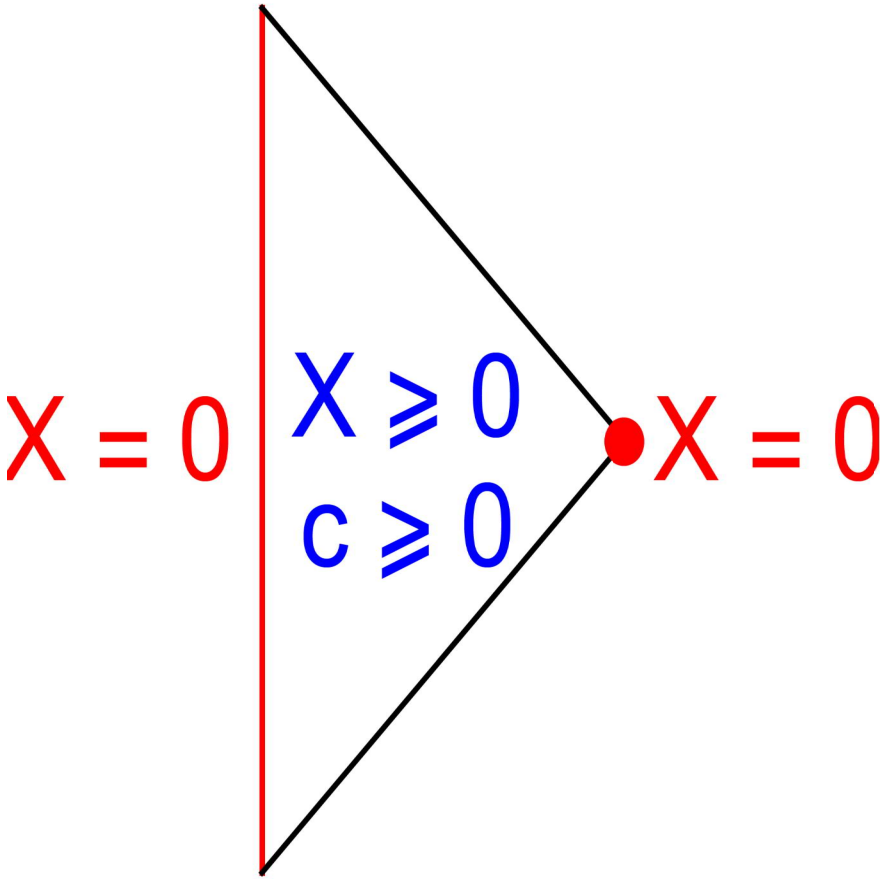


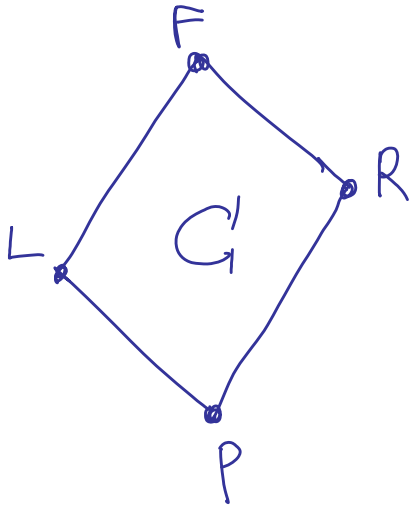
P + F – L –R = c



Positive Wave Equation

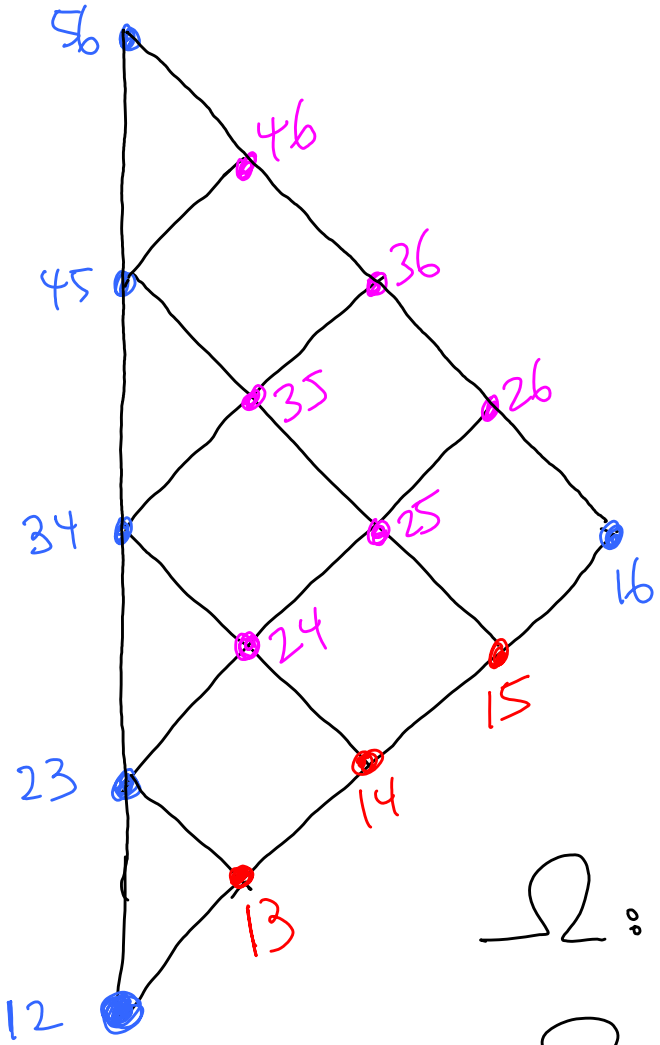
$$\partial_u \partial_v X = c$$

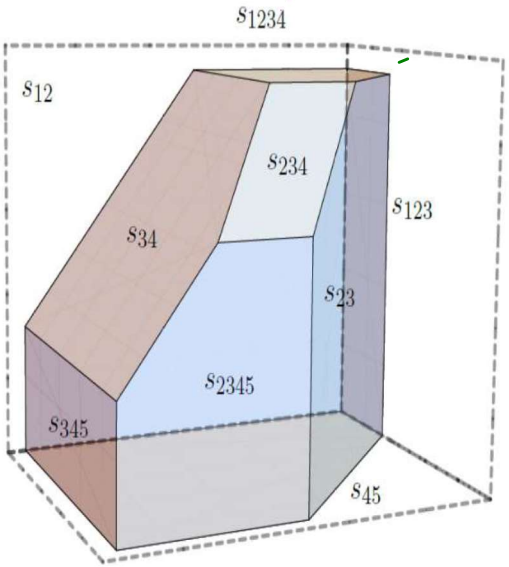




$$X_P + X_F - X_L - X_R = C'$$

Positive WE ⟶ ABHY Associahedron in Kinematic Space.

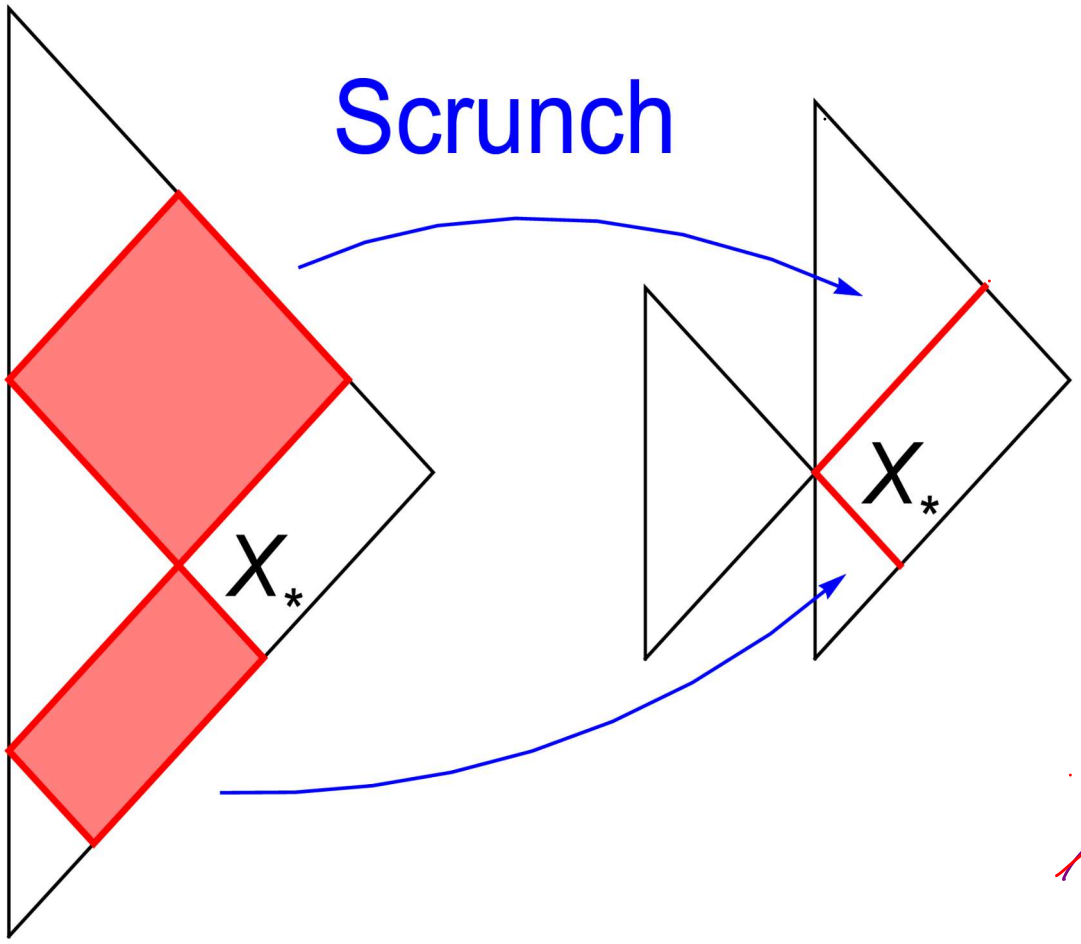




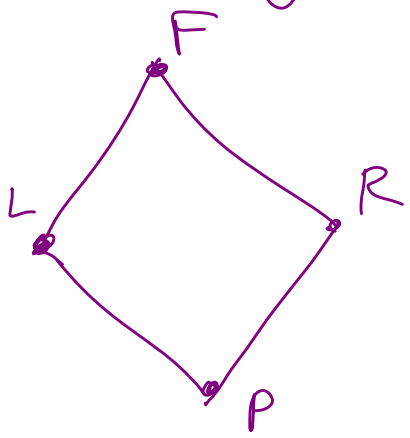
Ω: Fully determined by

$\Omega|_{WE} = \omega^{can}[\text{Assoc.}]$

Factorization!



Crucial point:

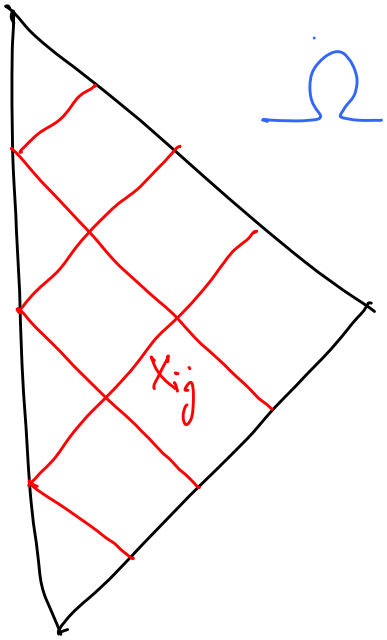


can't put $P=0, F=0$.

~~$P+F$~~ $-L-R=c$

neg = pos X

Scattering Form as Canonical Form



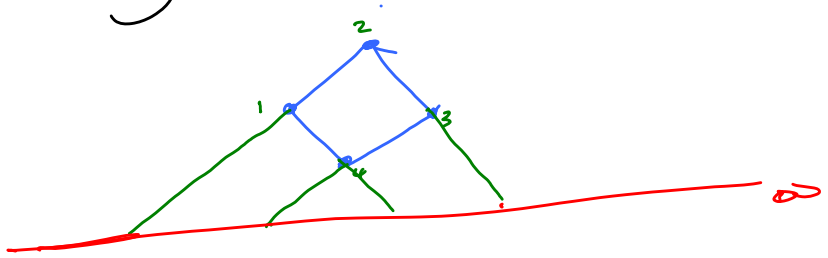
Ω fixed by : $\Omega\big|_{\substack{\text{Wave}\\ \text{Eqn}}} = \omega^{can}[\text{Assoc}]$

* Trivially Projective Invariant

* Since $\det[g_{a_1}, \cdots, g_{a_{n-3}}]$ of g-vectors in a cluster cone $= \pm 1$,

$\Omega\big|_{WE} = \omega^{can}[\text{Assoc}] = (dVol) \times \text{Amplitude}!$

* Feynman diagrams are one particular triangulation:

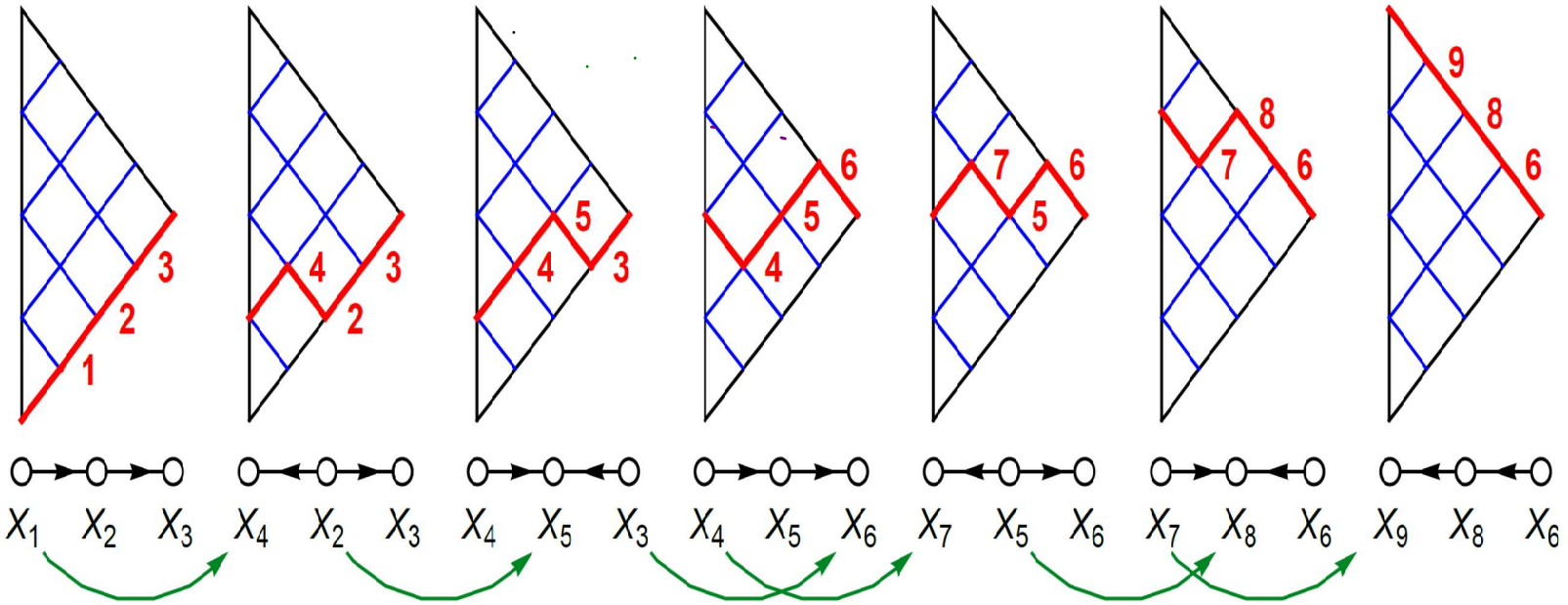


Spurious poles @ ∞ break proj. invariance term-by-term

* Many other triangulations! Manifest proj. invariance term-by-term, vastly more efficient than Feynman Diagrams

* Another general method for computing can. forms, applied here, yields string amplitudes

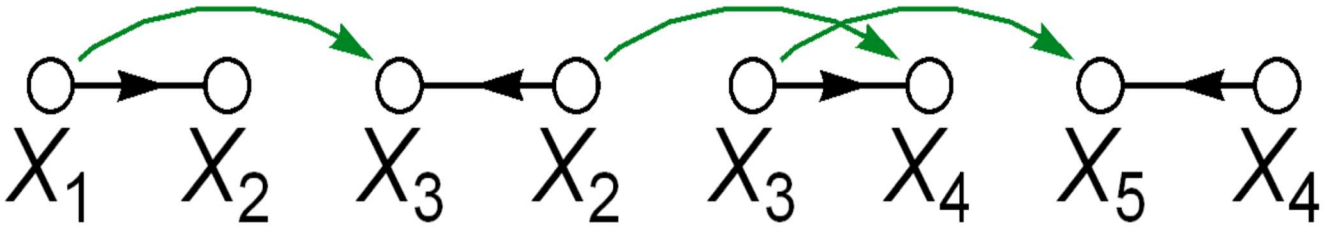
"Time Evolution" in Kinematic Space

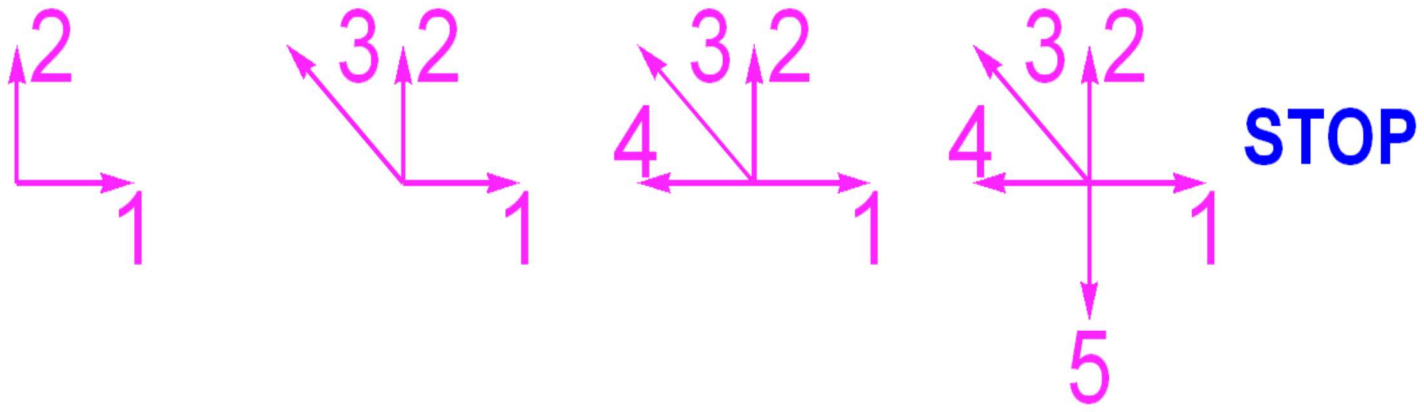


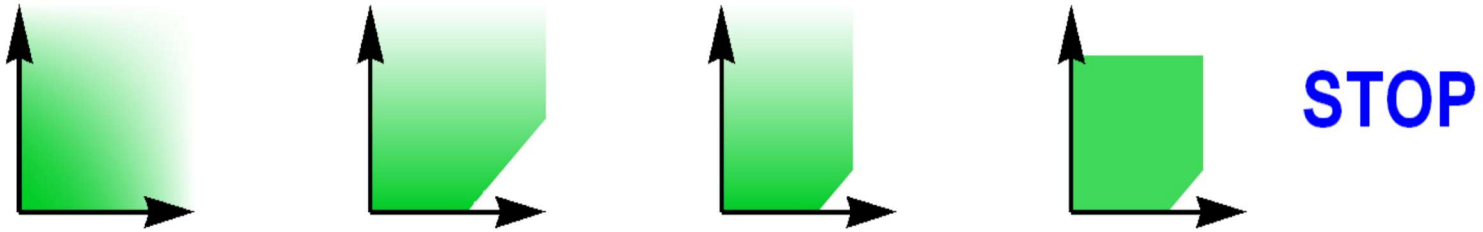
"Walk" by mutating only on sources

$$X_v + X_v' - \sum_{v \to w} X_w = c_v$$

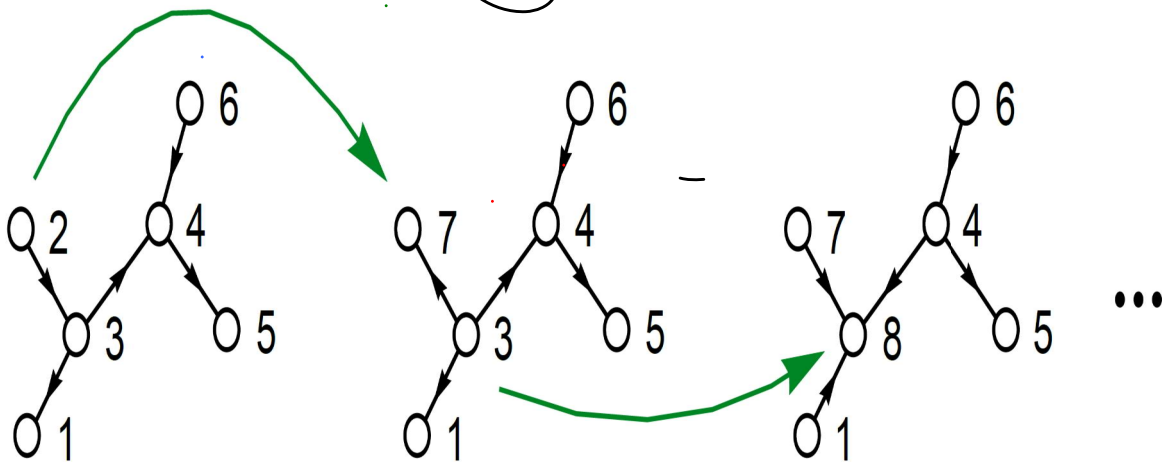
(~ "tropicalized mutation")


X1
X2
X3
X2
X3
X4
X5
X4


2
1
3
2
1
3
2
4
1
3
2
4
1
5
STOP


STOP

Abstract Away "Time Evolution"



$$X_v + X'_v - \sum_{v \to w} X_w = c_v$$

(Auslander–Reiten Walk)

Can Define For Any Acyclic Quiver

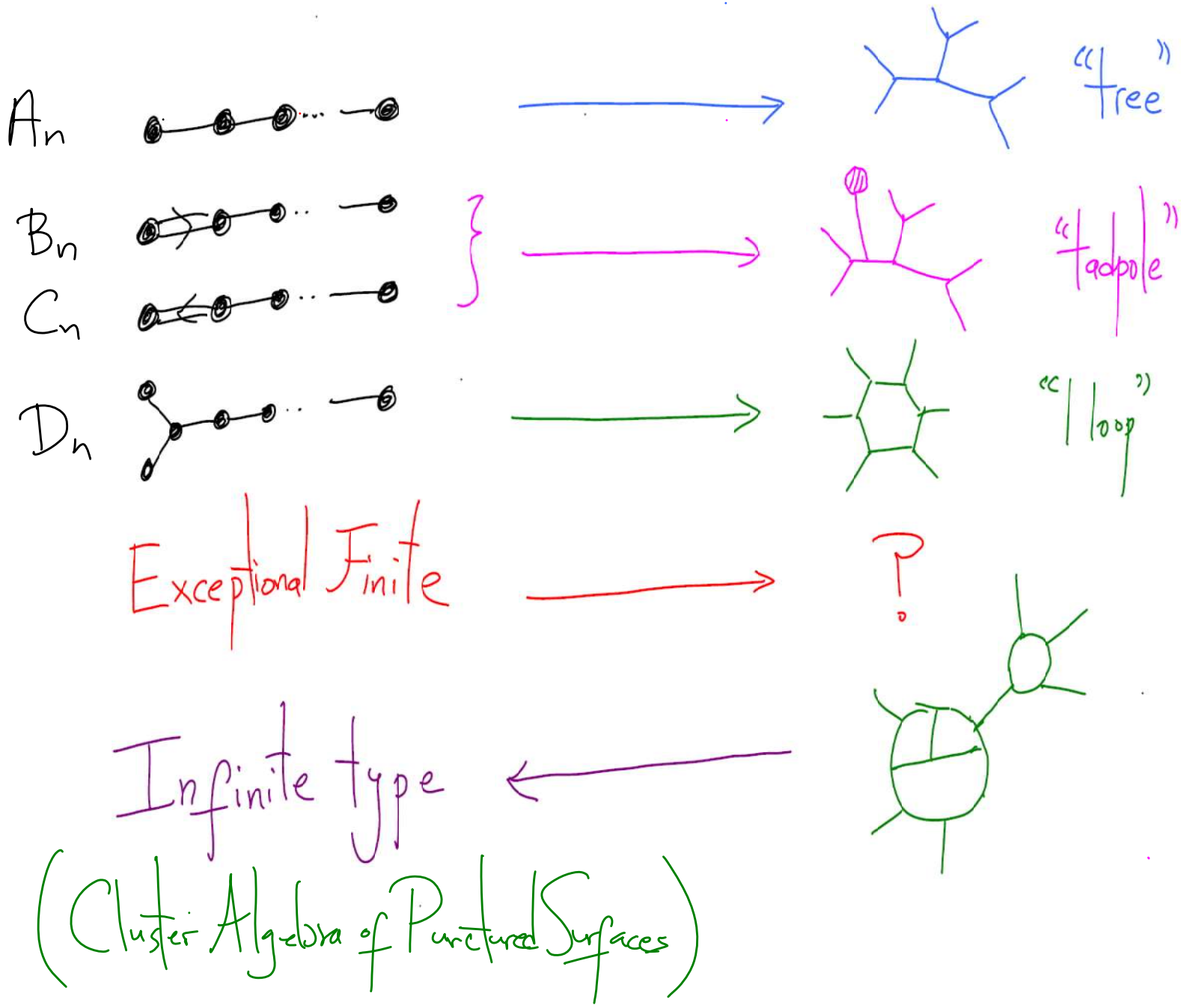
$Q \xrightarrow[@i]{\text{mutate}} Q' \Longrightarrow X' = A_i X$

A_i generators of Coexeter Group of Q

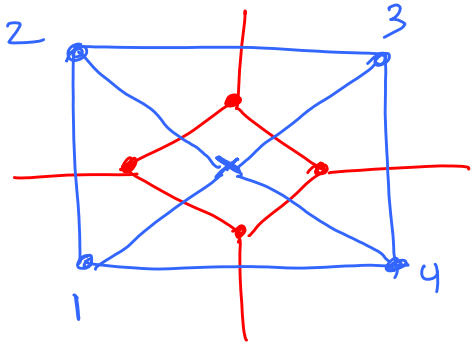
ABHY Polytope Finite $\longrightarrow$ CoexterGrpFinite

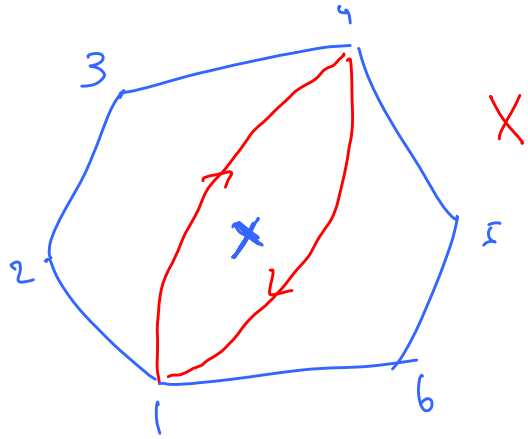
$\longrightarrow$ Dynkin Classification $\longrightarrow$ ABHY Real.

of all Finite Cluster Algebra Polytopes

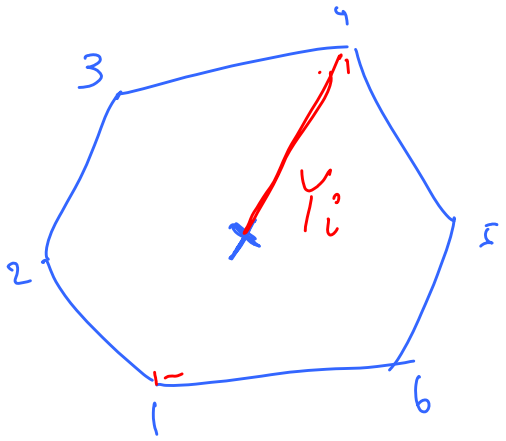

An
Bn
Cn
Dn
"tree"
"tadpole"
"loop"
Exceptional Finite
?
Infinite type
(Cluster Algebra of Punctured Surfaces)

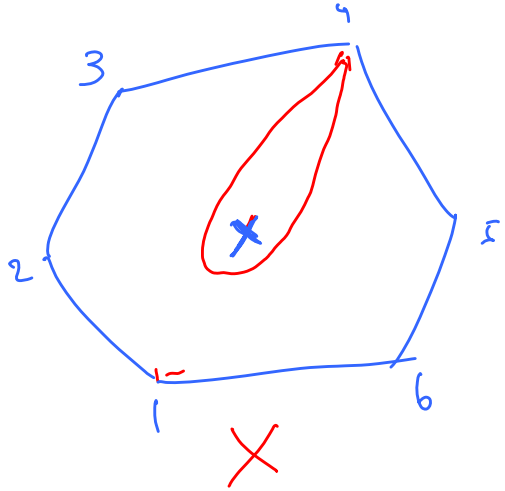
Variables for 1-loop Integrand



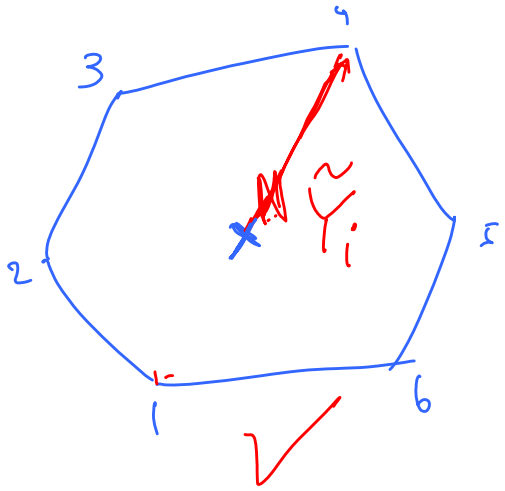


$X_{ij} \neq X_{ji}$



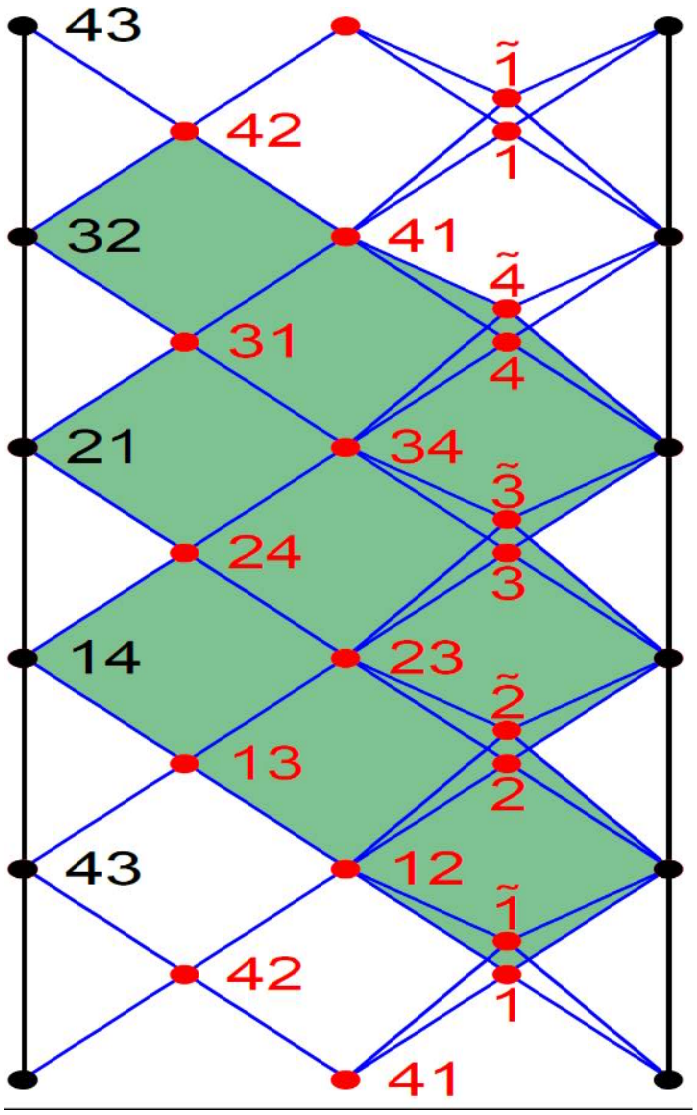


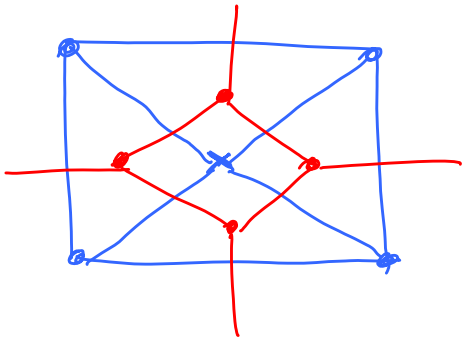
X

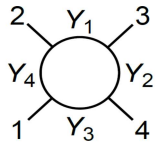


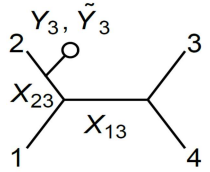
✓

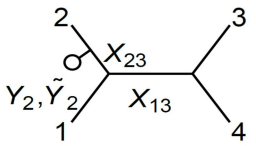
Kinematic "Spacetime" @ 1-loop

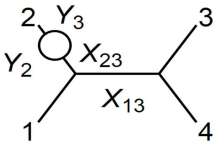


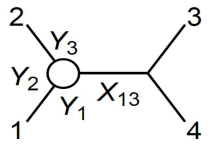


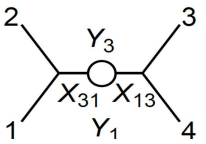


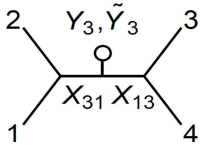


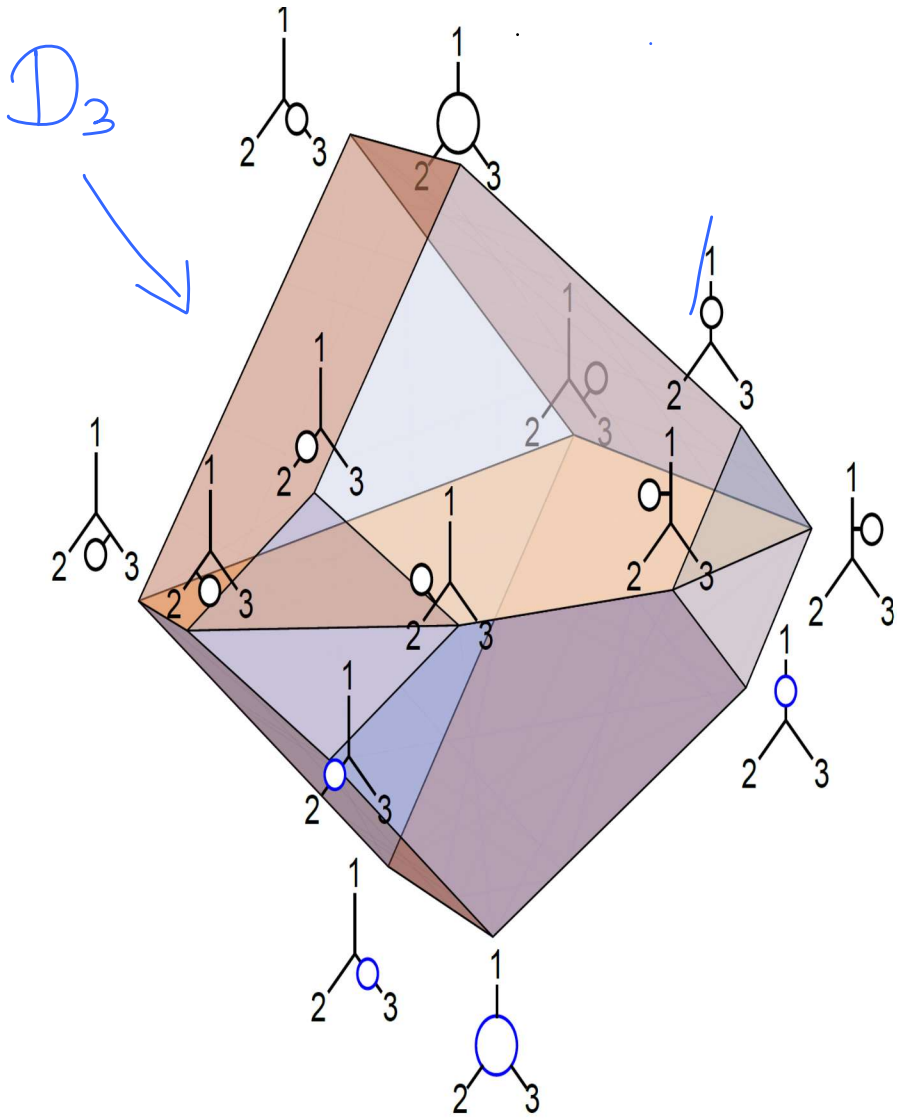


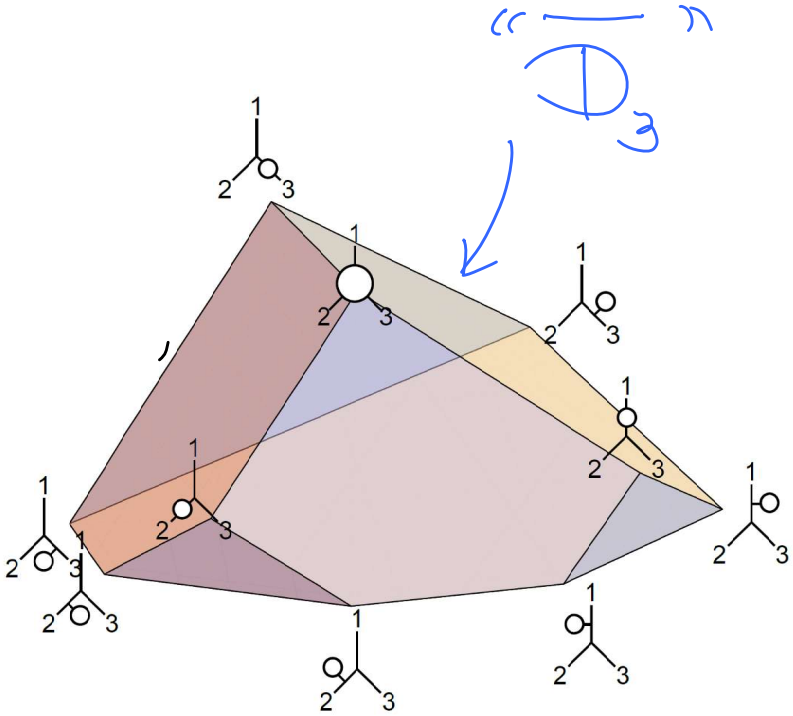


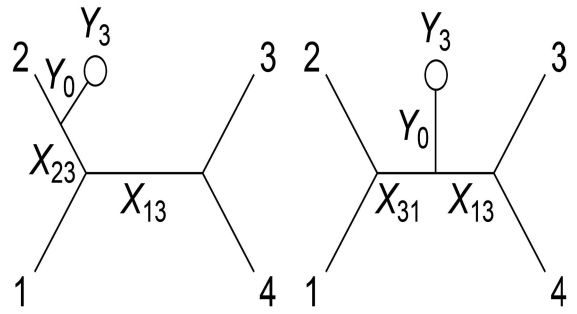




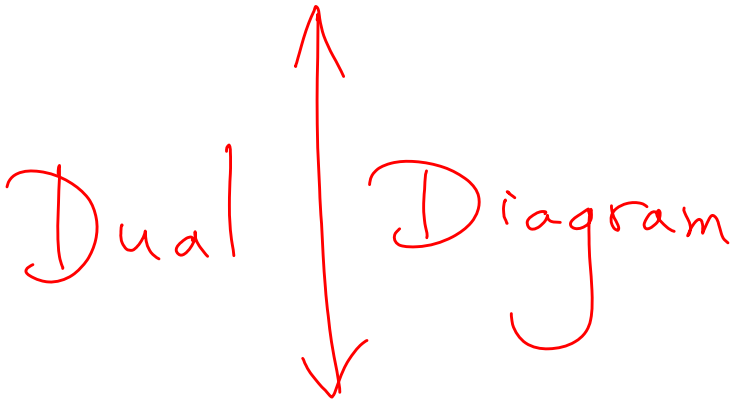



D_3

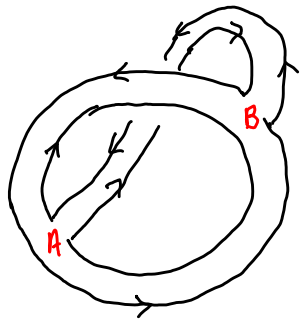

"D_3"


Y_3
Y_0
X_{23}
X_{13}
Y_3
Y_0
X_{31}
X_{13}

h-trace Feynman Diagrams, genus g, L-loops

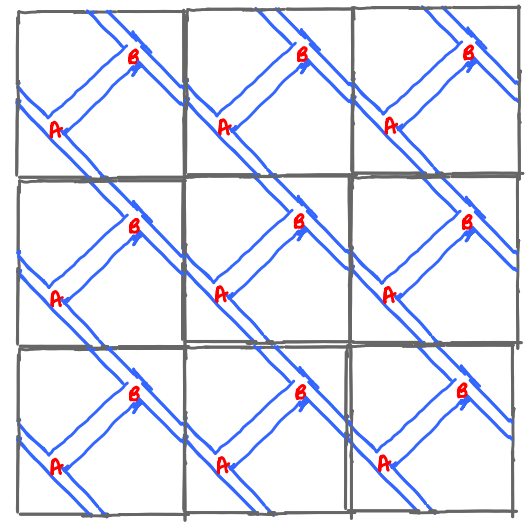


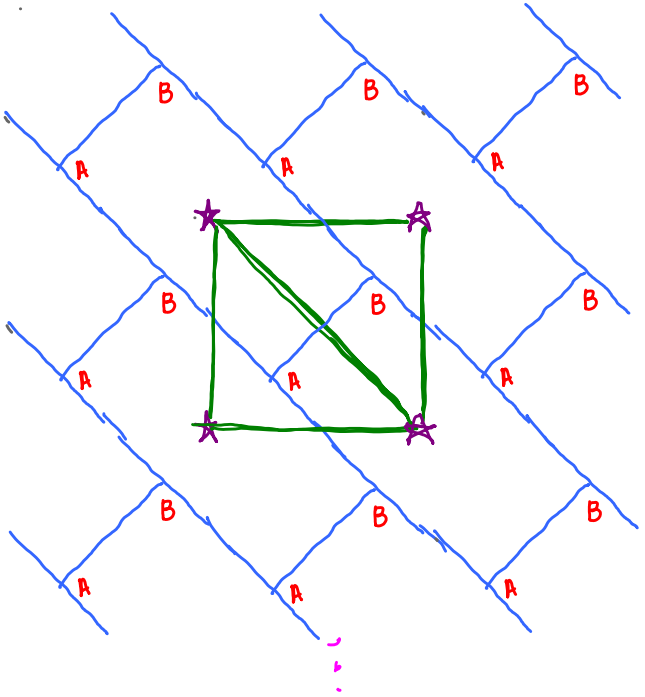
Triangulation of surface with h bndries, genus g, and $p = L + l - 2g - h$ punctures



$L = 2, h = 0,$

$g = 1$



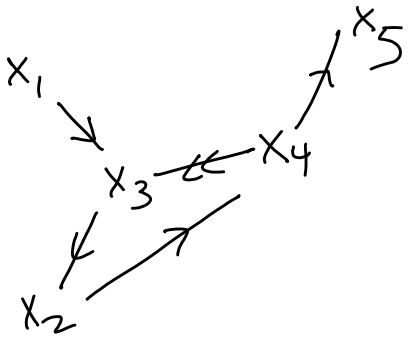


$p = 1,$

Triangulation of once-punctured torus

Some Cluster Algebra Basics

Quiver, "cluster"



- "Mutation" @ v
  1) Reverse all arrow
  2) If $a \to v \to b$ add $a \to b$.
  3) Delete 

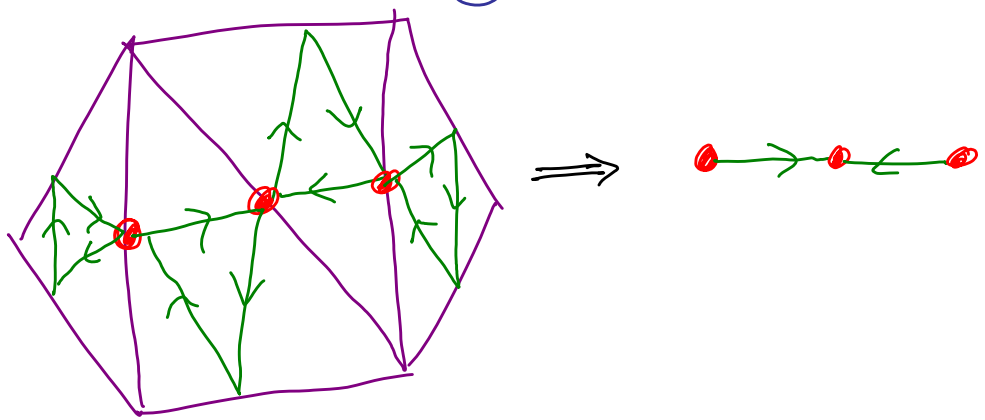
- $x_v x_{v'} = \prod_{w \to v} x_w + \prod_{w \leftarrow v} x_w$

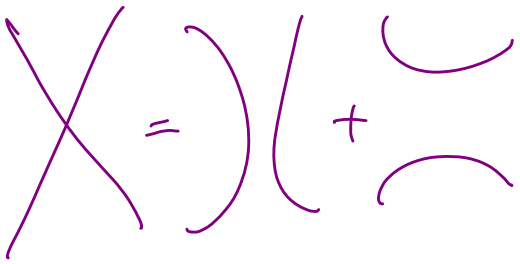
- Also "tropicalized" pair:

$$\left\{ X_v + X_{v'} - \sum_{w \leftarrow v} X_w \,,\; X_v + X_{v'} - \sum_{w \to v} X_w \right\}$$

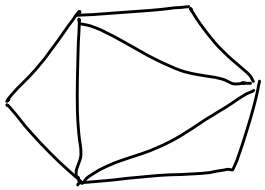
Surface Cluster Algebras

* Quivers from triangulations of surface

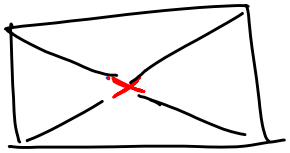


* Mutation $\longleftrightarrow$ Skein relations : 

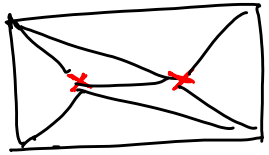
All-loop, all orders in $1/N$ ϕ^3_{BA}



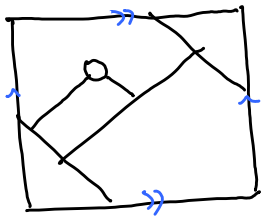
trees ⟶  A_{n-3}

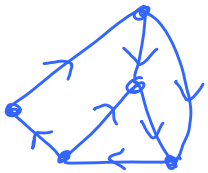


1-loop ⟶  D_n



2-loop ⟶  $\widetilde{D}_{n+2}$



2-loop, leading $1/N$ [orientable] ⟶ 

} "Infinite Type"

Seeking "Clusterhedra"

g-vectors

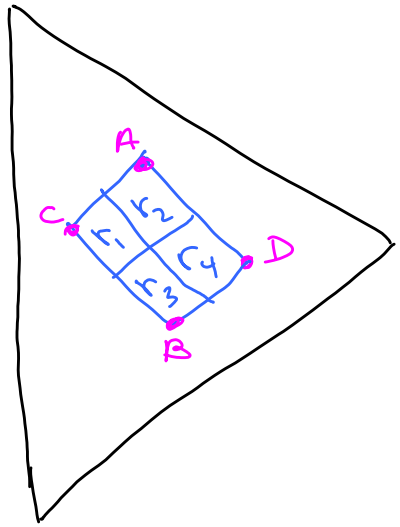
- Canonical way of labelling cluster variables, given choice of initial cluster $(x_1, \dots, x_n)$.
- For each mutation μ, there is one of $\{X_v + X_{v'} - \sum_{w \to v} X_w,\ X_v + X_{v'} - \sum_{w \leftarrow v} X_w\}$ that is satisfied by (+ determines) g-vectors.
- We'll call this set of relations [again, dep. on choice of initial cluster], $\{R_\mu\}$.

Emergence of "Kinematic Spacetime" from Cluster Algebra.

Consider all "$X+X-X-\ldots$" relations $\{R_\mu\}$.

There is a tiny subset of them R_{μ_*} that generate all the rest by positive lin combination:

$R_\mu = \sum c_{\mu\mu_*} R_{\mu_*}$. R_{μ_*} are "pixels" or "meshes" in "Kin Spacetime".



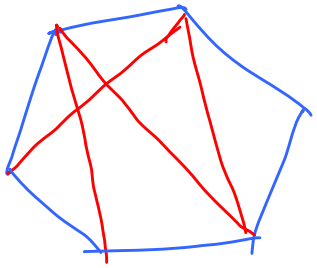
Mut.rel. $A+B-C-D$

$=$

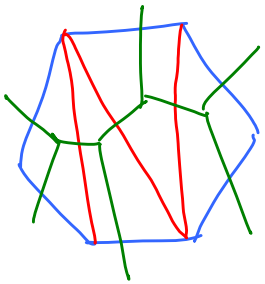
$r_1+r_2+r_3+r_4$

Understood for all finite-type cluster algebras

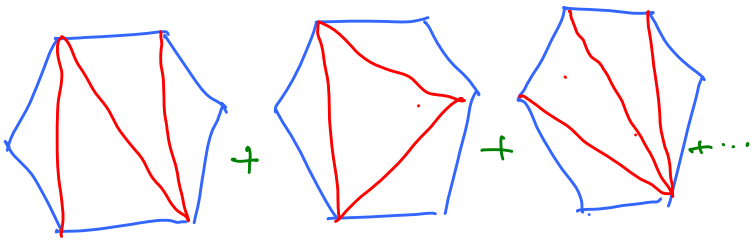
Spacetime + QM $\leftrightarrow$ Cluster g-vector Cones + Fan



Propagates X_{ij}

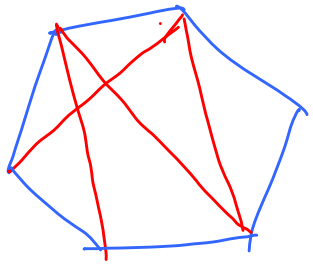


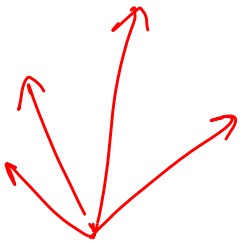
Spacetime process



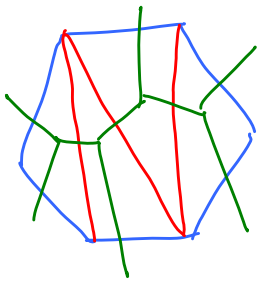
Add an amplitude for all processes with "+1", because of QM

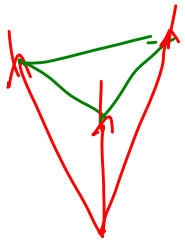
Spacetime + QM $\leftrightarrow$ Cluster g-vector Cones + Fan



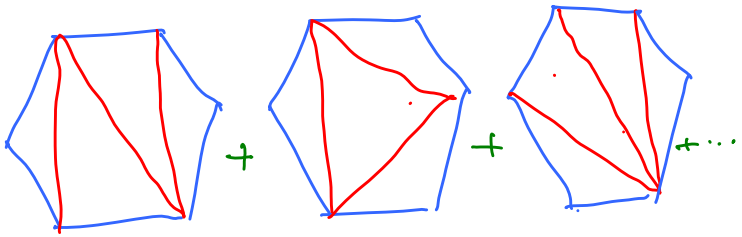


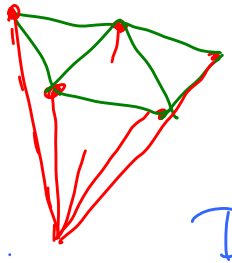
g-vectors.





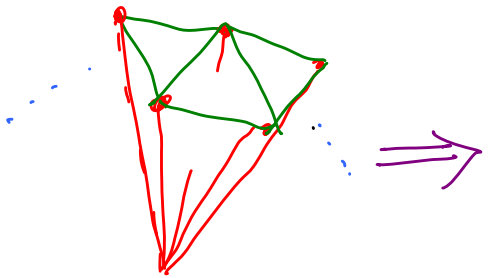
g-vector cone





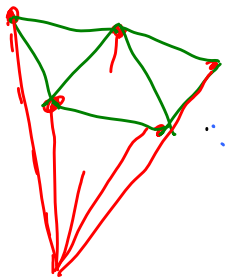
g-vector cones tile a fan

Amazing + Deep Fact!



$$\Rightarrow \quad \Omega = \sum \operatorname{sgn} \begin{vmatrix} g_{a_1} \cdots g_{a_n} \end{vmatrix} \, d\log X_{a_1} \cdots d\log X_{a_n}$$

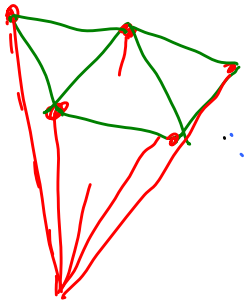
is projective invariant



$$\Rightarrow \quad \Omega\Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i\right)$$

$$\times \sum_{\text{diag}} (+1) \prod \frac{1}{X's}$$

Amplitude

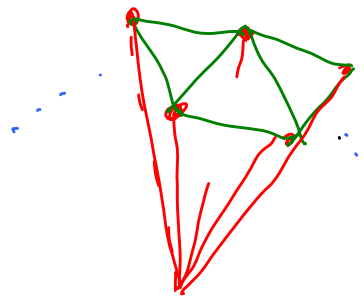


$$\Rightarrow \quad \Omega\Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i\right)$$

$$\times \sum_{\text{diag}} +1 \cdot \prod \frac{1}{X's}$$

Because

$$\left| g_{a_1} \cdots g_{a_m} \right| = \pm 1 \; !$$



$\Rightarrow$ $\Omega\Big|_{X_a = g_a^i X_i + k_a} = \left(\prod_i dX_i\right)$

$$\left[\begin{array}{l}\text{g-vectors are normal}\\ \text{to facets of polytope}\end{array}\right] \times \sum_{\text{diag}} \; +1 \; \prod \frac{1}{X's}$$

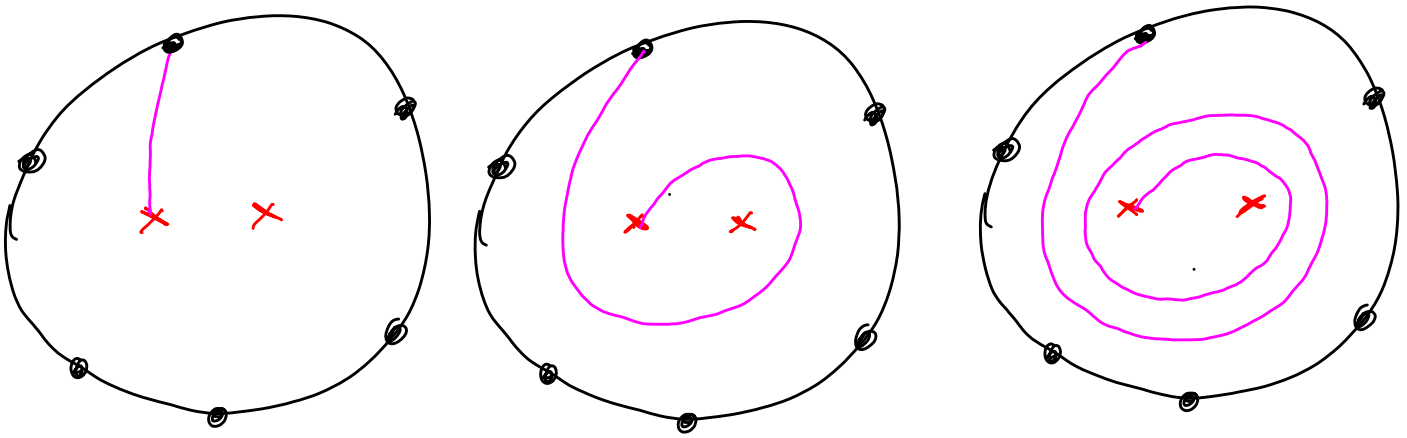
ABHY construction automatically orders k_a such that Polytope given by $X_A \geq 0$ has correct shape!

* Much of this carries over to infinite-type cluster algebras —

except that g-vector fan is infinite but doesn't cover all of space

. . .

Infinite Winding



* Infinite action of Mapping class groups, Braid L
* Also, g-vector cones still fit snugly together, but leave lower-codim. holes in the space

[⟹ Projective invariance works as "telescopic cancellation" but we have to "compactify" the infinite space, somehow].

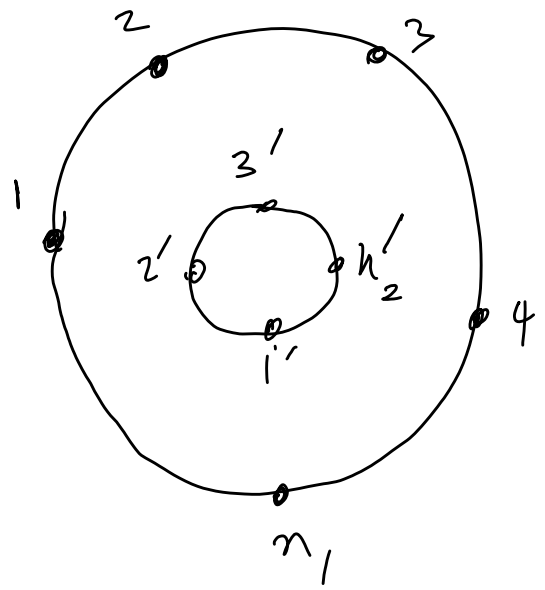
"Clusterhedra"

Are the polytopes where:

* Vertices unambiguously associated w/ diagrams/clusters.
* g-vectors are (among) facet normals.
* Projective [trivial]
* Pull back to amplitude with all +1's
  (= all facet normal dets. $= \pm 1$, non-trivial!)

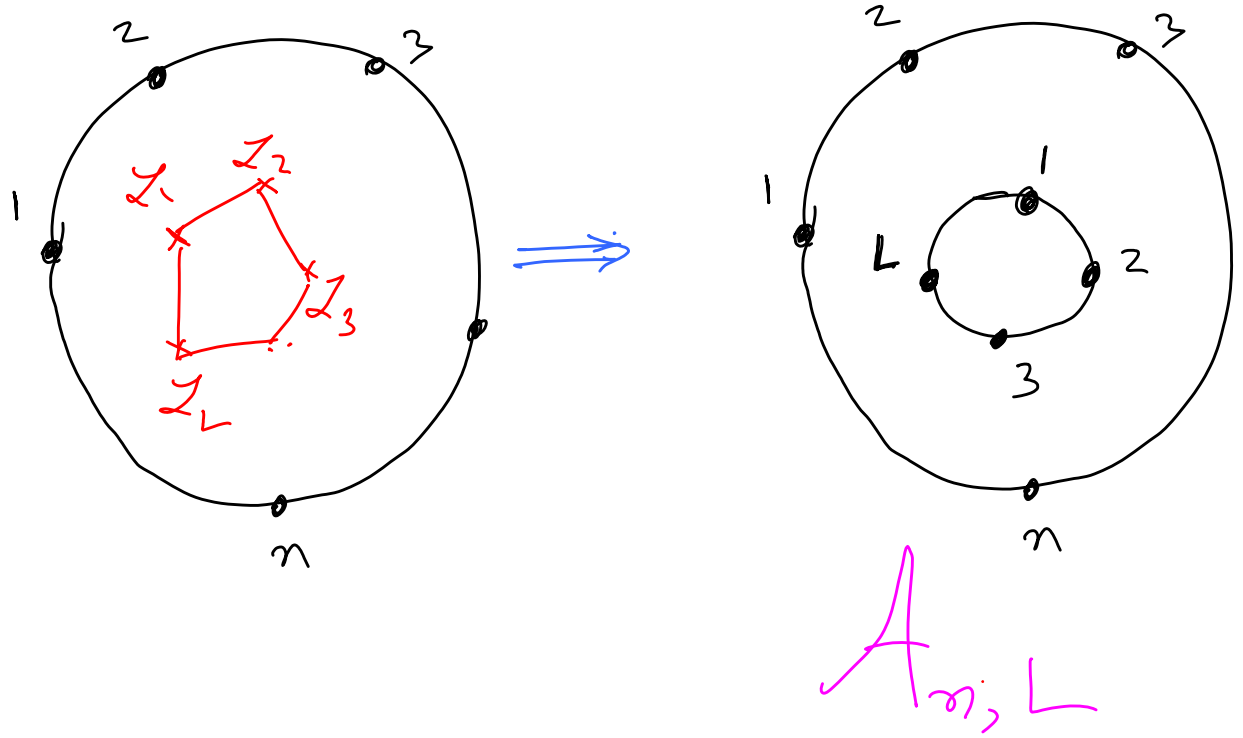
"Kinematical Spacetime" richer @ multiloop level +
we will find Clusterhedra naturally living there

Double-trace, 1-loop

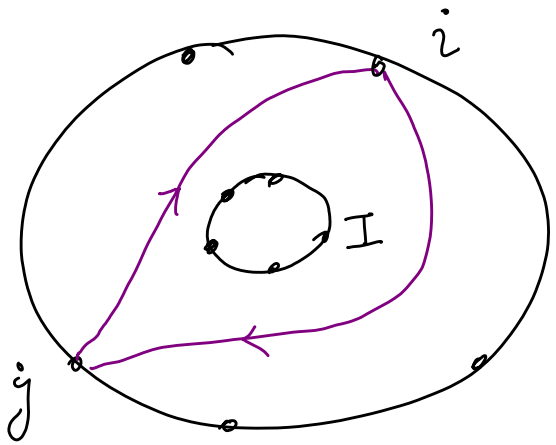


A_{n_1, n_2}

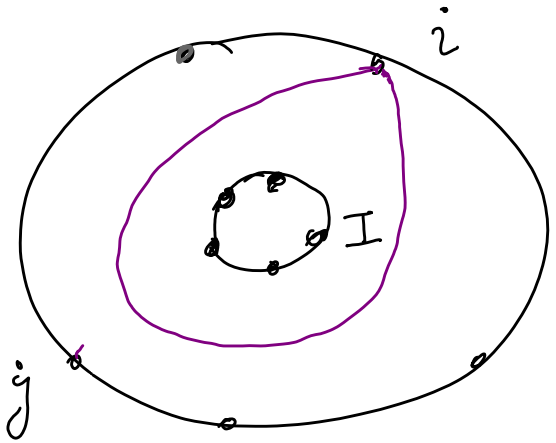
Class of All-loop Cuts



Variables for $A_{n,L}$



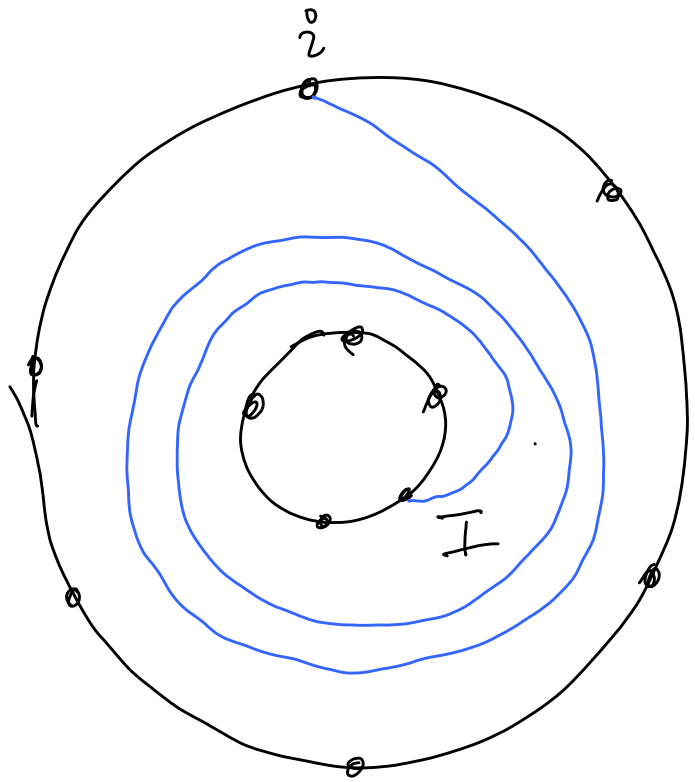
$X_{ij} \neq X_{ji}$



X_{ii+n}

"Out/Out" or "J_n/J_n"

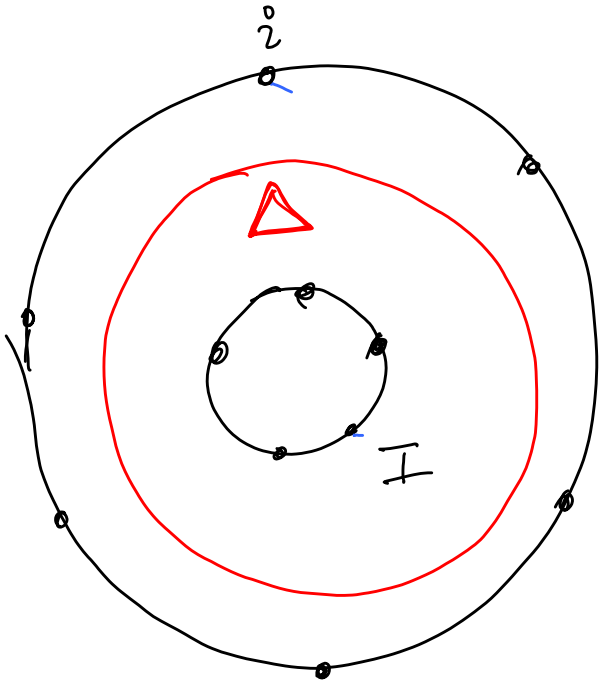
Variables for $A_{n,L}$



X^{ω}_{iI}

"In - Out"

Variables for $A_{n,L}$

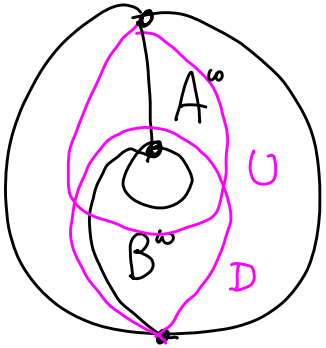


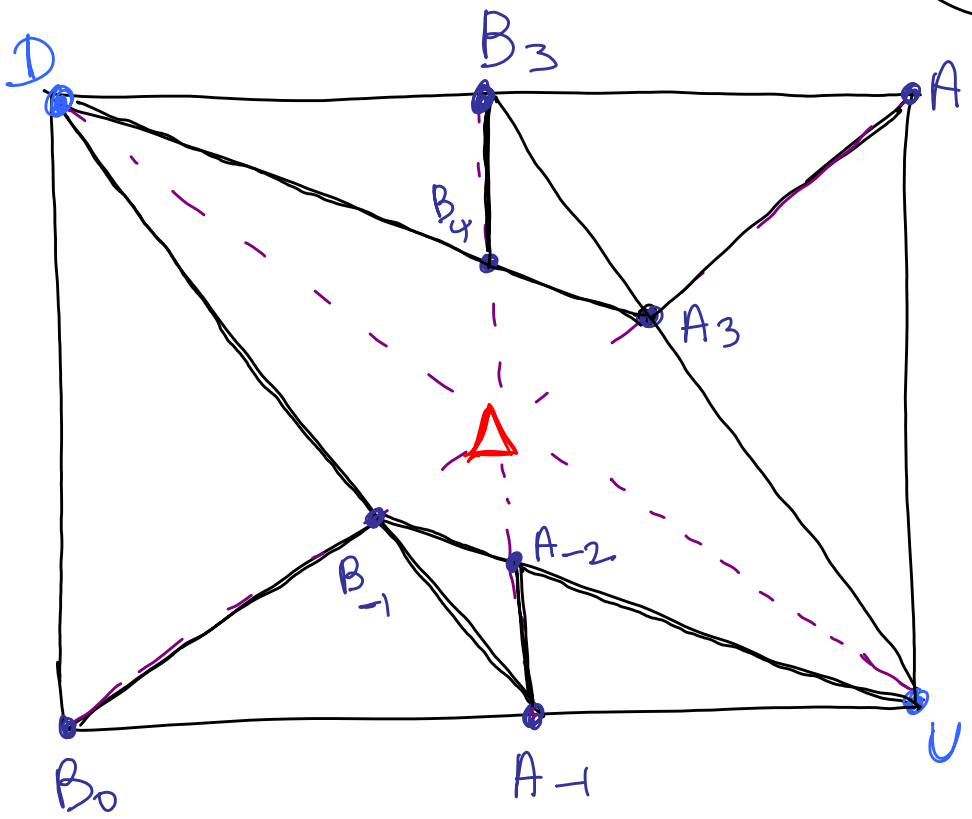
$$\begin{cases} X^{tw} + \Delta = X^{tw+1} \\ X^{-w} + \Delta = X^{-w-1} \end{cases}$$

Δ : Not Cluster Variable,
but crucial part of story.

The q-vectors asymptote to Δ

For example in $A_{2,1}$:

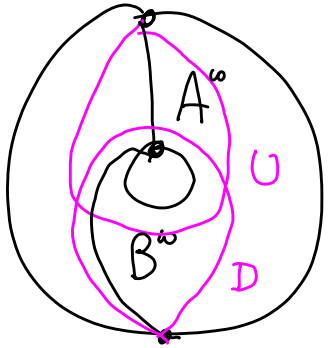


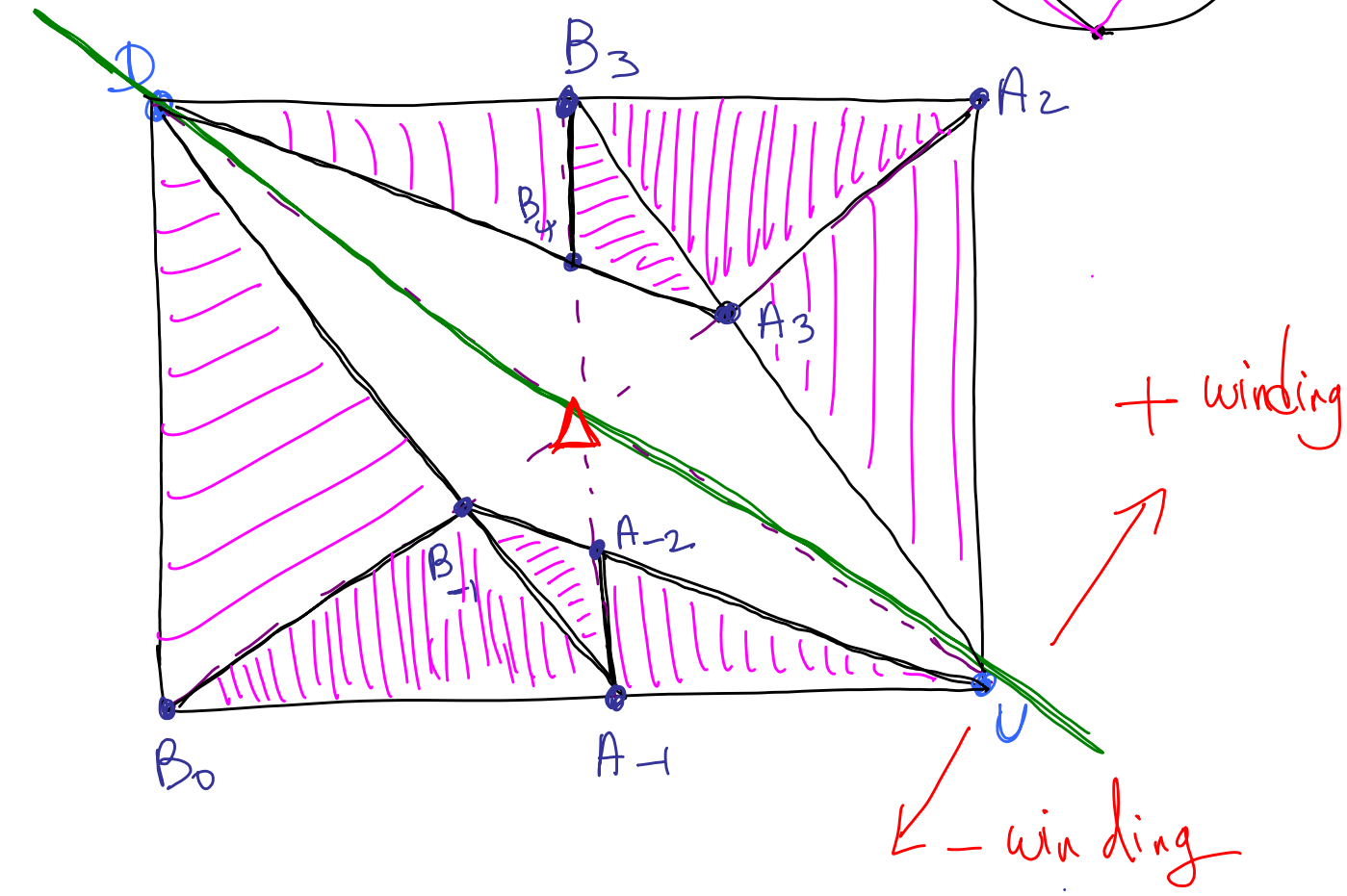


(2-dim projection of 3-d q-vectors)

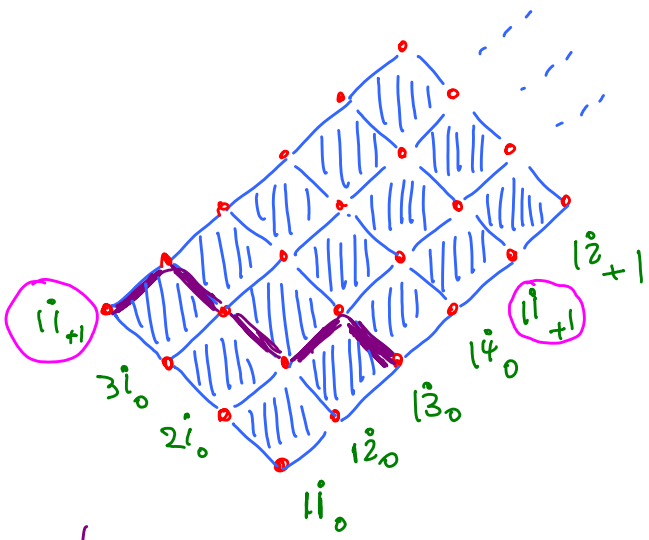
The q-vectors asymptote to Δ

For example in $A_{2,1}$:

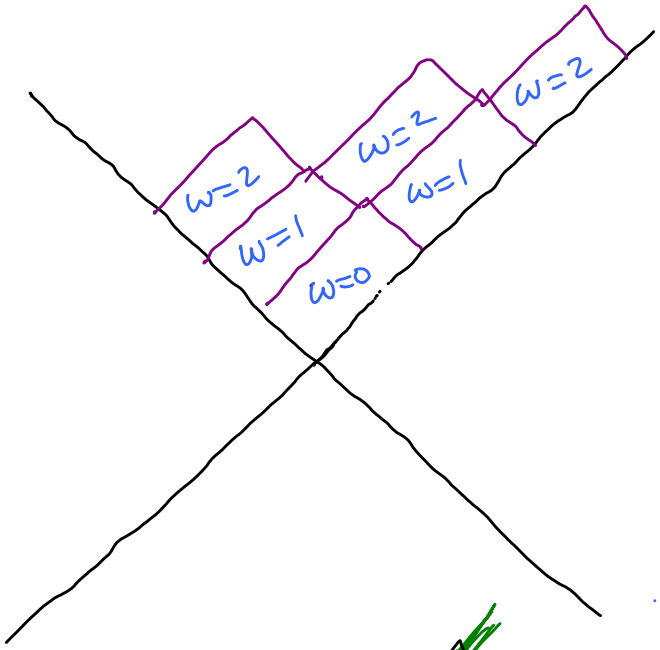


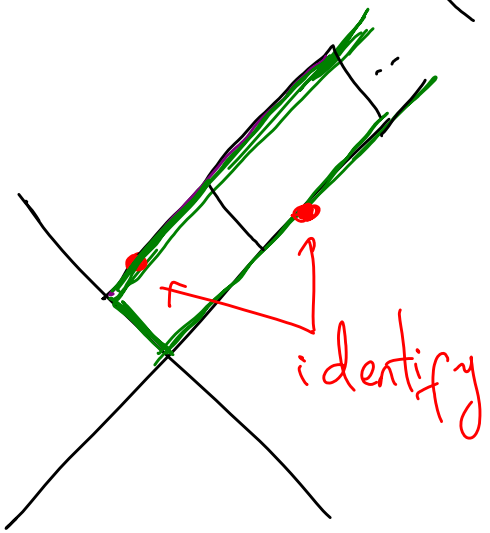


In-Out "Kinematical Spacetime"



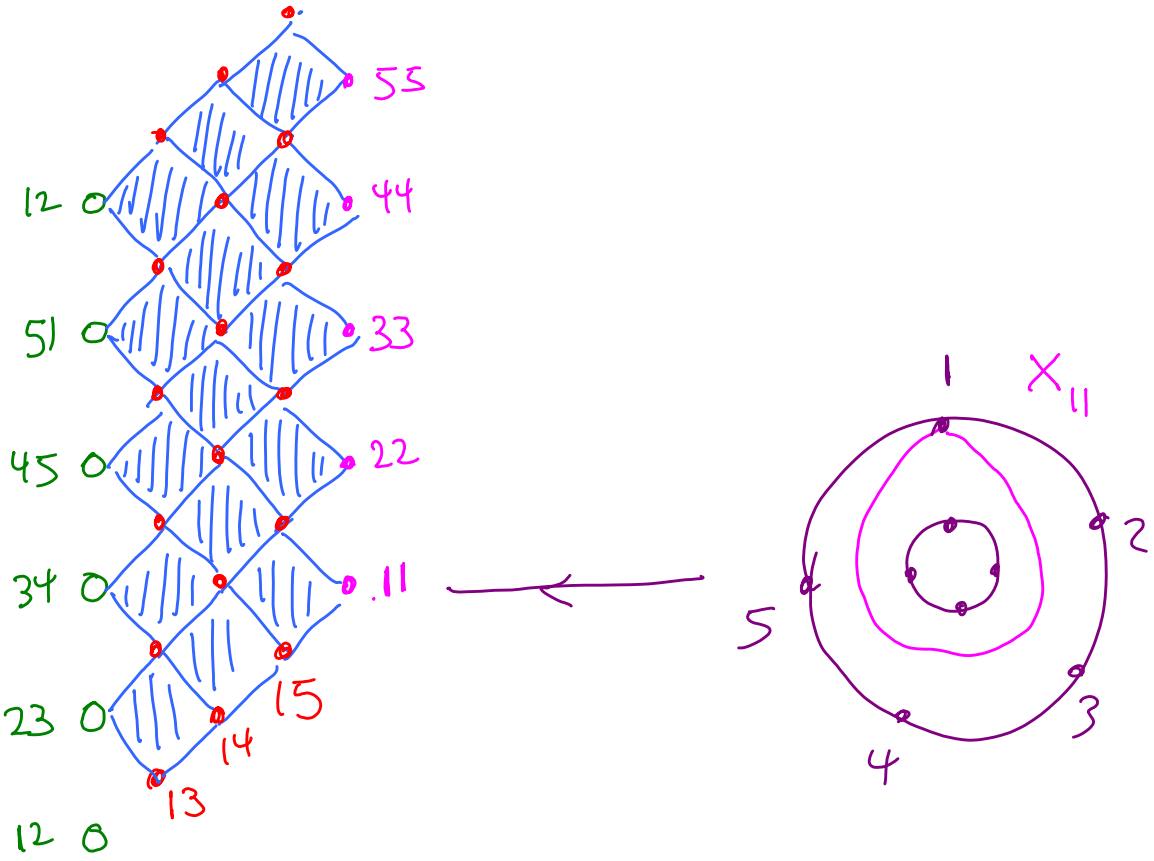
Initial "Cauchy Surface" set by initial cluster



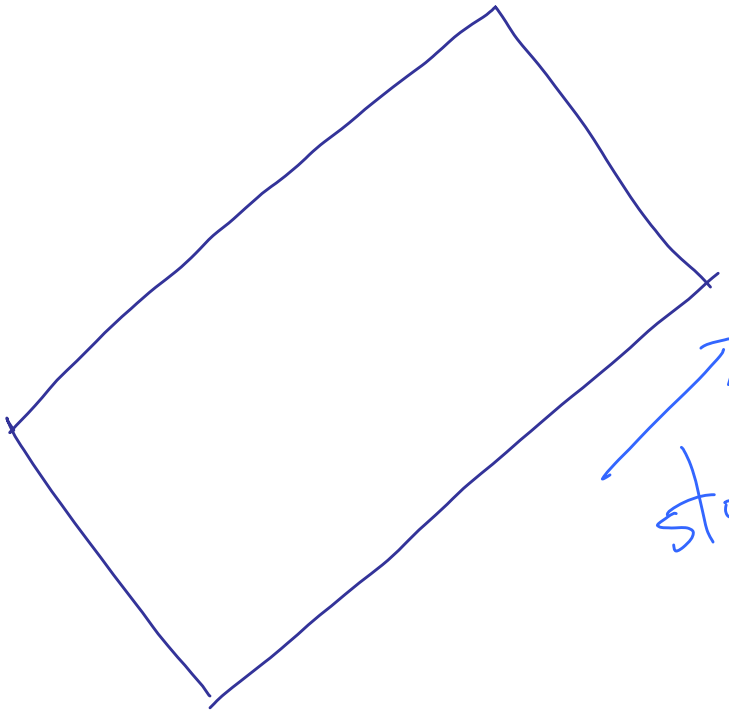


* So, we have an infinite cylinder.

In-In / Out-Out "Kinematic Spacetime"



"Compactifying" winding with Δ



stop @ w^+_{max}

Relation $X_w + X_{w'} - X_{\cdots}$

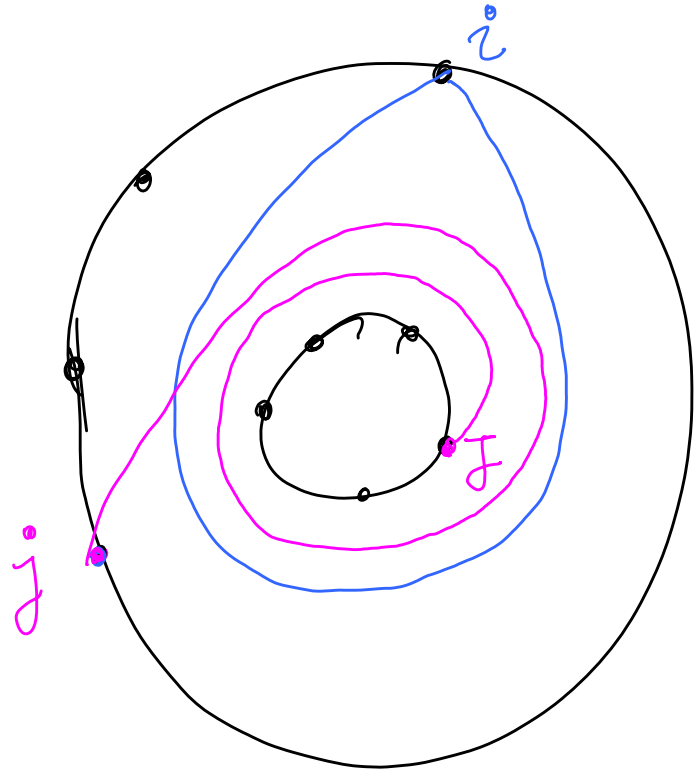
missing!

⇓ Replace with.

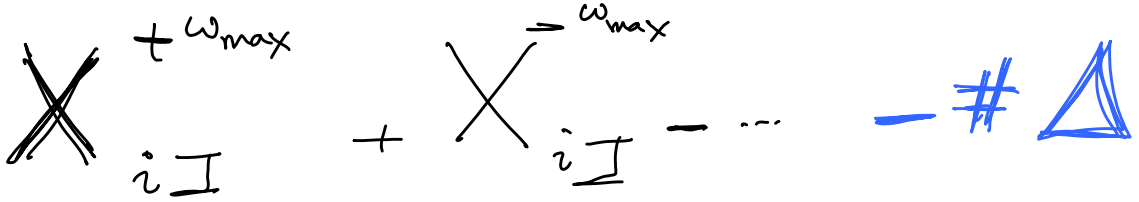
$X_w + \Delta - X_{w+1}$

Gluing "In-In/Out-Out" to "In-Out"

The "mesh" relations are $\left\{ \begin{matrix} X_{ii} \\ X_{II} \end{matrix} + X_{jI}^{\omega_{max}^{+},\omega_{min}^{-}} - \cdots \right\}$

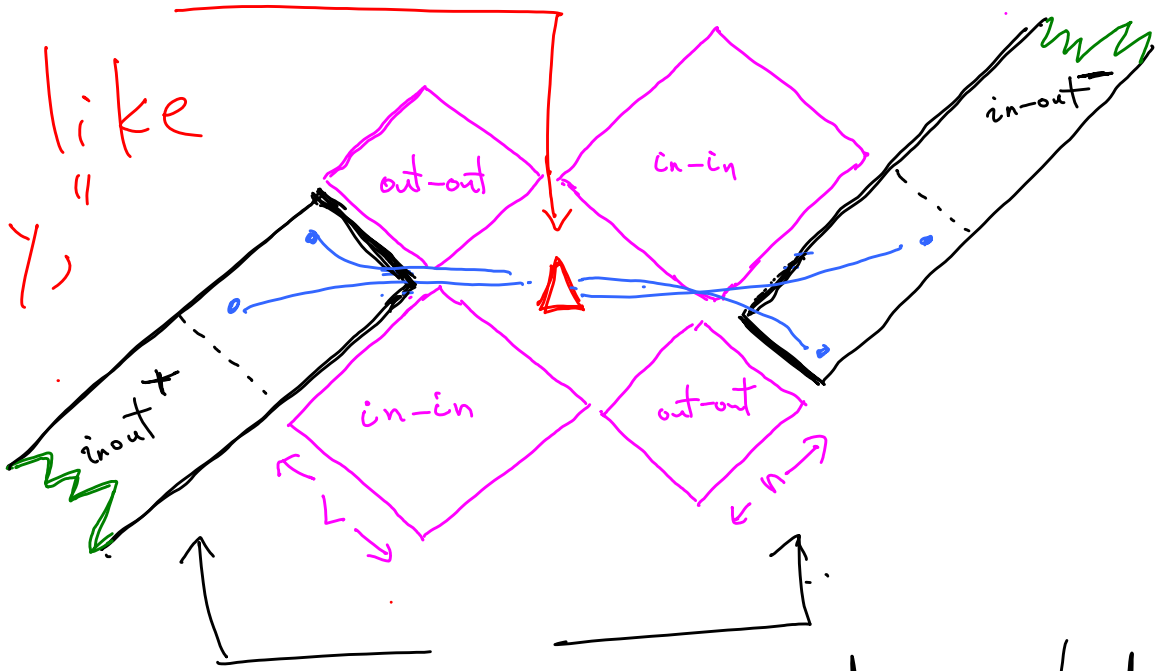


Gluing + to − winding



* Existence of these relations is non-trivial → surface cluster structure telling us to use "Δ" for "compactification" of cluster complex

The Final "Kinematical Spacetime" for $A_{\eta,L}$



Δ is like "gravity," coupling all sectors

The relations $\{R_\mu\}$ of this "Kin. Spacetime" positively generate all the rest!

Affine Clusterhedra.

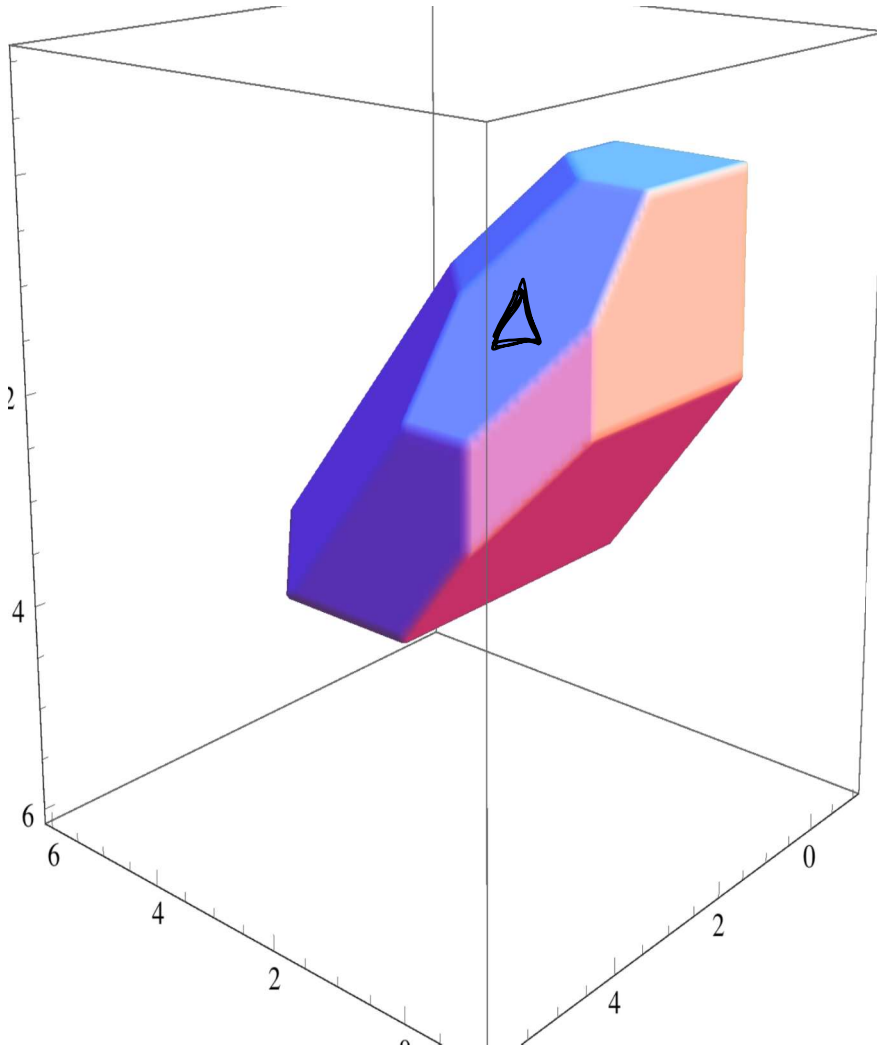
* Exactly the same ideas work for $\tilde{D}_{n+2}$ = 2-loop. So, we have clusterhedra through 2-loops and all L cut of $A_{n,L}$.

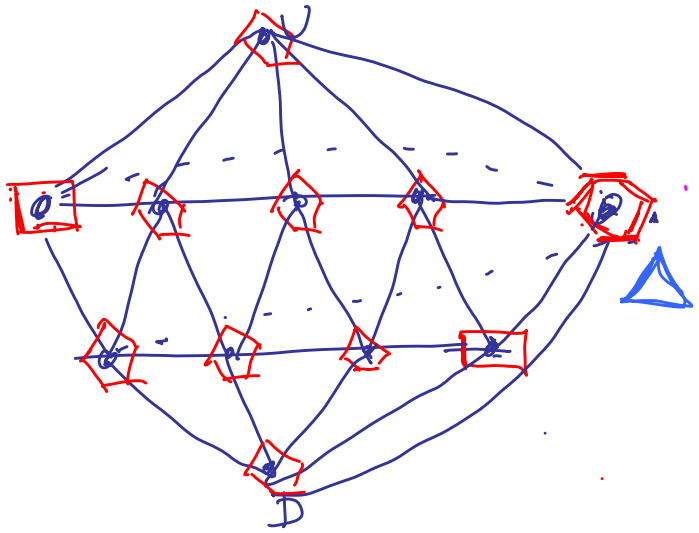
* $\triangle$ is "mutation partner" for every missing var; all vertices $\Longrightarrow$ all vertices unamb. assoc. w/diagrams.

* All non-$\triangle$ vertices carried same # of times

* All normal facet det's involving $\triangle$ are ± 1, get all $+1$'s for amplitude upon pullback

$A_{2,1}$ Clusterhedron





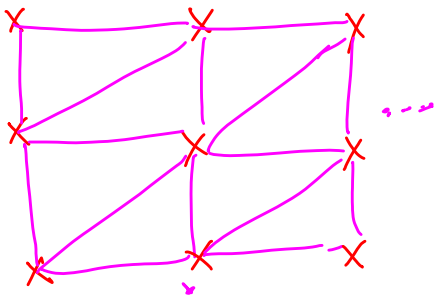
So we are done through 2-loops + some all-loop "cuts". On our way to all loops / all orders in $1/N$.

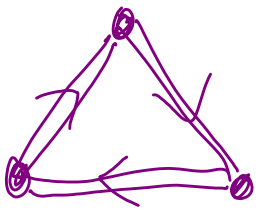
Understand decoupled "Topological Sectors," closed loop "Δ"s should "gravitationally" couple all sectors.

Novelty beyond Affine-Type: Kinematic Space is "Cosmological"

e.g. once punctured torus

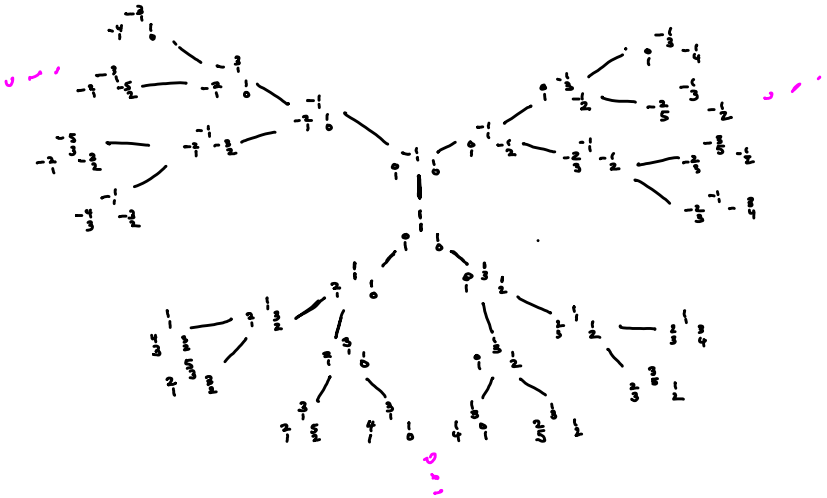
+ $SL(2,\mathbb{Z})$ for gen triang-

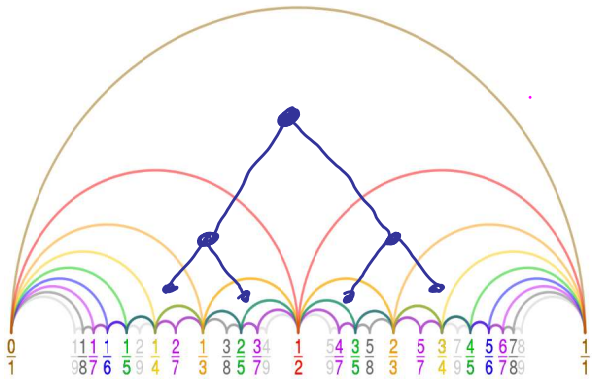


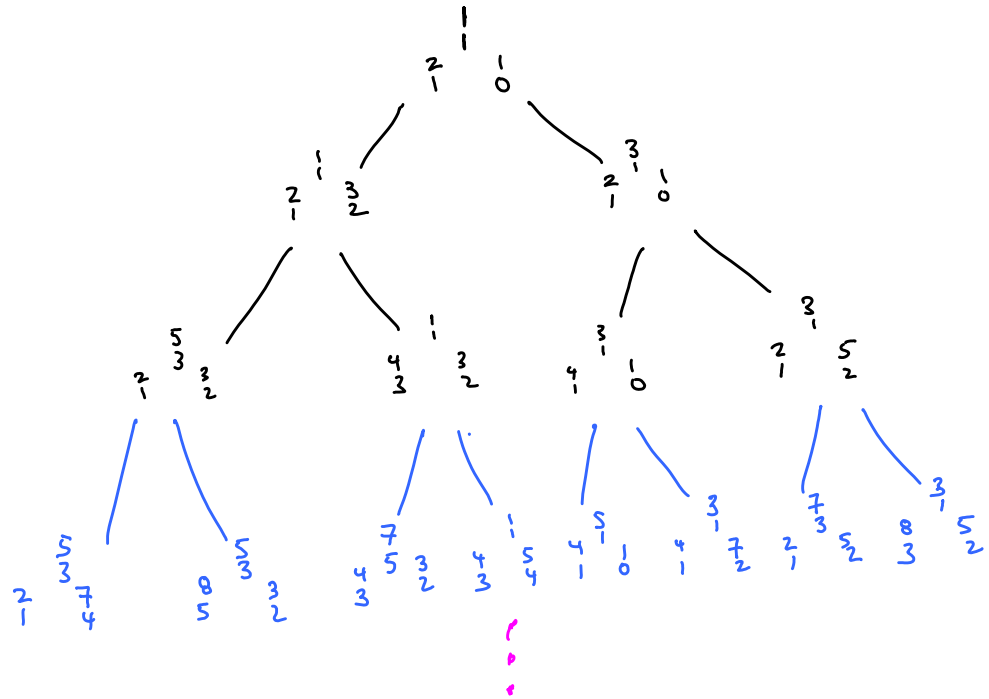


Markov Quiver

- Cluster variables: lines of slope (p/q) connecting punctures
- Δ's: lines of slope (p/q) missing punctures

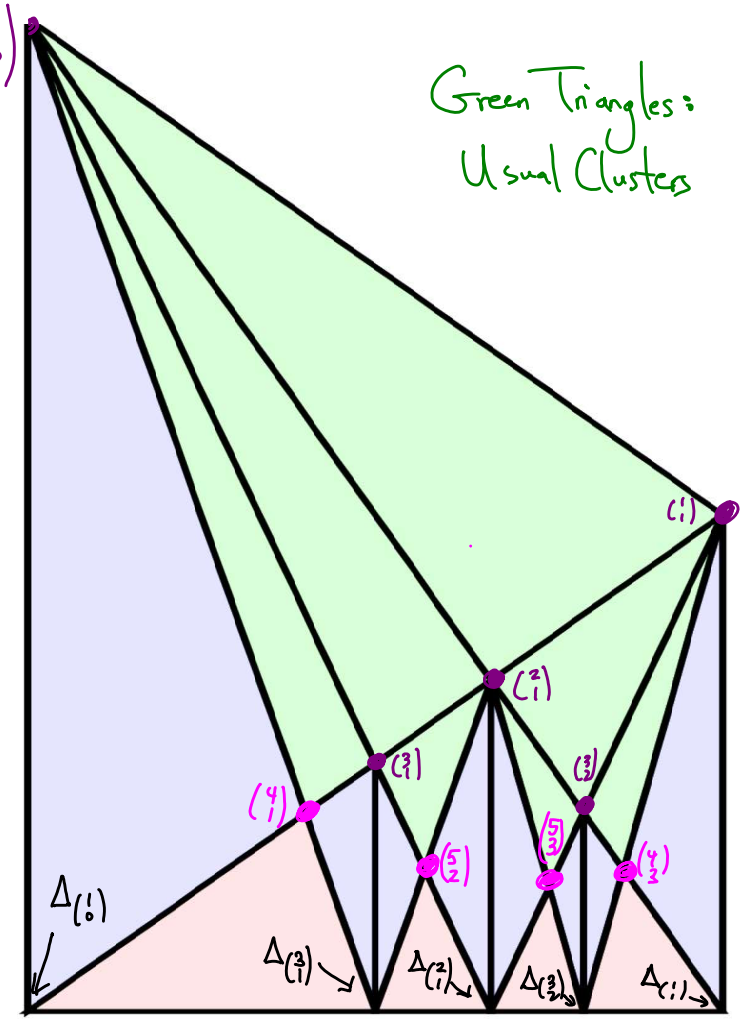






"dS"
in Kin. Space

Markov Clusterhedron (Fan)



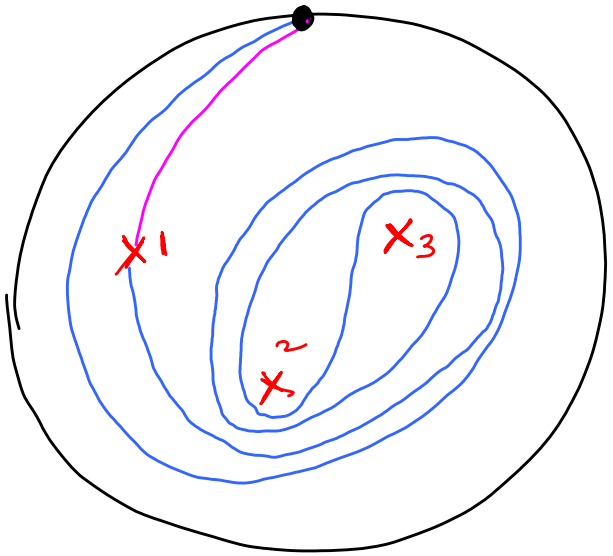
Green Triangles:
Usual Clusters

Blue triangles:
(Δ, g, g)

Red triangles:
$(\Delta, \Delta g)$

Again: Δ's
"compactify",
giving
clusterhedron

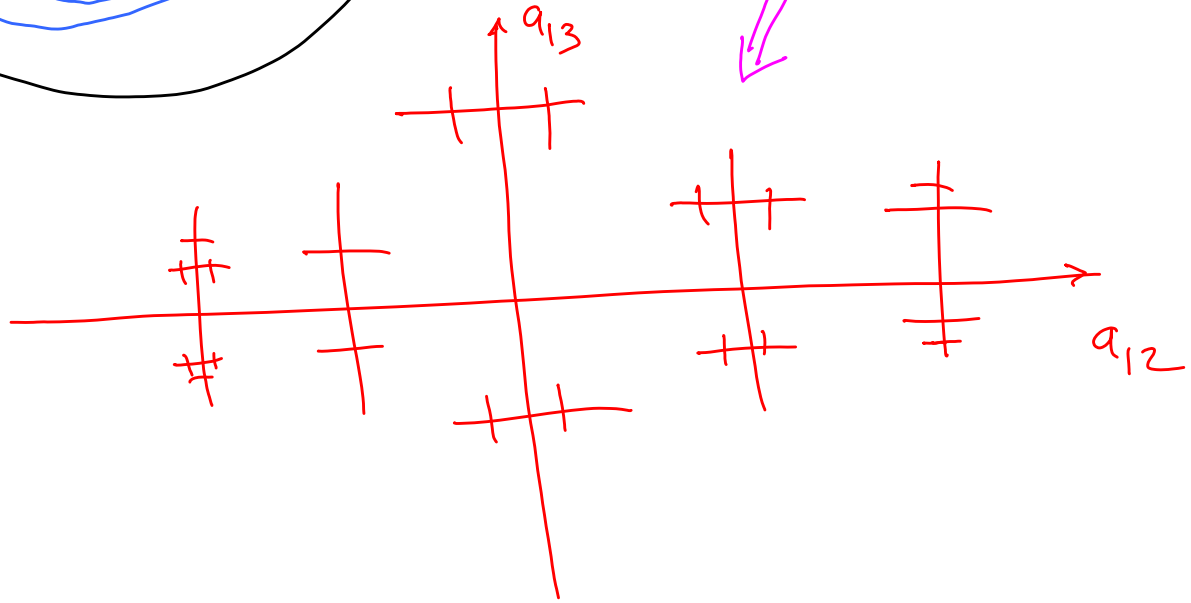
$L \geq 3$ punctures $\longrightarrow$ "Eternal Inflation"



Arcs labelled by

$\text{Braid}_3 / \text{Braid}_2^4$

$\text{Free}_2(a_{12}, a_{13})$



Outlook

* Over the past few years, we have been seeing concrete examples where rules of spacetime + quantum mechanics — locality + unitarity — are seen to emerge from simple new structures in Kinematical space @ infinity

* All-loop planar $\mathcal{N}=4$ SYM $\longrightarrow$ Amplituhedra

* Non-SUSY ϕ^3_{BA}, heading towards all-loop, all orders in $1/N$ $\longrightarrow$ Clusterhedra

* These ideas are directly related to real-world physics in reasonable approximations, expose hidden symmetries, and describe universal features of wide class of theories

* How are they related? Relation to String theory? $\Delta \leftrightarrow$ gravity more than analogy? More general theories?