Koszul duality & twisted holography for asymptotically flat spacetimes

Western Hemisphere Colloquium
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A confluence of progress in a few different subfields
points of contact: symmetry, universality

Twisted holography

Celestial holography

Bootstrap (CFTs, S-matrix,...)

\[ A = A_z d\bar{z} + A_{\bar{w}_1} d\bar{w}_1 + A_{\bar{w}_2} d\bar{w}_2 \]

\[ \int \Omega \wedge CS(A) \]

\[ \eta \in \Omega^{2,1}(M) \]

\[ \frac{1}{2} \int (\bar{\nabla}^{-1} \eta)(\bar{\nabla} \eta) + \frac{1}{6} \int \eta^3 \]

\[ SO^+(1,3) \cong SL(2,\mathbb{C})/\mathbb{Z}_2 \]
Today: we will start to flesh out some of those connections

From this perspective:

- Form factors which reproduce some 4d (YM) scattering amplitudes
- Correlators of a certain 2d chiral algebra

Ultimately, we will obtain a top-down example of (twisted) holography in an asymptotically flat spacetime, using open/closed duality of the topological string
A brief twistor primer

Analogy for 2d:

\[
(z, \bar{z}) \in \mathbb{C}^2 \cong \mathbb{C} \mathbb{M}_2
\]

\[
\frac{\partial^2 \phi}{\partial z \partial \bar{z}} = 0
\]

complex analytic solution

\[
\phi = f(z) + g(\bar{z})
\]

solution to 2d wave equation on \(\mathbb{M}_2\)

solution to Laplace equation on \(\mathbb{E}_2\)

Further:

Fix \(\phi, \nabla \phi\) on \(S \subset \mathbb{C} \mathbb{M}_2\), \(D(S) = \{(z, \bar{z}) : \text{both null lines through } (z, \bar{z}) \text{ meet } S\}\)

\[
S \subset \mathbb{E}_2
\]

\[
D(S) \subset \mathbb{C} \mathbb{M}_2 \text{ to which } \phi \text{ can be analytically extended}
\]

\[
S \subset \Sigma
\]

\[
D(S) \subset \mathbb{M}_2 \text{ usual domain of dpdce}
\]
Twistor space is this gadget for four dimensions

dpds on \textit{conformal structure} \{ 
\begin{itemize}
  \item In 2d space w/ inner product, choice of orientation \rightarrow\text{ unique complex structure}
  \item Every harmonic function on $\mathbb{E}_2$ the real part of a holomorphic function
  \item In 4d, orientation + conformal structure don't pick out a unique complex structure
\end{itemize}

$x \in \mathbb{E}_4$

\begin{align*}
S^2_{x, \pm} \\
\mathbb{P} \mathbb{T} \cong \mathbb{R}^4 \times \mathbb{C} \mathbb{P}^1 \cong \mathcal{O}(1) \oplus \mathcal{O}(1)
\end{align*}

Penrose transform $H^{0,1}(\mathbb{P} \mathbb{T}, \mathcal{O}(2h - 2))$

hol'c massless fields on $\mathbb{C}^4$

solutions to massless field equations

harmonic functions

\[z \in \mathbb{C} \mathbb{P}^1 \quad \mathbb{C} \mathbb{P}^1\]

[Penrose, Atiyah, Ward, Wodehouse, Mason, Lebrun,...]
Twistor space is good for:

- computing classical solutions to nonlinear (massless) field equations
- making symmetries manifest (twistor gauge transformations ARE chiral mode algebra)
- computing amplitudes (esp. integrands, so we don't have to worry about symmetry-breaking regulators)

Spinor helicity variables

\[ P^\mu, (P^0)^2 - (P^1)^2 - (P^2)^2 - (P^3)^2 = 0 \]

\[ P^{a\tilde{a}} = \begin{pmatrix} P^0 + P^3 & P^1 - iP^2 \\ P^1 + iP^2 & P^0 - P^3 \end{pmatrix} =: \lambda^\alpha \tilde{\lambda}^{\tilde{\alpha}} \]

\[ \lambda^\alpha = \begin{pmatrix} 1, P^1 + iP^2 \\ P^0 + P^3 \end{pmatrix} \equiv (1, z) \]

incidence relation:

\[ Z^A = (\mu^{\dot{a}}, \lambda^\alpha) \]

\[ \mu^{\dot{a}} = x^{\alpha\dot{\alpha}} \lambda^\alpha \]

hol'c coord. of celestial sphere, “space” of null momenta

2d OPE limits ↔ 4d collinear limits

\[ S^2 \simeq \mathbb{CP}^1 \]
Today I’d like to discuss work on connections between 4d physics and 2d chiral algebras.

In work with Costello (2204.05301, 2201.02595), we showed that if a 4d theory admits a lift to a local holomorphic theory on twistor space, a chiral algebra can also control collinear singularities in its scattering amplitudes at loop-level.

Failures of associativity in the chiral algebra at the quantum level are tied to gauge anomalies in twistor space. The 4d theory isn’t inconsistent: this is like an obstruction to integrability.
We focused on self-dual Yang-Mills, coupled to an axion with a quartic kinetic term. Similar considerations apply to self-dual gravity, or to, e.g., SD SU(Nc) YM w/ Nf=Nc flavors.

\[
\int_{\mathbb{P}T} \text{Tr}(\mathcal{B} \mathcal{F}^{(0,2)}(\mathcal{A})) \mapsto \int_{\mathbb{R}^4} \text{Tr}(BF(A))
\]

\[
\mathcal{B} \in \Omega^{3,1}(\mathbb{P}T, \mathfrak{g}) \quad B \in \Omega^2(\mathbb{R}^4, \mathfrak{g})
\]

\[
\mathcal{A} \in \Omega^{0,1}(\mathbb{P}T, \mathfrak{g})
\]

\[
g = su(2), su(3), so(8), e_{6,7,8}
\]

\[
\frac{1}{2} \int (\partial^{-1} \eta)(\tilde{\partial} \eta) + k\hat{\lambda}_g \int \eta \text{tr} (\mathcal{A} \partial \mathcal{A}) \mapsto \frac{1}{2} \int (\Delta \rho)^2 + k'\hat{\lambda}_g \int \rho (F \wedge F)
\]

6d: free “closed string” (BCOV) sector
**Self-dual YM**

\[ A \quad B \]

+ −

form factors:

\[ \text{tr}(B^2)(x) \]

L loops, N insertions → N-L+1 (-) helicity, arbitrary (+) helicity gluons in QCD (integrand)

+ axion

\[ + \]

\[ \rho \]

effectively 1-loop by Green-Schwarz

4d OPE:

\[
\text{Tr}B^2(0)\text{Tr}B^2(x_1)\ldots\text{Tr}B^2(x_{n-1}) \sim \sum_{i} F_i(x_1, \ldots, x_{n-1}) \mathcal{O}^i(0)
\]

rational, constrained by associativity

\[
\text{tr}(B^2)(0) \text{tr}(B^2)(x) \sim \frac{1}{\|x\|^2} B_{\alpha_1\beta_1}^a B_{\alpha_2\beta_2}^b B_{\alpha_3\beta_3}^c f_{abc} e^{\beta_1} e^{\beta_2} e^{\beta_3} e^{\alpha_1}.
\]
The associated chiral algebra (conformally soft modes on celestial sphere, governing collinear singularities) can be obtained from Koszul duality approaches on twistor space.

\[ \mathcal{J}[r, s](z_i) \leftrightarrow \mathcal{A} = \delta_{z=z_i}(\tilde{\lambda}^1)^r(\tilde{\lambda}^2)^s \]

This is a very large, non-unitary algebra.

\[ \mathbb{CP}^1 \]

Conformal primary states on twistor space of neg. weight (on-shell gauge theory states) are related to the on-shell background field localized on \( \mathbb{CP}^1 \).

<table>
<thead>
<tr>
<th>Generator (J[m, n], m, n \geq 0)</th>
<th>Spin (1 - (m + n)/2)</th>
<th>Weight ((m - n)/2)</th>
<th>(SU(2)_+) representation ((m + n)/2)</th>
<th>Field (A)</th>
<th>Dimension (-m - n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{J}[m, n], m, n \geq 0)</td>
<td>(-1 - (m + n)/2)</td>
<td>((m - n)/2)</td>
<td>((m + n)/2)</td>
<td>(B)</td>
<td>(-m - n - 2)</td>
</tr>
<tr>
<td>(E[m, n], m + n &gt; 0)</td>
<td>(- (m + n)/2)</td>
<td>((m - n)/2)</td>
<td>((m + n)/2)</td>
<td>(\rho)</td>
<td>(-m - n)</td>
</tr>
<tr>
<td>(F[m, n], m, n \geq 0)</td>
<td>(- (m + n)/2)</td>
<td>((m - n)/2)</td>
<td>((m + n)/2)</td>
<td>(\rho)</td>
<td>(-m - n - 2)</td>
</tr>
</tbody>
</table>

Table 1: The generators of our 2d chiral algebra and their quantum numbers. Dimension refers to the charge under scaling of \( \mathbb{R}^4 \).
There is a prescription for obtaining the OPEs, not just classically but including possible deformations:

\[
P\text{Exp} \sum_{r,s \geq 0} \int_{\mathbb{CP}^1_z} (\partial^r \lambda^{\lambda_1} \lambda^{\lambda_2} \bar{B}_{z}^a) \tilde{\mathcal{J}}_a[r, s](z)
\]

\[
P\text{Exp} \sum_{r,s \geq 0} \int_{\mathbb{CP}^1_z} (\partial^r \lambda^{\lambda_1} \lambda^{\lambda_2} \bar{A}_{z}^a) J_a[r, s](z)
\]

gauge inv't couplings to arbitrary defect
→ Hom from Koszul dual algebra into defect algebra

Quantum deformations:

OPEs among currents on defect by imposing gauge (BV-BRST) invariance

→ universal or “Koszul dual” algebra
Tree level: recover current algebra for gauge symmetry

BRST variation of all diagrams at given loop order must be zero
A brief interlude on Koszul duality
(for associative algebras)

for more details, see NMP-Williams: 2110:10257
Koszul duality: Coupling an order-type defect to a Lagrangian theory
Focus on line defects

\[ S = S_A + S_B + S_{AB} \]

generalized Wilson line

What are the constraints on \( S_{AB} \) (actually PExp) imposed by gauge (BRST) invariance?

\[ S_{AB} \leftrightarrow \alpha \in \text{MC}(\mathcal{A} \otimes \mathcal{B}) \]

Koszul dual \( \alpha : \mathcal{A}^! \rightarrow \mathcal{B} \)

\[ \delta \alpha + \alpha \star \alpha = 0 \]

Claim: \( \mathcal{A}^! \) is the algebra of operators on the “universal line defect” for theory A
The concrete setting: twisted theories

Begin w/ a supersymmetric field theory on flat space, Euclidean signature

General twisted theories: \( \delta \rightarrow \delta + Q =: \delta_Q \)

\[ Q^2 = 0 \]
choice of nilpotent supercharge

Compute cohomology with respect to the twisted BRST differential

cf.
\[ \phi : \text{Spin}(d) \rightarrow G_R \]
\[ T^\phi = \{ Q, \ldots \} \]
to obtain topological field theories
\[ \rightarrow E_d\text{-algebras} \]

\[ \left\{ Q, Q_\mu \right\} = iP_\mu \]

\begin{itemize}
  \item A- and B-twist of 2d \( \mathcal{N} = (2,2) \) theories
  \item Half-twist of 2d \( \mathcal{N} = (0,2) \) theories (CDOs, cdR,..)
  \item Holomorphic twist of 4d \( \mathcal{N} = 2 \) theories (chiral algebra)
  \item Holomorphic-topological twist of 3d \( \mathcal{N} = 2 \) theories
  \item \( \vdots \)
\end{itemize}
$E_1/A_\infty$-algebra: topological theory in 1d
$L_\infty, E_2$-algebra: topological theory in 2d
vertex/chiral algebra: holomorphic theory in 2d

- higher brackets/products on algebras of local ops
- generate couplings/deformations

\[
\left\{ Q, Q_\mu \right\} = iP_\mu
\]

\[
\mathcal{O}^{(k)} := \frac{1}{k!} Q_{\mu_1}(\ldots Q_{\mu_k}(\mathcal{O})) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}
\]

\[
Q(\mathcal{O}^{(k)}) = d\mathcal{O}^{(k-1)} \rightarrow \int_{\gamma_k} \mathcal{O}^{(k)}
\]

is Q-closed for $\gamma_k$ a cycle

$\mathcal{O}^{(k)}$ is Q-closed for $\gamma_k$ a cycle

[Beem-Ben-Zvi-Bullimore-Dimofte-Neitzke]

[Hol’c-top’l: [Yagi, Costello-Dimofte-Gaiotto]]
The Maurer-Cartan equation & bulk/defect couplings

\[ \left\{ Q, \hat{Q} \right\} = iP_t \]

translations along \( \mathbb{R}_t \) are cohomologically trivial

\[ \therefore \mathcal{A} \] (homotopy) associative algebra

Add in to a top'l QM, thy B with op alg \( \mathcal{B} \) (homotopy associative)

\[ S_A + S_B \]

Claim: This "generalized Wilson line"/bulk-defect coupling is BRST invariant to all orders in perturbation theory iff

\[ \delta_Q \mathcal{O} + \mathcal{O} \star \mathcal{O} = 0 \] [Costello-Paquette, Paquette-Williams]

\[ \delta_Q \mathcal{O} + \mu_2(\mathcal{O}, \mathcal{O}) + \mu_3(\mathcal{O}^3) + \ldots = 0 \] [Gaiotto-Oh]
Koszul Duality & the Universal Top’l Line Defect

\[ \alpha \in \text{MC}(\mathcal{A} \otimes \mathcal{B}) \]
\[ \text{s.t.} \]
\[ (\epsilon \otimes 1_{\mathcal{B}})\alpha = 0 \in \mathcal{B} \]

\[ \phi : \mathcal{A}^! \to \mathcal{B} \]

\[ \mathcal{A}^! := \mathbb{R}\text{Hom}_{\mathcal{A}}(\mathbb{C}_{\epsilon}, \mathbb{C}_{\epsilon}) \]

\[ \epsilon : \mathcal{A} \to \mathbb{C} \]
\[ \text{augmentation} \]
\[ \text{(choice of massive vacuum} \]
\[ \text{s.t. compactified thy trivial)} \]

\[ \mathbb{R}_t \times M_\epsilon \to \mathbb{R}_t \]

Require coupling preserves vacuum

\[ \alpha \in g^* \otimes \mathcal{B} \subset C^*(g) \otimes \mathcal{B}[1] \]

\[ \alpha_{\text{univ}} \in \mathcal{A} \otimes \mathcal{A}^! \]
\[ \alpha = \tilde{\phi}_\alpha(\alpha_{\text{univ}}) \]

any other coupling can be obtained from the universal coupling by application of an algebra-preserving map \[ \tilde{\phi}_\alpha : \mathcal{A}^! \to \mathcal{B} \]

\[ g \xrightarrow{\phi_\alpha} \mathcal{B} \]
\[ U(g) \]

example commuting diagram
Curved Koszul duality & (Twisted) AdS/CFT

For vertex algebras: no math formalism, but proceed via BRST invariance of order-type holomorphic defects

D-branes, even twisted, can **backreact** on the geometry. In at least some special examples, one can fully incorporate this effect.

e.g. B-model string theory:
- holomorphic CS theory
- Kodaira-Spencer “supergravity”

\[ S \rightarrow S + S_{BR} \]
\[ \delta_{gauge} S_{BR} \neq 0 \text{ on } \mathbb{C}z \]

\[ S_{BR} = \frac{1}{2} \int_{\mathbb{C}^3} \mu \mu_{BR} \mu dz dw_1 dw_2 \sim N \int_{\mathbb{C}^3} \frac{e^{i\ell \bar{w}_i dw_j}}{(w_1 \bar{w}_1 + w_2 \bar{w}_2)^2} \mu \partial_\mu dz dw_1 dw_2 \]

- coupling to identity element of vertex algebra
- total BRST variation, including this contribution, must cancel
- curved Koszul duality (source term in MC equation)

In classical (\( \mathcal{N} \rightarrow \infty \)) limit, this was used to compute central extensions of boundary chiral algebra in an example of twisted holography (AdS3/CFT2)

Loop corrections for these theories are in progress [Costello-Paquette-Williams]

see also [Costello, Costello-Li, Costello-Gaiotto, …]
End of Interlude
A more bottom-up perspective: the OPEs are related to 4d collinear singularities, which are known in detail in Yang-Mills.

Compute the associator (say at tree or 1-loop order) and see if it vanishes!

\[ J^a[r, s](0)J^b[t, u](z) \sim \frac{1}{z} f^{ab}_{c} J^c[r + t, s + u](0) \]
\[ J^a[r, s](0)\tilde{J}^b[t, u](z) \sim \frac{1}{z} f^{ab}_{c} \tilde{J}^c[r + t, s + u](0) \]

**Tree-level**

\[ J^a[r, s](0)E[t, u](z) \sim \frac{1}{z} \left( ts - ur \right) J^a[t + r - 1, s + u - 1](0) \]
\[ J^a[r, s](0)F[t, u](z) \sim -\frac{1}{z} \partial_z J^a[r + t, s + u](0) - \frac{1}{z^2} \left( 1 + \frac{r + s}{t + u + 2} \right) J^a[r + t, s + u](0) \]
\[ J^a[r, s](0)J^b[t, u](z) \sim \frac{1}{z} K^{ab}(ru - st) F[r + t - 1, s + u - 1](0) \]
\[ + \frac{1}{z^2} K^{ab}(r + s + t + u) E[r + t, s + u](0) - \frac{1}{z^2} K^{ab}(r + s + t + u) E[r + t, s + u](0). \]

**Failure of associativity in pure SDYM theory in one-loop**

Axion field necessary for its restoration

\[ J_a[1, 0](0)J_b[0, 1](z) \]
\[ = -\frac{1}{2\pi i z} CK f^e_c (f^{ab}_{ae} f^{d}_{bf} + f^{d}_{ae} f^{c}_{bf}) : J_c[0, 0] \tilde{J}_d[0, 0] : \]

Quantum deformation

\[ + \frac{1}{2\pi i z} D f^e_c \partial_z \tilde{J}_c(0) + \frac{1}{2\pi i z^2} D f^e_c \tilde{J}_c(0). \]

**Includes level-o Kac-Moody algebra for Maps(\(\mathbb{C}^2, g\))**

[Bern-Dixon-Kosower]

[Split^{[1]}(a^+, b^+) = - \frac{N_c}{96\pi^2} \frac{[ab]}{\langle ab \rangle^2}]

C, D are known & fixed by anomaly coefficient in 6d

[Bern-Dixon-Kosower]

[Maps(\(\mathbb{C}^2, g\))]

[Bern-Dixon-Kosower]
For self-dual YM, no further collinear singularities at higher loops. Costello and I further showed that a 4d theory with a twistorial uplift has form factors which are isomorphic to chiral correlators of the 2d theory. In the case of Yang-Mills, this leads to some cute 2d expressions for certain 4d amplitudes.

\[
\langle tr(B^2) | \tilde{J}^a(z_1) \tilde{J}^b(z_2) J^c(z_3) \rangle = \frac{z_{12}^3}{z_{13}z_{23}} f^{abc}
\]

\[
P^{a\tilde{a}} =: \lambda^a \tilde{\lambda}^\tilde{a}
\]

\[
\lambda^a \equiv (1, z)
\]

Momentum eigenbasis:

\[
\langle ij \rangle = z_i - z_j
\]

\[
J(\tilde{\lambda}, z) = \sum_{r,s} \omega^{r+s} (\tilde{\lambda}^1)^r (\tilde{\lambda}^2)^s \frac{r! s!}{r! s!} J[r, s](z)
\]

\[
\langle tr(B^2) | J^{a_1}(z_1) \cdots \tilde{J}^{a_2}(z_i) \cdots \tilde{J}^{a_j}(z_j) \cdots J^{a_n}(z_n) \rangle = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} tr(t^{a_1} \cdots t^{a_n}) + \text{permutations}
\]
4d form factors as computed by 2d chiral correlators

\[ P^{\alpha\bar{\alpha}} =: \lambda^\alpha \tilde{\lambda}^{\bar{\alpha}} \]
\[ \lambda^\alpha \equiv (1, z) \]

**Tree-level**

\[ \langle tr(B^2) | \tilde{J}(z_1) \tilde{J}(z_2) J(z_3) \ldots J(z_n) \rangle \leftrightarrow \text{color ordered MHV amps} \quad \text{[Parke-Taylor]} \]

\[ \frac{1}{|x|^2} \langle tr(B^3) | \tilde{J}(z_1) \tilde{J}(z_2) \tilde{J}(z_3) J(z_4) \ldots J(z_n) \rangle \leftrightarrow \text{NMHV amps in CSW form} \quad \text{[Cachazo-Svrcek-Witten]} \]

**1-loop (axion comes in)**

\[ \langle (\Delta \rho)^2 | J_{a_1}(\tilde{\lambda}_1, z_1) \ldots J_{a_n}(\tilde{\lambda}_n, z_n) \rangle \leftrightarrow \text{all-(+)} \text{ one-loop in SDYM/QCD} \quad \text{[Mahlon, Bern et. al.,...]} \]

\[ \langle tr(B^2) | \tilde{J}_{a_1}(\tilde{\lambda}_1, z_n) J_{a_2}(\tilde{\lambda}_2, z_2) \ldots J_{a_n}(\tilde{\lambda}_n, z_n) \rangle = \]
\[ \frac{1}{192\pi^2} \sum_{2 \leq i < j \leq n} \frac{[ij] \langle 1i \rangle^2 \langle 1j \rangle^2}{\langle ij \rangle \langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \text{Tr}(t_1 \ldots t_n) + \text{perms in } S_{n-1} \quad \text{[Costello-NP]} \]

\[ \text{cf. [Dixon-Glover-Khoze] for QCD} \]
SDYM + axion isn’t the only 4d theory with a nice twistorial uplift…
Self-dual gravity version of the previous analysis, see recent work by Bittleston
WZW4 with $G = \text{SO}(8) + \text{quartic “Kahler scalar” field}$
Costello-Li show this follows from a topological string analogue of Green-Schwarz mechanism for type I string

Again, a special, integrable 4d theory. But I claim this is the seed for a nice toy model of holography in asymptotically flat spacetimes!
(Work w/ Costello & Sharma, 2208.14233 + to appear)

Recall [Donaldson, Nair, Losev-Moore-Nekrasov-Shatashvili]:

$$g : M \to \text{SO}(8)$$

$$\mathcal{L} = \frac{N}{8\pi^2} \int_M \partial \bar{\partial} K \wedge \text{tr}(g^{-1} \partial g \wedge g^{-1} \bar{\partial} g) - \frac{N}{24\pi^2} \int_{M \times [0,1]} \partial \bar{\partial} K \wedge \text{tr}(\bar{g}^{-1} d\bar{g})^3$$

$N \in \mathbb{Z}_+$

4d analogue of KM level

Classically, a gauge-fixed formulation of SDYM w/ $A = - \bar{\partial} gg^{-1}$

``Closed string/gravitational” sector is the theory of a scalar controlling perturbations of the Kahler potential
(See Costello’s Strings 2021 talk)

e.o.m. $R(K + \rho) = 0$ $K \mapsto K + \rho$

Again, a special, integrable 4d theory. But I claim this is the seed for a nice toy model of holography in asymptotically flat spacetimes!
(Work w/ Costello & Sharma, 2208.14233 + to appear)
Let’s use the origin of the 6d anomaly cancellation from topological type I open+closed string theory on twistor space.

Add additional \( N \) D1-branes on top of \( \mathbb{C}P^1 \) and “backreact”, study resulting open/closed duality a la twisted holography.

In type I Kodaira-Spencer theory, “backreaction” is a deformation of complex structures.

\[
Z_0 = \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1
\]
\[
\begin{align*}
&\notag \mu^1 \quad \mu^2 \\
&\notag z
\end{align*}
\]
\[
\begin{align*}
z &\mapsto \frac{1}{z} \implies \mu^\alpha \mapsto \frac{\mu^\alpha}{z}
\end{align*}
\]
\[
\bar{\partial} = d\bar{z} \frac{\partial}{\partial \bar{z}} + d\bar{\mu} \cdot \frac{\partial}{\partial \bar{\mu}}
\]
\[
\bar{\partial}V + \frac{1}{2} [V, V] = (2\pi)^2 N \delta^2(\mu) \ z^2 \frac{\partial}{\partial z}
\]

\[
V = N \ \frac{\bar{\mu}^1 d\bar{\mu}^2 - \bar{\mu}^2 d\bar{\mu}^1}{||\mu||^4} \ z^2 \frac{\partial}{\partial z}
\]

\[
\mathcal{L}_V \Omega_0 = 0 \quad \text{away from } \mu^\alpha = 0
\]

\[
\Omega_0 = \frac{dz \ d\mu^1 \ d\mu^2}{z^2}
\]

\[
u^\alpha = (u^1, u^2) \in \mathbb{C}^2, \quad ||u||^2 = |u^1|^2 + |u^2|^2
\]

\[ds^2 = ||du||^2 + \frac{|u^1 du^2 - u^2 du^1|}{||du||^4}\]

4d: Burns metric
Moreover, worldvolume theories on D1-branes in the topological string are well known. Proceeding a little more carefully and putting the pieces together ultimately gives us:

Note: Burns space is asymptotically flat

\[ \mathbb{C}P^1 \]

\( N \) D1 branes in twistor space

\[ \mathcal{B}_N \]

Burns space

4d, Euclidean signature, asymptotically flat

Large \( N \) duality

\( \beta \gamma \) system with

\( \text{Sp}(N) \) gauge symmetry

+ \( \text{SO}(8) \) flavour

+ defects at 0, \( \infty \)

WZW4 for \( G=\text{SO}(8) \)

+ self-dual scalar-flat gravity

Here: dual chiral algebra understood at finite \( N \)

In principle, exact description of collinear limits of 4d theory at finite coupling, from 2d chiral algebra at finite \( N \)
To start, we have checked 2 & 3-pt funs in this proposed duality when $N \to \infty$
For example, matching states between part of chiral algebra to the WZW4 sector in the bulk:

Chiral worldvolume actions follow from Witten's prescriptions. [Witten '95]

$N$ D1 branes at $\mu^x = 0$ 4 space-filling ``D5'' branes (+ O-plane)

$g = \exp(\phi t)$

linearized field eqn

$\Delta \phi = 0$

closed form sol'n for "momentum eigenstates"

(Recovers $e^{ik \cdot x}$ when $N = 0$)
• Dictionary between soft modes of states and symmetry currents in the dual CFT

\[ \tilde{\lambda}_\alpha = \omega(1, \tilde{z}) \]

Soft expansion

\[ \phi(\omega, z, \tilde{z}) = \frac{1}{z} \sum_{p=0}^{\infty} (i \omega)^p \sum_{k+l=p} \frac{\tilde{z}^l}{k! l!} \phi[k, l](z) \]

Dictionary

\[ \frac{1}{z} \phi[k, l](z) \quad t_{rs} \quad \langle I_r, X_1^{(k} X_2^{l)} I_s \rangle(z) \]

Examples of soft modes

\[ \phi[0, 0] = 1, \quad \phi[1, 0] = u^1 - z \bar{u}^2, \quad \phi[0, 1] = u^2 + z \bar{u}^1 \]

\[ \phi[1, 1] = \phi[0, 1] \phi[1, 0] + \frac{N z}{2} \frac{|u|^2 - |w|^2}{|u|^2 + |w|^2} \]

• chiral algebra OPE computations easily done in planar limit by Wick contractions

• In bulk, Euclidean amplitudes computed via on-shell effective action, as in standard AdS/CFT computations

\[
J_a[\tilde{\lambda}_1](z_1) J_b[\tilde{\lambda}_2](z_2) \sim \frac{f_{ab}^c}{z_{12}^2} J_c[\tilde{\lambda}_1 + \tilde{\lambda}_2](z_2) \\
- \frac{[1 \, 2]}{z_{12}^2} \int_0^1 d\omega_1 \int_0^1 d\omega_2 \, J_c[\omega_1 \tilde{\lambda}_1 + \omega_2 \tilde{\lambda}_2](z_2) \\
\phi_1 \cdot \phi_2 \sim \frac{f_{a_1 a_2}^c}{z_{12}^2} \phi_c(z_2, \tilde{\lambda}_1 + \tilde{\lambda}_2) \\
- \frac{[1 \, 2]}{z_{12}^2} \int_0^1 d\omega_1 \int_0^1 d\omega_2 \, \phi_c(z_2, \omega_1 \tilde{\lambda}_1 + \omega_2 \tilde{\lambda}_2) + O([1 \, 2]^2). \]

Our toy top-down example of asymptotically-flat holography has passed standard checks.
Next: Go beyond the planar limit, study states of dim’n \( \Theta(\sqrt{N}), \Theta(N) \), fully flesh out embedding in physical string, etc.
This gives a concrete toy-model of a "celestial holography"-type correspondence, analogous to the supersymmetric sectors of AdS/CFT we have been studying in twisted holography program.

Unfortunately, I only know how to build associative chiral algebras using these methods for theories that are integrable/self-dual in 4d. Relatedly, the closed-string sector of the bulk theory, which only captures Kahler potential fluctuations, means our gravitational sector is rather poor.

Perhaps the first step towards 4d asymptotically flat holography in more physically interesting setups would be to find a chiral algebra dual for self-dual Einstein gravity (perhaps coupled to SDYM).

We know how to cancel the twistorial anomaly there, thanks to work of Bittleston-Sharma-Skinner.

Note that Burns space can be viewed as an Einstein-Maxwell instanton…

non-unitarity, operator product associativity, integrability, etc. are all connected, insight for how to move beyond the twisted realm?
Thank you!
To start, we have checked 2 & 3-pt funs in this proposed duality when $N \rightarrow \infty$

For example, matching states between part of chiral algebra to the WZW4 sector in the bulk:

$$g = \exp(\phi \cdot t)$$
$$\Delta \phi = 0$$

"Momentum eigenstates" 

$$\psi = \sqrt{\frac{1}{([u \tilde{\lambda}] + z[\tilde{u} \tilde{\lambda}]) + \frac{4N}{||u||^2} [u \tilde{\lambda}] [\tilde{u} \tilde{\lambda}]}$$

$N$ D1 branes at $\mu^\alpha = 0$ 4 space-filling "D5" branes (+ O-plane)

Chiral worldvolume actions follow from Witten’s prescriptions. [Witten ’95]
[Costello, Gaiotto ’18]