Mirror symmetry for Langlands dual Higgs bundles at the tip of the nilpotent cone

based on joint work with Nigel Hitchin arxiv: 2101.0853

§1 Mirror symmetry for Higgs bundles

- C complex smooth projective curve of genus $g > 1$
- G complex reductive group (= complexification of compact Liegroup)

  G^L its Langlands dual group, e.g. $GL_n^L = GL_n$, $SL_n^L = PGL_n = SL_n / \mathbb{Z}_n$
- $\mathcal{M}_{Dol}(G)$ moduli of $\underbrace{G\text{- Higgs bundles on } C}_{(E, \Phi)}$

  E principal G-bundle, $\Phi \in H^0(C; ad(E) \otimes K)$

  Higgs field

$h_G : \mathcal{M}_{Dol}(G) \to \mathcal{A}_G \cong \bigoplus_i H^0(C, K^{d_i})$ Hitchin map

Completely integrable Hamiltonian system, gen. fibers $\cong$ Abelian varieties (compact tori)

- $\mathcal{M}_{DR}(G)$ moduli of flat G-connections on C
- $\mathcal{M}_{DR}(G) \cong_{diff} \mathcal{M}_{Dol}(G)$ two complex structures in natural hyperkähler structure on $(\mathcal{M}_{Hit}(G), I, J, K, \omega_I, \omega_J, \omega_K)$

solution space to Hitchin self-duality equations

$\mathcal{M}_{Dol}(G)$ $\mathcal{M}_{DR}(G)$

Three aspects of mirror symmetry for Higgs bundles

1) SYZ (Strominger-Yau-Zaslow) for $\mathcal{M}_{DR}(G)$ & $\mathcal{M}_{DR}(G^L)$

$$\begin{tikzcd}
\mathcal{M}_{DR}(G) \arrow[dr, "h_G"'] & & \mathcal{M}_{DR}(G^L) \arrow[dl, "h_{G^L}"] \\
& \mathcal{A}_G = \mathcal{A}_{G^L} &
\end{tikzcd}$$

for $G = SL_n$ (Hausel-Thaddeus 2002)

for general G (Donagi-Pantev 2012)

such that generic fibers special Lagrangian tori

2) Topological mirror symmetry for $\mathcal{M}_{DR}(G)$ & $\mathcal{M}_{DR}(G^L)$ (Hausel-Thaddeus 2002)

Hodge numbers of $\mathcal{M}_{DR}(SL_n)$ = Hodge numbers of $\mathcal{M}_{DR}(PGL_n)$

— " — " $\mathcal{M}_{Dol}(SL_n)$ = — " — $\mathcal{M}_{Dol}(PGL_n)$

proved by (Gröchenig, Wyss, Ziegler 2020a) using p-adic integration

(— " — 2020b) ⇒ Ngo's cohomological Fundamental Lemma

3) Homological mirror symmetry

(Kapustin-Witten 2006) A-model of $\mathcal{M}_{DR}(G)$ ≅ B-model of $\mathcal{M}_{DR}(G^L)$

$$S: \mathrm{Fuk}(\mathcal{M}_{DR}(G), \omega_J) \simeq D^b_{Coh}(\mathcal{M}_{DR}(G^L), J)$$

(Donagi-Pantev 2012) $S_{sc}: D^b_{Coh}(\mathcal{M}_{Dol}(G)) \cong D^b_{Coh}(\mathcal{M}_{Dol}(G^L))$

"classical limit" of S, generically Fourier-Mukai transform

more generally (KW 2006) ⇒ BAA-branes on $\mathcal{M}_{Hit}(G) \longleftrightarrow$ BBB branes on $\mathcal{M}_{Hit}(G^L)$

e.g. hol. Lagrangian on $(\mathcal{M}_{Dol}(G), \omega_{\mathbb{C}} = \omega_J + i\omega_K)$ $\longleftrightarrow$ hyperholomorphic vector bundle

§ 2. Bialynicki-Birula decomposition for $\mathcal{M}$

§ 2.1 BB decomposition for semiprojective variety

- X (smooth), complex semiprojective $\Leftrightarrow$ $\mathbb{T} := \mathbb{C}^\times \curvearrowright X$ s.t.
  - linear
  - $X^{\mathbb{T}}$ is projective
  - $\lim_{\lambda \mapsto 0} \lambda \cdot x$ exists $\forall\, x \in X$

e.g. cotangent bundles, Higgs moduli

Nakajima quiver varieties

<u>note</u> we expect GAGA for X smooth semiprojective

<u>Definition</u> $\alpha \in X^{\mathbb{T}}$ $\quad W_\alpha^+ := \{ x \in X \mid \lim_{\lambda \to 0} \lambda x = \alpha \}$ <u>upward flow</u> "stable manifold"

$W_\alpha^- := \{ x \in X \mid \lim_{\lambda \to \infty} \lambda x = \alpha \}$ <u>downward flow</u> "unstable manifold"

Theorem (BB 1973) $W_\alpha^\pm \cong_{\mathbb{T}} T_\alpha^\pm X$

semi-projective $\Rightarrow$ $\quad X = \coprod_{\alpha \in X^{\mathbb{T}}} W_\alpha^+ \quad$ BB-partition

$\mathcal{C} := \coprod_{\alpha \in X^{\mathbb{T}}} W_\alpha^- \quad$ (projective) <u>Core</u> of X

(Thm $\mathcal{C} \sim X$)

<u>Remark</u> when (X, ω) hol. symplectic

$\lambda^* \omega = \lambda \omega$ homogeneity $1 \Rightarrow W_\alpha^+$ & $\mathcal{C}$ are Lagrangian

<u>Definition</u> α, W_α^+ <u>very stable</u> $\Leftrightarrow$ $W_\alpha^+ \cap \mathcal{C} = \{\alpha\}$

<u>Theorem (HH 2021)</u> W_α^+ is very stable $\Leftrightarrow$ $W_\alpha^+ = \overline{W_\alpha^+}$ in X

<u>Definition</u> <u>partial order</u> on $X^{\mathbb{T}}$ $\quad \alpha \leq \beta \Leftarrow \beta \in \overline{W_\alpha^+} \Leftrightarrow \alpha \in \overline{W_\beta^-}$

<u>Observation</u> very stable $\Leftrightarrow$ maximal in partial order

§ 2.2 Moduli of Higgs bundles

- C complex smooth projective curve $g > 1$; $G = GL_n$
- Higgs bundle : (E, Φ) E rank n vector bundle $\Phi \in H^0(C, \mathrm{End}\, E \otimes K)$
- $\mathcal{M} := \mathcal{M}_{Dol}(GL_n)$ moduli space of semistable rank n "Higgs field" degree d Higgs bundles
- $\mathbb{T} \curvearrowright \mathcal{M}$ $(E, \Phi) \mapsto (E, \lambda \Phi)$
- $h : \mathcal{M} \to \mathcal{A} := H^0(K) \times \dots \times H^0(K^n)$

  $(E, \Phi) \mapsto \det(x - \Phi)$

Hitchin map : proper $\mathbb{T}$-equivariant $\Rightarrow$ $\mathcal{M}$ semiprojective

\+ Completely integrable Hamiltonian system

e.g. fibers Lagrangian $h^{-1}(a) = \mathrm{Jac}(C_a)$ for generic $a \in \mathcal{A}$

$h^{-1}(0)$ = "nilpotent cone" $h^{-1}(0)_{red} = \mathcal{C}$

Lagrangian (Laumon 1987)

def^n $\quad \mathcal{E} = (E, \Phi) \in \mathcal{M}^T$ very stable $\Leftrightarrow$ $W^+_{\mathcal{E}}$ very stable $\Leftrightarrow$

$$W^+_{\mathcal{E}} \cap \mathcal{T} = W^+_{\mathcal{E}} \cap h^{-1}(0) = \{\mathcal{E}\} \Leftrightarrow \overline{W^+_{\mathcal{E}}} = W^+_{\mathcal{E}}$$

when $\Phi = 0$ $\quad \mathcal{N} = \{(E,0)\} \subset \mathcal{M}$ moduli space of stable bundles

Lagrangian + $\mathcal{T}$ Lagrangian $\Rightarrow$ there exist very stable $(E,0)$

(Laumon 1987)

§ 2.3 Very stable Higgs bundles of type $(1, \dots, 1)$

- $M_0, M_1, \dots, M_{n-1}$ line bundles on C & $b_i : M_{i-1} \to M_i K \Leftrightarrow b_i \in H^0(M_{i-1}^{-1} M_i K)$

let $\quad \delta_i = \mathrm{div}(b_i)$ effective divisor for $i > 0$

let $\quad M_0 = \mathcal{O}(\delta_0)$, fix $\delta := (\delta_0, \delta_1, \dots, \delta_{n-1})$

$$\Phi_\delta := \begin{pmatrix} 0 & & & \\ b_1 & 0 & & 0 \\ & b_2 & \ddots & \\ & & b_{n-1} & 0 \end{pmatrix}$$

$$\mathcal{E}_\delta : M_0 \oplus M_1 \oplus \dots \oplus M_{n-1} \longrightarrow (M_0 \oplus M_1 \oplus \dots \oplus M_{n-1}) K$$

assume it is stable

<u>Theorem (HH)</u> $\mathcal{E}_\delta$ is very stable $\Leftrightarrow$ $\delta_1+\dots+\delta_{n-1}$ is reduced

i.e. zeroes of b_i are single and distinct

<u>proof of Thm</u> by Hecke transformation

<u>def^n</u> $(E,\Phi)\in\mathcal{M}$, $c\in C$, $k=0,\dots,n-1$ $\quad V\subset E_c$ $\dim V=k$ s.t. $\Phi_c(E_c)\subset E_cK$

$W:=E_c/V$

<u>(n-k)th fundamental Hecke transform</u> of (E,Φ) at V is (E',Φ'):

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "\Phi'"] & E \arrow[r] \arrow[d, "\Phi"] & W\otimes\mathcal{O}_c \arrow[r] \arrow[d, "\Phi_c"] & 0 \\ 0 \arrow[r] & E'K \arrow[r] & EK \arrow[r] & W\otimes\mathcal{O}_cK \arrow[r] & 0 \end{tikzcd}$$

proof by induction on $|\delta_1+\dots+\delta_{n-1}|$

<u>Step 1</u> $|\delta_1+\dots+\delta_{n-1}|=0 \Rightarrow \delta_1=\dots=\delta_{n-1}=0$ i.e. b_i has no zero i.e. $=1$

if $\delta_0:=0 \Rightarrow \mathcal{E}_0$ canonical uniformising Higgs bundle

$W_0^+ := W_{\mathcal{E}_0}^+$ is <u>Hitchin section</u> of h $\Rightarrow$ very stable step 2   can reach any $\mathcal{E}_\delta$ with repeated fundamental Hecke transforms from $\mathcal{E}_0$

- when $\delta_1 + \dots + \delta_{n-1}$ reduced Hecke transformation preserves closedness
- when $\delta_1 + \dots + \delta_{n-1}$ non-reduced we can produce Hecke curve in $W^+_{\mathcal{E}_\delta} \cap Z$. $\blacksquare$

§ 3 Minor of type $(1,\dots,1)$ flows

Observation: Labelling of $\mathcal{E}_\delta$ $\delta = (\delta_0, \underbrace{\delta_1, \dots, \delta_{n-1}}_{\text{effective}})$ divisors on C

$\Leftrightarrow$ $\mu : C \to P^+$ dominant weights of GL_n + finite support

$\mu = (\mu_{c_1}, \dots, \mu_{c_j}) \in (P^+ \setminus 0)^j$ $c_1, \dots, c_j \in C$ distinct

$\mathcal{E}_\mu := \mathcal{E}_\delta$

$\mathcal{E}_\mu$ is very stable $\Leftrightarrow$ $\operatorname{im}(\mu) \subset P^+_{\min}$ minuscule

minimal w.r.t. partial order $\mu_1 \leq \mu_2$

<u>Conjecture (HH)</u> when $\mathrm{im}(\mu) \subset P^+_{\min}$

$$S(\mathcal{O}_{W^+_\mu}) = \bigotimes_{i=1}^{j} \rho_{\mu_{c_i}}(\mathbb{E})_{c_i} := \Lambda_\mu$$

when $(\mathbb{E}, \Phi) \to \mathcal{M} \times C$
universal Higgs bundle

<u>Thm (HH)</u> - true on generic fiber $h^{-1}(a)$

- Λ_μ is hyperholomorphic

- $h_*(\mathcal{O}_{W^+_\mu})$ is a Π-equivariant vector bundle on $\mathcal{A}$

"equivariant multiplicity" $\chi_\Pi(h_*(\mathcal{O}_{W_\mu})_0) = \chi_\Pi(\Lambda_\mu|_{E_0})$

<u>Remark</u> this last property is a consequence of a symmetry of (KW 2006)

(KW 2006) $\Rightarrow$ $c \in C$

$$\mu \in X_*^+(G) \cong X^{*+}(G^L) \cong \mathrm{Irrep}(G^L)$$

$$\mathcal{H}_\mu \circlearrowright D^b_{Coh}(\mathcal{M}_{Dol}(G))$$ Hecke transformation at c

"t' Hooft operator"

$$W_\mu \circlearrowright D^b_{Coh}(\mathcal{M}_{Dol}(G^L))$$ "Wilson operator"

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \rho_\mu(\mathbb{E})\big|_{\mathcal{M}_{Dol}(G^L) \times c}$$

where $(\mathbb{E}, \Phi) \to \mathcal{M}_{G^L} \times C$ universal G-Higgs bundle

then

$$S \circ \mathcal{H}_\mu \circ S^{-1} = W_\mu \qquad \boxed{W_\mu \circlearrowright D^b_{Coh}(\mathcal{M}_{DR}(G))}$$

apply this to $\mathcal{O}_{\mathcal{M}_{G^L}}$ and restrict to Hitchin section

$$S \circ \mathcal{H}_\mu \circ S^{-1}(\mathcal{O}_{\mathcal{M}_{G^L}})\Big|_{W_{\mathcal{E}_0}^L} = S \circ \mathcal{H}_\mu(\mathcal{O}_{W^+_{\mathcal{E}_0}})\Big|_{W_{\mathcal{E}_0}^L} = h_{G*}(\mathcal{H}_\mu(\mathcal{O}_{W^+_{\mathcal{E}_0}}))$$

thus $\quad h_{G*}(\mathcal{H}_\mu(\Theta_{W_0^+})) = \Lambda_\mu\big|_{W_{\mathcal{E}_0^L}}.$

<u>Conjecture</u> for any $\mu \in X_*^+(G)$ we expect

- $\chi_\pi\left(h_{G*}(\mathcal{H}_\mu(\Theta_{W_0^+}))_0\right) = \chi_\pi(\Lambda_\mu|_{\mathcal{E}_0^L}) = \mathrm{IP}_{t^{1/2}}(\overline{Gr_\mu})$

Geometric Satake

$\lambda \in X_*^+(G)$

- $\chi_\pi(\mathcal{H}_\mu(\Theta_{W_0^+})|_{\mathcal{E}_\lambda}) = m_\mu^\lambda(t)$

Lusztig's t-analogue of weight multiplicity

a Kazhdan-Lusztig polynomial