What is Floer homotopy?

Monday, October 3, 2022 11:41 AM

1. Review of Morse & Floer homology
2. Goal of Floer homotopy
3. Construction
4. Applications

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1. Morse homology

M - compact, smooth $\quad f: M \to \mathbb{R}$ Morse-Smale

$C_k = \mathbb{Z}\langle Crit(f)\rangle \qquad \partial x = \sum n_{xy}\, y$

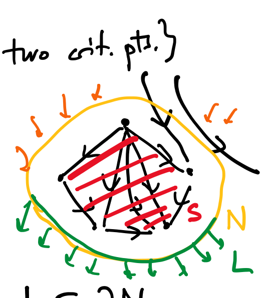
$H_*(C_k) = H_k(M) \qquad$ # gradient flow lines $x \to y$

What if M – noncompact?

$S = \{$pts. on gradient flow lines b/w two crit. pts.$\}$

Assume S - compact

$\rightsquigarrow C_k \rightsquigarrow H_*(C_k) = \widetilde{H}_k(\underbrace{\text{Conley index}}_{N/L})$



$L \subset \partial N$
exit set

Floer homology

= Morse homology in ∞ dim.

$\mathcal{B}$ = Banach mfld. $\qquad F: \mathcal{B} \to \mathbb{R}$

$x \in Crit(F) \qquad Hess_x(F) \rightsquigarrow E^+ \oplus E^- = T_x\mathcal{B}$

can still define $gr(x,y) \in \mathbb{Z}$ $\quad$ both ∞-diml.
(sometimes)

$\to C_k \xrightarrow{H_*} HF_*$

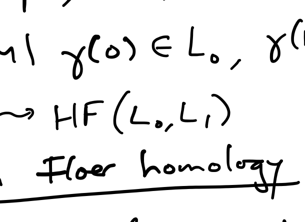
Ex.: Symplectic geometry

(a) Hamiltonian Floer homology ✓ ($\exists$ "Floer htpy type")

(M,ω) - sympl. $\qquad H: S^1 \times M \to \mathbb{R}$ Hamiltonian

$\to \omega(X_t, -) = dH_t(-)$ $\quad$ (X_t vr. field) $\rightsquigarrow$ flow $\Phi_t: M \to M$

$\mathcal{B} = \{\alpha: S^1 \to M,\ \hat{\alpha}: D^2 \to M\}/\sim \qquad \hat{\alpha} \sim \hat{\alpha}'$ if $[\omega], c_1(TM) = 0$ on $\hat{\alpha} \cup \hat{\alpha}'$



$F: \mathcal{B} \to \mathbb{R}$

$F(\alpha) = \int_{S^1} H_t(\alpha(t)) - \int_D \hat{\alpha}^*\omega \qquad$ area functional

$Crit(F)$ = closed orbits of Φ_1
flow lines = pseudohol. cylinders $\qquad \rightsquigarrow HF_* = H_*(M)$

(b) (M,ω) - sympl., $L_0, L_1 \subset M$ Lagrangians

$\mathcal{B} = \{\gamma: [0,1] \to M \mid \gamma(0) \in L_0,\ \gamma(1) \in L_1\}$

$\to F = \text{area} \rightsquigarrow HF(L_0, L_1)$

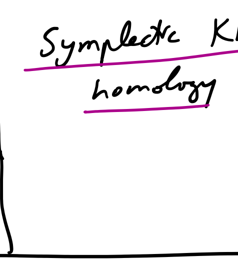
Lagrangian Floer homology $\quad \rightsquigarrow$ WIP Abouzaid-Blumberg

(c) periodic Floer homology, contact homology, etc.

Low Dimensional Topology

Y^3 or $(Y^3, K \subset Y)$ (knot) $\rightsquigarrow HF_*$

Floer homologies:

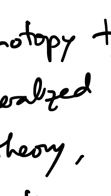
Instanton (Yang-Mills) = $\mathcal{B} = \{SU(2)$ conns. on $Y\}/$gauge $F = CS = \frac{1}{4\pi}\int tr(A \wedge dA + \frac{2}{3} A\wedge A \wedge A)$?	Symplectic instanton ?  U_0, U_1, Y, Σ $\rightsquigarrow (M,\omega)$ $\cup$ L_0, L_1 $HF(L_0, L_1)$
✓ Monopole (Seiberg-Witten) = $\mathcal{B} = \{(A,\Phi) \mid A = U(1)$ conn., Φ - spinor$\}$ $F = CSD = -\int A \wedge dA + \int \langle \Phi, D_A\Phi\rangle$	Heegaard Floer ? $\rightsquigarrow M = Sym^g(\Sigma)$ L_0, L_1 knots ✓
✓ Khovanov homology = - combinatorial for $K \subset S^3$ Witten: conjectural interpretation w/ gauge theory (5d Haydys-Witten)	Symplectic Khovanov homology ?

2. Floer homotopy

Goal: Given (M, ω, H_t) or (M, ω, L_0, L_1)
or Y^3 or (Y, K) $\rightsquigarrow$ (ideally)
construct a space X with $H_*(X) = HF_*$

X = invariant up to stable homotopy equivalence

"Floer (stable) homotopy type"

$X \rightsquigarrow \Sigma X$  $\qquad [X,Y] = \{f: X \to Y\}/\text{htpy}$

stable htpy: $\quad \{X, Y\} = \operatorname{colim}_{n\to\infty} [\Sigma^n X, \Sigma^n Y]$

More generally: X could be a spectrum
(i.e. a sequence of spaces $\{X_n\}$, $f_n: \Sigma X_n \to X_{n+1}$) $\quad$ e.g. $X_n = \Sigma^n X$

or pro-spectrum or twisted parametrized spectrum

Floer homotopy type $X \rightsquigarrow h_*(X)$ more info than $H_*(X) = HF_*$

h_* = generalized homology like
K-theory, KO, bordism, stable homotopy

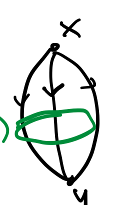
3. Constructions

Cohen, Jones, Segal (1995): use a flow category

$\mathcal{B},\ F: \mathcal{B} \to \mathbb{R} \qquad CF_* = \mathbb{Z}\langle Crit(F)\rangle$
∂ counts flow lines

flow category $\mathcal{C}$: objects = $Crit(F)$

$Hom(x,y) = \overline{\mathcal{M}}(x,y) = \{$broken flow lines $x \to y\}$



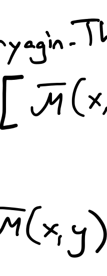
$$\boxed{\partial \overline{\mathcal{M}}(x,y) = \bigcup_z \overline{\mathcal{M}}(x,z) \times \overline{\mathcal{M}}(z,y)} \quad (*)$$

+ framings (of the normal bundles of $\overline{\mathcal{M}}(x,y) \hookrightarrow \mathbb{R}^N \times [0,\infty)^k$)

+ compatibility

$\rightsquigarrow$ framed flow category $\xrightarrow{CJS}$ "space" X

idea: say $Crit(F) = \langle x, y\rangle$
degrees: $m+k$, m $\qquad m \gg 0$



$\dim \underbrace{\overline{\mathcal{M}}(x,y)}_{\text{smooth mfld + framed}} = k-1$

$X = \underbrace{pt \cup D^m}_{S^m} \cup D^{m+k}$ (x, y) $\quad$ attach by $\tau: S^{m+k-1} \to S^m$

$\tau \in \pi^{st}_{m+k-1}(S^m) = \pi^{st}_{k-1}(S^0) \ni [\overline{\mathcal{M}}(x,y)] =: \tau$ $\quad$ Pontryagin-Thom

In practice: difficult b/c
- need compatible smooth structures on $\overline{\mathcal{M}}(x,y)$
- bubbles $\to (*)$ may not hold
- framings may not exist.

What is actually done:

(a) Hamiltonian Floer homology (Abouzaid-Blumberg 2021)

$\overline{\mathcal{M}}(x,y)$ - topological orbifolds w/ corners
- not framed, but at least stably almost complex

$\rightsquigarrow$ not a Floer htpy type, but we get complex oriented generalized Floer homologies
(e.g. K-theory, complex bordism, Morava K-theories)

(b) construct the framed flow category by hand, inductively on dim.:
combinatorial data (knot $K \subset S^3$) $\rightsquigarrow \overline{\mathcal{M}}(x,y) \rightsquigarrow$ htpy type
- Khovanov htpy type (Lipshitz-Sarkar, 2011)
- knot Heegaard Floer htpy type (M.-Sarkar, 2021)

(c) Use finite diml. approximation:
in Seiberg-Witten theory
(M. - 2001 for Y^3, $b_1(Y) = 0$
+ Kronheimer-M., Khandhawit-Lin-Sasahira, Sasahira-Stoffregen, Behrens-Hedenlund-Kragh (WIP) $\to$ all Y^3)
- no bubbles

$\mathcal{B}$ = Hilbert space $\qquad F: \mathcal{B} \to \mathbb{R}$, $F = CSD$

replace $\mathcal{B}$ by $\ldots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset \ldots \subset \mathcal{B}$
$\mathcal{B}_n$ - finite diml. $\qquad \mathcal{B} = \widehat{\bigcup \mathcal{B}_n}$

$F|_{\mathcal{B}_n}: \mathcal{B}_n \to \mathbb{R} \rightsquigarrow$ Conley index X_n

$\rightsquigarrow$ spectrum $\{X_n\}$ = SW Floer htpy type

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4. Applications

a) Thm. (M, 2013): $\exists$ non-triangulable manifolds in each dim ≥ 5.
(known in dim. 4 by Casson/Freedman)

Galewski-Stern, Matumoto '70s: reduced the problem to a question in 3+1 D:

$$\boxed{\nexists [Y] \in \Theta^3_{\mathbb{Z}},\ 2[Y] = 0,\ \mu(Y) = 1.}$$

$\Theta^3_{\mathbb{Z}} = \{Y^3 \quad H_*(Y) = H_*(S^3)\}/\sim$
homology cobordism $\qquad Y_0$ W Y_1 $\quad H_*(W, Y_i) = 0$

$\mu: \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$ Rokhlin

$Y^3 \rightsquigarrow$ SW eqs. have $Pin(2)$ symmetry

$\rightsquigarrow SWFH^{Pin(2)}_*(Y) \quad$ $Pin(2)$-equiv. Floer homology
take Floer htpy X, then $H^{Pin(2)}_*(X)$

$\rightsquigarrow \beta: \Theta^{\mathbb{Z}}_3 \to \mathbb{Z} \qquad \beta(-Y) = -\beta(Y)$
μ: $\Theta^{\mathbb{Z}}_3 \to \mathbb{Z}/2$ (through $\mathbb{Z} \to \mathbb{Z}/2$)

$2[Y] = 0 \Rightarrow \beta(-Y) = -\beta(Y) = \beta(Y) \Rightarrow \beta(Y) = 0$
$\Rightarrow \mu(Y) = 0.$

b) Arnol'd Conjecture mod p
(Abouzaid-Blumberg, 2021)

Hamiltonian Floer homology (M,ω) H_t
$\to HF_*$ - only defined over $\mathbb{Q}$ (b/c of bubbles)
$HF_* \cong H_*(M;\mathbb{Q})$
$CF_* = \mathbb{Z}\langle$closed orbits of time-1-flow$\rangle$

$\to$ Arnol'd Conj.: # closed orbits $\geq \dim H_*(M;\mathbb{Q})$

A-B: define cx. oriented gen. Floer homology
Morava K-theory $\quad p \geq 2$ prime, $n \geq 1$ $\quad K_p(n)$

$HF(M, \{H_t\}; K_p(n))$ over $\mathbb{F}_p[v, v^{-1}]$, $\quad |v| = 2(p^n - 1)$
$\cong H^*(M; \mathbb{F}_p)$ for $n \gg 0$

$$\Rightarrow \boxed{\#\text{ closed orbits} \geq \dim_{\mathbb{F}_p} H_*(M; \mathbb{F}_p)}$$