Strong mass gap implies quark confinement

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Lattice gauge theories

- Quantum Yang–Mills theories are constituents of the Standard Model of quantum mechanics that describe interactions between elementary particles.
- Lattice gauge theories are discrete Euclidean versions of quantum Yang–Mills theories.
- A lattice gauge theory may be coupled with a Higgs field, or it may be a pure lattice gauge theory.
- We will only deal with pure lattice gauge theories in this talk.
- A pure lattice gauge theory is characterized by its gauge group (usually a compact matrix Lie group), the dimension of spacetime, and a parameter known as the coupling strength.
- These theories on their own, even without passing to the continuum limit or constructing the quantum theory, can yield substantial physically relevant information about features of elementary particles and the interactions between them.

Mass gap

- Two very important open questions have lattice gauge theoretic formulations.
- The first is the question of Yang–Mills mass gap.
- In lattice gauge theories, mass gap is equivalent to exponential decay of correlations.
- Mass gap has not yet been mathematically proved in the most important cases.
- Huge Monte Carlo studies, as well as physical heuristics based on renormalization group methods, indicate that the conjecture is correct, and give correct physical predictions.

Quark confinement

- The second big open question is the problem of quark confinement.
- It is an enduring mystery why quarks are never observed freely in nature.
- The problem of quark confinement has received a lot of attention in the physics literature, and yet the current consensus seems to be that a satisfactory theoretical explanation does not exist.
- Wilson (1974) argued that quark confinement is equivalent to showing that the relevant lattice gauge theory satisfies what's now known as Wilson's area law.
- A number of deep results are known about the area law (will discuss soon), but the main question remains open.

Content of this talk

- Main result: *If the gauge group is compact, connected, and has a nontrivial center, then the presence of exponential decay of correlations under arbitrary boundary conditions implies that Wilson's area law holds.*
- The nontriviality of the center is known as center symmetry.
- The exponential decay assumption is stronger than usual mass gap, which — quite subtly — means exponential decay under a specific boundary condition. I call it strong mass gap.

Physical viewpoint

- There is a longstanding belief in physics, originating in the work of 't Hooft (1978), that mass gap plus unbroken center symmetry implies confinement.
- Note: 'Unbroken center symmetry' is not a rigorously defined concept.
- The result from the previous slide shows that *strong mass gap plus center symmetry implies confinement*.
- The physical explanation is that strong mass gap prevents the breaking of center symmetry.
- There are lattice gauge theories where the gauge group has a nontrivial center, which are believed to be gapped and yet non-confining. The reason, in view of our result, must be that there are some boundary conditions under which correlations do not decay exponentially in these models.

Is strong mass gap valid at weak coupling?

- The main interest is in 4D $SU(3)$ lattice gauge theory, which is the theory of the strong force, which governs interactions between quarks.
- This theory is supposed to be both gapped and confining at all coupling strengths.
- Since $SU(3)$ has a nontrivial center, our theorem implies that if it has strong mass gap at all coupling strengths, then it is also confining at all coupling strengths. If true, this would solve the quark confinement problem.
- But does it have strong mass gap at all coupling strengths?
- I don't know. All I can say is that there is as much evidence for mass gap (which everyone believes to be true) as for strong mass gap. So there is no particular reason to believe that one is true and the other is not.

Some forthcoming results

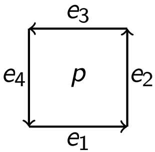
- In a forthcoming revision of my preprint, I will include a rigorous formulation of the conjecture that mass gap plus unbroken center symmetry implies quark confinement, along with a proof.
- In the revision, I will also present a toy example — a modification of $SU(n)$ lattice gauge theory — which exhibits strong mass gap at all coupling strengths.

Notation

- Let $n \geq 1$ and $d \geq 2$ be two integers.
- Let G be a closed connected subgroup of the unitary group $U(n)$.
- Let $B_N := [-N, N]^d \cap \mathbb{Z}^d$.
- Let E_N be the set of positively oriented nearest-neighbor edges of B_N.
- Let Ω_N be the set of all functions from E_N into G, that is, the set of all assignments of group elements to positively oriented edges. This is our space of configurations.
- If $\omega \in \Omega_N$ and e is a negatively oriented edge, we define $\omega_e := \omega_{e^{-1}}^{-1}$, where e^{-1} is the positively oriented version of e.

Plaquettes

- A plaquette in $\mathbb{Z}^d$ is a set of four directed edges that form the boundary of a square.
- Let P_N be the set of all plaquettes in B_N.
- Given some $p \in P_N$ and $\omega \in \Omega_N$, we define ω_p as follows.
- Write p as a sequence of directed edges e_1, e_2, e_3, e_4, each one followed by the next.



- Let $\omega_p := \omega_{e_1}\omega_{e_2}\omega_{e_3}\omega_{e_4}$.
- Although there are ambiguities in this definition about the choice of e_1 and the direction of traversal, that is not problematic because we will only use the quantity $\Re(\mathrm{Tr}(\omega_p))$, which is not affected by these ambiguities.

Definition of lattice gauge theory

- Define the Hamiltonian
$$H_N(\omega) := -\sum_{p\in P_N} \Re(\mathrm{Tr}(\omega_p)).$$
- The pure lattice gauge theory on B_N with gauge group G, and coupling parameter β, is the probability measure $\mu_{N,\beta}$ on Ω_N defined as
$$d\mu_{N,\beta}(\omega) = Z_{N,\beta}^{-1} e^{-\beta H_N(\omega)} d\lambda_N(\omega),$$
where λ_N is the product Haar measure on Ω_N and $Z_{N,\beta}$ is the normalizing constant.
- Here $\beta = 1/g_0^2$, where g_0 is the coupling strength.
- Given a measurable function $f : \Omega_N \to \mathbb{C}$, the expected value of the function under the above lattice gauge theory is the quantity
$$\langle f\rangle := \int_{\Omega_N} f(\omega) d\mu_{N,\beta}(\omega),$$

Boundary conditions

- Let $\mu_{N,\beta}$ be the lattice gauge theory defined in the previous slide.
- Let ∂E_N denote the set of positively oriented boundary edges of B_N.
- Let $\partial \Omega_N$ denote the set of all functions from ∂E_N into G.
- An element of $\partial \Omega_N$ is called a boundary condition.
- Let δ be a boundary condition. The restriction of $\mu_{N,\beta}$ to the set of all configurations which agree with δ on the boundary is called the lattice gauge theory with boundary condition δ.
- We will henceforth always assume the presence of a boundary condition.

Area law

- Let π be a finite-dimensional unitary representation of the group G, and let χ_π be the character of π.
- Let ℓ be a closed loop in B_N, with directed edges $e_1, \ldots, e_k$.
- Given a configuration ω, the Wilson loop variable $W_\ell(\omega)$ is defined as

$$W_\ell(\omega) := \chi_\pi(\omega_{e_1}\omega_{e_2}\cdots\omega_{e_k}).$$

- The lattice gauge theory is said to satisfy Wilson's area law for the representation π if

$$|\langle W_\ell \rangle| \le C_1 e^{-C_2\,\mathrm{area}(\ell)}$$

for any rectangular loop ℓ, where C_1 and C_2 are positive constants that depend only on G, β, d and π, and $\mathrm{area}(\ell)$ is the area enclosed by ℓ.

Why does area law imply quark confinement?

- Let $V(R)$ be the potential energy of a static quark-antiquark pair separated by distance R.
- QFT calculations imply that for a rectangular loop ℓ of side-lengths R and T in the continuum limit of 4D $SU(3)$ theory and a suitable representation π, $\langle W_\ell \rangle$ should behave like $e^{-V(R)T}$.
- So if the area law holds, then $V(R)$ grows linearly in the distance R between the quark and the antiquark. By the conservation of energy, this implies that the pair will not be able to separate beyond a certain distance.
- Renormalization group arguments predict that β has to be sent to infinity as the lattice spacing goes to zero to obtain the continuum limit of 4D non-Abelian theories. This indicates that we need the area law to hold at arbitrarily large values of β in 4D $SU(3)$ theory for it to imply confinement of quarks.

Area law: Basic facts

- It is easy to show that the area law holds at all β in any 2D theory, since gauge fixing can be used to reduce any 2D theory to a 1D model.
- Seiler (1978) proved an area law lower bound for any theory at any β.
- Simon and Yaffe (1982) proved a perimeter law upper bound for any theory at any β.

Area law: Deeper results

- Osterwalder and Seiler (1978) showed that the area law holds at small enough β (strong coupling) for any theory in any dimension.
- Guth (1980) and Fröhlich and Spencer (1982) showed that for 4D $U(1)$ theory, area law breaks down at large enough β; instead, perimeter law holds. This is known as the deconfinement transition for this theory.
- The deconfinement transition was physically expected, because 4D $U(1)$ theory is related to photons, which are not confined.
- Göpfert and Mack (1982) showed that the area law holds at all β for 3D $U(1)$ theory. This is still the only nontrivial case where the area law has been established at large β.

Area law: Various other theorems

- Fröhlich (1979) showed that confinement holds in $SU(n)$ theory if it holds in the corresponding $\mathbb{Z}_n$ theory.
- Durhuus and Fröhlich (1980) showed that confinement in a d-dimensional pure lattice gauge theory holds if there is exponential decay of correlations in a $(d-1)$-dimensional nonlinear σ model.
- Borgs and Seiler (1983) investigated confinement in lattice gauge theories at nonzero temperature, building on technology developed by Brydges and Federbush (1980).
- A toy model exhibiting a sharp transition from the confining to the deconfining regime was studied by Aizenman, Chayes, Chayes, Fröhlich and Russo (1983).

Area law: Recent progress

- Area law at small β for arbitrary loops (where area(ℓ) is the minimal surface area enclosed by ℓ) was established in large N limit of $SO(N)$ and $SU(N)$ theories by Chatterjee (2019), Jafarov (2016), and Chatterjee and Jafarov (2016).
- Chatterjee (2020a) computed the exact leading order behavior of Wilson loop expectations in 4D $\mathbb{Z}_2$ theory at large β.
- This result was extended to all 4D theories with finite Abelian gauge groups by Forsström, Lenells and Viklund (2020), and to all 4D theories with finite gauge groups by Cao (2020).

Towards the main result: Assumptions

- Center symmetry: Assume that the center of G is nontrivial, and there is an element g_0 in the center such that $\pi(g_0) = cI$ for some $c \neq 1$, where I is the $m \times m$ identity matrix, m being the dimension of π.
- Strong mass gap: Say that two edges are neighbors if they both belong to some common plaquette. Say that a measurable map $f : \Omega_N \to \mathbb{R}$ is a *local function* supported on an edge $e \in E_N$ if $f(\omega)$ depends only on the values of ω_u for u that are neighbors of e. Given two local functions f and g, let $\mathrm{dist}(f, g)$ denote the Euclidean distance between the midpoints of their supporting edges. Assume that there are positive constants K_1 and K_2 depending only on G, β and d, and not on N or the boundary condition δ, such that for any local functions $f, g : \Omega_N^\circ \to [-1, 1]$,

$$|\langle fg\rangle - \langle f\rangle\langle g\rangle| \le K_1 e^{-K_2\,\mathrm{dist}(f,g)}.$$

Main result

Theorem (C., 2020)

Let G be a compact connected subgroup of $U(n)$ for some n, and let π be a finite-dimensional unitary representation of G. Take any $d \geq 2$ and $\beta \in \mathbb{R}$, and consider the lattice gauge theory on the cube B_N. Suppose that the center symmetry and strong mass gap assumptions are satisfied. Then there are positive constants C_1 and C_2 depending only on G, β, π and d, such that the following holds. Take any $N \geq 2$, any boundary condition δ on B_N, and any rectangular loop ℓ contained in $B_{N'}$ for some $N' \leq N/2$. Then

$$|\langle W_\ell \rangle| \leq C_1 e^{-C_2 \operatorname{area}(\ell)},$$

where area(ℓ) *is the area enclosed by ℓ. Moreover, there is a unique infinite volume Gibbs state, and the above bound holds for any rectangular loop ℓ if the expectation on the left is taken with respect to this Gibbs state.*

Proof sketch: Step 1

- Let the first coordinate in $\mathbb{R}^d$ denote time.
- Let ℓ be a rectangular loop with side lengths R and T, where the sides of length T are parallel to the time axis.
- We wish to show that $|\langle W_\ell \rangle| \le C_1 e^{-C_2 RT}$.
- It is not very difficult to prove the perimeter law upper bound $|\langle W_\ell \rangle| \le C_1 e^{-C_2(R+T)}$.
- Given this, it is not hard to see that the area law can be established by proving the weaker bound

$$|\langle W_\ell \rangle| \le C_1 e^{C_2(R+T) - C_3 RT},$$

so this is what we will aim for.

Proof sketch: Step 2

- Take one element from each of the matrices $\pi(\omega_e)$, $e \in \ell$, and let f be the product of these elements.
- Such a variable will be called a component variable of ℓ.
- Note that W_ℓ is a sum of $m^{2(R+T)}$ component variables, since

$$W_\ell = \mathrm{Tr}(\pi(\omega_{e_1}) \cdots \pi(\omega_{e_k})),$$

  where $e_1, \ldots, e_k$ are the edges of ℓ.
- Thus, it suffices to prove that for any component variable f,

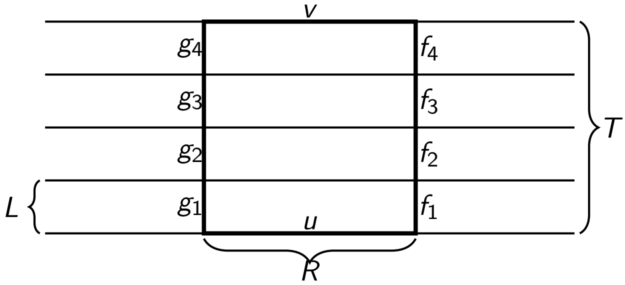
$$|\langle f \rangle| \le C_1 e^{C_2(R+T) - C_3 RT}.$$

Proof sketch: Step 3

- Partition spacetime into slabs of thickness L in the time direction, where L will be chosen later, depending on G, β and d.
- Let $r := T/L$. Then a component variable f can be decomposed as

$$f = uvf_1 f_2 \cdots f_r g_1 g_2 \cdots g_r,$$

as depicted in the following diagram.



Proof sketch: Step 4

- Let ω be a random configuration from the lattice gauge theory μ.
- Let μ' be the conditional probability distribution given ω_e for all e that belong to boundaries between slabs.
- Let $\langle f\rangle'$ denote the expected value of f under μ', so that $\langle f\rangle = \langle\langle f\rangle'\rangle$.
- So it suffices to prove the area law for $\langle f\rangle'$ instead of $\langle f\rangle$.
- Under μ', the ω_e's inside one slab are independent of the ω_e's inside another slab, and the ω_e's on the boundaries are not random.
- Thus, we get the identity

$$\langle f\rangle' = uv\langle f_1 g_1\rangle'\langle f_2 g_2\rangle' \cdots \langle f_r g_r\rangle'.$$

- So, to show $|\langle f\rangle'| \le C_1 e^{C_2(R+T)-C_3RT}$, it suffices to show that $|\langle f_i g_i\rangle'| \le C_4 e^{-C_5 R}$ for each i, since $r = T/L$ and L is a constant.

Proof sketch: Step 5

- Under μ', irrespective of the values of the ω_e's on the slab boundaries,
$$\langle f_i \rangle' = \langle g_i \rangle' = 0.$$
This holds because G has a nontrivial center.
- Thus, we need to show that
$$|\langle f_i g_i \rangle' - \langle f_i \rangle' \langle g_i \rangle'| \leq C_1 e^{-C_2 R}.$$
- This will follow if we can show that correlations decay exponentially *within each slab*, for any given boundary condition on the slab.
- The key idea is to show that this holds if the thickness L is chosen to be sufficiently large.

Proof sketch: Step 6

- Take a large but finite slab

$$S = ([-N, N] \times [-M, M]^{d-1}) \cap \mathbb{Z}^d,$$

  where $M \gg N$.
- The set of all boundary edges of S that do not belong to either the top face or the bottom face will be called the spatial boundary of S.
- Consider lattice gauge theory in S with some given boundary condition.
- We will show that the influence of the spatial boundary near the center of S is exponentially small in M when N is fixed and $M \to \infty$, provided that N is sufficiently large, depending on G, β and d.
- This suffices to show exponential decay of correlations in a sufficiently thick slab.

Proof sketch: Step 7

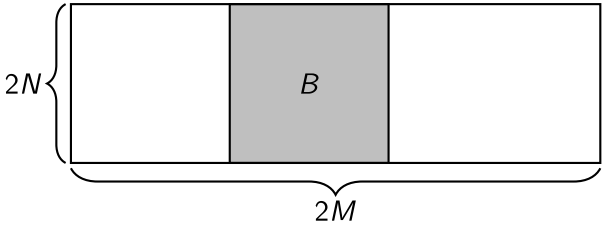
- Take two boundary conditions on S that differ only on the spatial boundary.
- Let μ and μ' be the probability measures defined by these two boundary conditions.
- A coupling of μ and μ' is a pair of random configurations (ω, ω') such that $\omega \sim \mu$ and $\omega' \sim \mu'$.
- Our goal would be achieved if we can construct a coupling such that for any edge e near the center of S, the chance of $\omega_e \neq \omega'_e$ is exponentially small in M.
- Such a coupling is constructed in three steps.

Construction of the coupling: Step 1

- First, we construct a coupling with similar properties in the cube B_N instead of the slab S, using the strong mass gap assumption and the coupling characterization of total variation distance between probability measures.
- The connectedness of the gauge group is used in this step, to pass smoothly from one boundary condition to another and bounding the rate of change using two-point correlations.

Construction of the coupling: Step 2

- Given any coupling (ω, ω') on S, we update the coupling to get a better coupling as follows.
- Choose a copy B of B_N inside S uniformly at random. Fixing ω_e and ω'_e outside the interior of B, replace the values in the interior by a coupled pair generated using the mechanism from the first step.



- The resulting pair of configurations is again a coupling of μ and μ', and is an 'improvement' of the original coupling. The 'amount of improvement' is quantified through an inequality.

Quantifying the improvement

- Let e be an edge in the slab S. Given a coupling γ of μ and μ', we quantify the performance of γ at the edge e using the quantity
$$\rho(\gamma, e) := \gamma(\{(\omega, \omega') : \omega_e \neq \omega'_e\}).$$
The smaller this is, the better the coupling is at e.
- Given a coupling γ, suppose that we improve it to $\widetilde{\gamma}$ using the procedure outlined in the previous slide.
- Let $U(e)$ be the set of all edges at distance $\le (\log N)^2$ from e, and let $V(e)$ be the set of all edges at distance $\le N$ from e.
- Then we show that
$$\rho(\widetilde{\gamma}, e) \le \left(1 - \frac{N^{d-1}}{M^{d-1}}\right)\rho(\gamma, e) + \frac{C_1 e^{-C_2(\log N)^2}}{M^{d-1}} \sum_{u \in V(e)} \rho(\gamma, u) + \frac{C_3 N^{d-2}}{M^{d-1}} \sum_{u \in U(e)} \rho(\gamma, u).$$

Construction of the coupling: Step 3

- The updating is repeated an infinite number of times to get a vastly improved coupling as a subsequential limit.
- Applying the inequality from the previous step, one can use this coupling to prove exponential decay of correlations in a slab if the thickness of the slab is large enough.
- The condition that the thickness has to be large is required to ensure that a certain parameter in the inequality is strictly less than one, which leads to the exponential decay.

Details for the last step

- Let $\{\gamma_n\}_{n\geq 1}$ be a sequence of couplings produced by successive improvements.
- Let γ be the weak limit of a subsequence γ_{n_k}. Then γ is also a coupling of μ and μ'.
- By standard results for weak convergence,

$$\begin{aligned}\rho(\gamma, e) &\leq \liminf_{k\to\infty} \rho(\gamma_{n_k}, e) \\ &\leq \limsup_{k\to\infty} \rho(\gamma_{n_k}, e) \leq \limsup_{n\to\infty} \rho(\gamma_n, e) =: p(e).\end{aligned}$$

- Thus, we only need to show that $p(e)$ decreases exponentially in the distance of e from the spatial boundary of the slab S.

Details, contd.

- The 'one-step improvement inequality' displayed earlier gives us

$$p(e) \le \left(1 - \frac{N^{d-1}}{M^{d-1}}\right) p(e) + \frac{C_1 e^{-C_2(\log N)^2}}{M^{d-1}} \sum_{u \in V(e)} p(u) + \frac{C_3 N^{d-2}}{M^{d-1}} \sum_{u \in U(e)} p(u).$$

- Rearranging this, we get

$$\begin{aligned} p(e) &\le \frac{C_1 e^{-C_2(\log N)^2}}{N^{d-1}} \sum_{u \in V(e)} p(u) + \frac{C_3}{N} \sum_{u \in U(e)} p(u) \\ &\le \frac{C_4 (\log N)^{2d}}{N} \max_{u \in V(e)} p(u). \end{aligned}$$

Details, contd.

- Choose N so large that $C_4(\log N)^{2d}/N < 1/2$. Then
$$p(e) < \frac{1}{2} \max_{u \in V(e)} p(u).$$
- Thus, for any e, there is some $e_1 \in V(e)$ such that $p(e) < p(e_1)/2$.
- But then, there is some $e_2 \in V(e_1)$ such that $p(e_1) < p(e_2)/2$.
- We can continue like this until we reach the spatial boundary.
- This shows that $p(e)$ is exponentially small in the distance of e from the spatial boundary, completing the proof.

Preprint

Chatterjee, S. (2020). A probabilistic mechanism for quark confinement. *Preprint.* Available at `https://arxiv.org/abs/2006.16229`.

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Thanks to everyone for listening!