Condensed Structures

Pyknotic

A Gentle Introduction
Slogan: Infinite algebraic structures are best understood when topologies are present.

Examples: topological groups, locally compact abelian groups, topological vector spaces, adic rings, etc., etc.
Example: Let $G$ be a compact group. A continuous unitary irreducible representation of $G$ is finite-dimensional. $G$ can be recovered from its continuous finite-dimensional representations:

$$\text{Rep}(G) \xrightarrow{\omega} \text{Vect} \quad G \approx \text{Aut}^\otimes(\omega)_{\sigma=1}$$
Issues

1. Statements like the Isomorphism Theorems are FALSE.

\[ 0 \to R \xrightarrow{\phi} R \to 0 \]

LCA is not abelian

quasi-abelian was some how alg.
2. Homological/homotopical algebra is **SUBTLE** and **FRAGILE**.

\( C^0(M, R) \) \( C^\infty(M, R) \)

BV-formalism

Appendix of Costello - Guillou

Differentiable vector spaces
3. Tensor products have poor formal properties.

If Banach, Fréchet is complex.
4. Categorification is mysterious. What even is a “topological higher category”?
Condensed/pyknotic structures try to repair these issues in a fundamental way, by "doing algebra" relative to a category that's "as good as" that of Sets, but of a topological nature.

Not looking for a conservative functor to Ab should not be a surjecti
Example.

If $X$ is a reasonable topological space, then it can be recovered from the functor $\text{Map}(\cdot, X)$:

$\text{Map}(\emptyset, X) = \ast$
Salient properties: \( \text{Map}(K \cup L, X) \)

1. \( X(K \cup L) = X(K) \times X(L) \)
2. \( X(\emptyset) = * \)

Sheaf condition

3. \( R \subseteq S \times S \) (closed equiv. relation)
Defn. A pyknotic set is a sheaf $X : \text{Comp}^{\text{op}} \to \text{Set}$ for the topology given by quotients.

$x \in C$

$\alpha_x : \text{C}^{\text{op}} \to \text{Set}

\text{YmsMap}(Y, X)$
Helpful observation

Every compactum is a quotient of a free compactum = compact Hausdorff space.

\[ S \xrightarrow{1} \beta(S) \xrightarrow{\text{cts}} K \xleftarrow{\text{cpt Haus}} \beta(S) = \text{Stone-Čech compactification} \]

The closure of open \( \beta \) still open.
\[ R^S(K) = \text{Map}(K, \mathbb{R}^S) = \text{locally constant fun. on } K \]

\[ R(K) = \text{Map}(K, \mathbb{R}) = \text{cts fun. on } K \]

\[ \text{cts fun. on } K \]

\[ \frac{\text{Map}(K, \mathbb{R})}{\text{Map}^K(K, \mathbb{R})} \]
Def \( n \) A pyknotic set is a functor \( X : \text{Free}^{op} \to \text{Set} \) such that:

1. \( X(K \cup L) = X(K) \times X(L) \)
2. \( X(\emptyset) = \ast \)

Def \( n \) The underlying set of \( X \) is \( X(\ast) \).

\[ \text{Pyh}(\text{Set}) \to \text{Set} \]
Example: Every reasonable topological space.

Example:

\[ 0 \to \mathbb{R}^s \to \mathbb{R} \to \mathbb{R}/\mathbb{R}^s \to 0 \]
Example: If $R$ is a pyknotic ring, then $\text{Mod}(R)$ is a pyknotic category.

\[
\begin{align*}
\text{Mod}(R) : \text{Free}^\text{op} & \to \text{Cat} \\
\text{Set} & \to \text{Set}
\end{align*}
\]
$TVS_R \rightarrow \text{Pyk}(\text{Vect}_R) \rightarrow \text{Fix}^0 \rightarrow \text{Vect}_R$

$\text{Pyk}(D\text{Vect}_R)$

$X : \text{scheme/variety}$

$\text{Cnst}_{\text{fr}}(X, D^b(C_{\mathcal{L}})) \sim \text{Fun}(\text{Exit}(X), D^b(C_{\mathcal{L}}))$