

Condensed Structures

pyknotic

A Gentle Introduction

Slogan: Infinite algebraic structures are best understood when topologies are present.

Examples: topological groups,
locally compact abelian groups,
topological vector spaces,
adic rings, etc., etc.

Example: Let G be a compact group.

A continuous unitary irreducible representation of G is finite-dimensional

G can be recovered from its continuous finite-dimensional representations:

$$\text{Rep}(G) \xrightarrow{\omega} \text{Vect} \quad G \cong \text{Aut}^{\otimes}(\omega)^{\sigma = \text{id}}$$

Issues

1. Statements like the **Isomorphism Theorems** are **FALSE**.

$$0 \rightarrow \mathbb{R}^8 \xrightarrow{\neq} \mathbb{R} \rightarrow 0$$

LCA is not abelian

quasiabelian in some
how. alg.

2. Homological/homotopical algebra
is SUBTLE and FRAGILE.

$C^\infty(M, \mathbb{R})$ $C^\omega(M, \mathbb{R})$

\mathbb{Q}

BV-formalism

Appendix of Costello-Gwilliam

Differentiable vector spaces \mathbb{R}

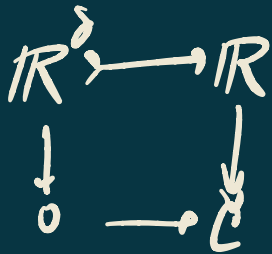
3. Tensor products have poor formal properties.

\otimes of Banach, Fréchet
is complex.

4. Categorification is mysterious.

What even is a "topological higher category"?

Condensed/pyknotic structures try to repair these issues in a fundamental way, by "doing algebra" relative to a category that's "as good as" that of Sets, but of a topological nature.



should not be a surjection

Not looking for a cosemantics functor to \mathbf{Ab}

Example.

$$\text{Map}(\phi, X) = *$$

If X is a reasonable topological space, then it can be recovered from the functor

$$\text{Map}(-, X) : \text{Comp}^{\text{op}} \rightarrow \text{Set}$$

$$K \rightsquigarrow \text{Map}(K, X)$$

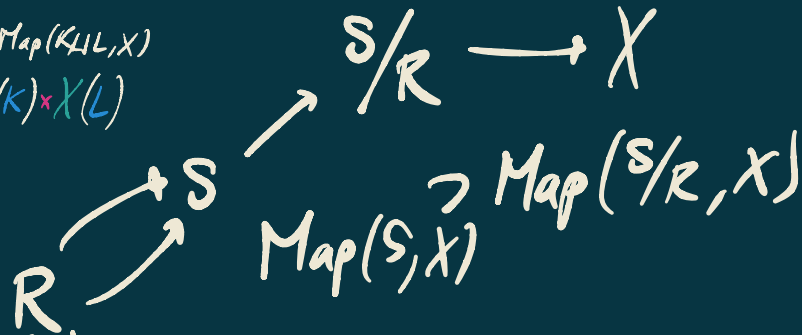
$$X(-)$$

$$X(\mathbb{R})$$

Salient properties: $\text{Map}(K \sqcup L, X)$

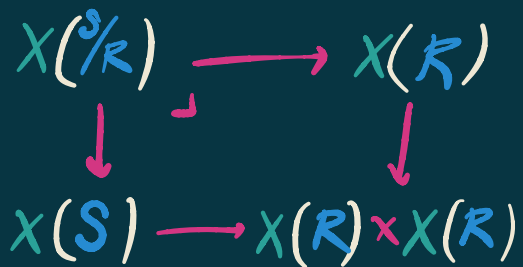
① $X(K \sqcup L) = X(K) \times X(L)$

② $X(\emptyset) = *$



sheaf condition

③ $R \subseteq S \times S$
closed equiv.
relation



Defⁿ A pyknotic set is a sheaf

$$X : \underline{\text{Comp}}^{\text{op}} \rightarrow \text{Set}^{X(K)}$$

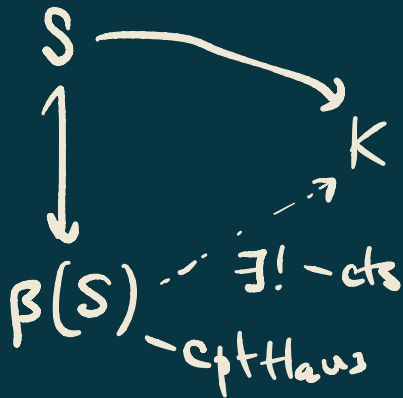
for the topology given by
quotients.

$$X \in \mathcal{C}$$

$$\alpha_X : \mathcal{C}^{\text{op}} \rightarrow \text{Set} \\ Y \mapsto \text{Map}(Y, X)$$

Helpful observation $\xrightarrow{\quad} =$ compact Hausdorff space.

Every compactum is a quotient of a free compactum



$\beta(S) =$ Stone-Čech
 $\overline{\text{compactification}}$
 \downarrow
 the closure of opens
 \mathbb{R} still open.

$$\mathbb{R}^S(K) = \text{Map}(K, \mathbb{R}^S) = \text{locally constant} \\ \text{fns. on } K$$

$$\mathbb{R}(K) = \text{Map}(K, \mathbb{R}) = \text{cts fns on } K$$

$$\text{cts} \\ \text{Map}(K, \mathbb{R}) / \text{Map}^k(K, \mathbb{R})$$

Defⁿ A *proknotic set* is a *functor*

$$X: \text{Free}^{\text{op}} \rightarrow \text{Set}$$

such that

$$(1) X(K \cup L) = X(K) \times X(L)$$

$$(2) X(\emptyset) = *$$

Defⁿ The *underlying set* of X \square

$$X(*)$$

$$\text{Pyh}(\text{Set}) \longrightarrow \text{Set}$$

Example: Every reasonable topological space.

Example:

$$0 \longrightarrow \mathbb{R}^s \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{R}^s \longrightarrow 0$$

Example: If R is a pyknotic ring, \dagger
 $\text{Mod}(R)$ is a pyknotic category.

$$\text{Free}^{\text{op}} \longrightarrow \text{Set}$$
$$\sqcup \qquad \qquad X$$

$$\text{Mod}(R): \text{Free}^{\text{op}} \longrightarrow \text{Cat}$$

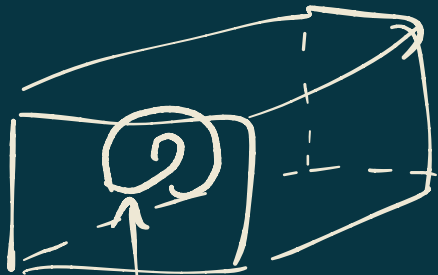
$$\text{TVS}_{\mathbb{R}} \xrightarrow{\quad} \text{Pyl}(\text{Vect}_{\mathbb{R}}) \quad \text{Free}^{\text{op}} \xrightarrow{\quad} \text{Vect}_{\mathbb{R}}$$

$$\text{Pyl}(\text{DVect}_{\mathbb{R}})$$

X : Scheme / Variety

$$\text{Cnstr}(X, \mathcal{D}^b(\overline{\mathbb{Q}}_e)) \simeq \text{Func}^{\text{cts}}(\text{Exit}(X), \mathcal{D}^b(\overline{\mathbb{Q}}_X))$$

Spec \mathbb{Z}



Spec \mathbb{F}_p

Spec \mathbb{Q}_p



$X^{FF}(R)$