Condensed Structures

pyknotic

A Gentle Introduction

Slogan: Infinite algebraic structures are best understood when topologies are present.

Examples:

topological groups,
locally compact abelian groups,
topological vector spaces,
adic rings,

etc., etc.

Example: Let G be a compact group.

A continuous unitary irreducible representation of G is finite-dimensional

G can be recovered from its continuous finite-dimensional representations:

$$\mathrm{Rep}(G) \xrightarrow[\omega]{} \mathrm{Vect} \qquad G \cong \mathrm{Aut}^{\otimes}(\omega)^{\sigma = \mathrm{id}}$$

Issues

1. Statements like the Isomorphism Theorems are FALSE.

$$0 \to \mathbb{R}^{\delta} \xrightarrow[\not\cong]{} \mathbb{R} \longrightarrow 0$$

LCA is not abelian

quasiabelian in some hom. alg.

2. Homological / homotopical algebra

is SUBTLE and FRAGILE.

$C^\infty(M,\mathbb{R})$ $\quad$ $C^\omega(M,\mathbb{R})$ $\quad$ $\mathbb{Q}_p$

BV-formalism

Appendix of Costello-Gwilliam

Differentiable vector spaces $/\mathbb{R}$

3. Tensor products have poor formal properties.

$\otimes$ of Banach, Fréchet

is complex.

4. Categorification is mysterious.

What even is a "topological higher category"?

Condensed/pyknotic structures try to repair these issues in a fundamental way, by "doing algebra" relative to a category that's "as good as" that of Sets, but of a topological nature.

Not looking for a conservative functor to Ab

$$\begin{tikzcd} \mathbb{R} \arrow[r, tail, "\delta"] \arrow[d] & \mathbb{R} \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

should not be a surjection

$$\mathrm{Map}(\emptyset, X) = *$$

Example. If X is a reasonable topological space, then it can be recovered from the functor

$$\mathrm{Map}(-, X) : \mathrm{Comp}^{op} \longrightarrow \mathrm{Set}$$

$$K \rightsquigarrow \mathrm{Map}(K, X)$$

$$\| $$

$$X(-)$$

$$X(R)$$

Salient properties: $\mathrm{Map}(K \sqcup L, X)$

① $X(K \sqcup L) = X(K) \times X(L)$

② $X(\emptyset) = *$

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & S \arrow[r] & S/R \arrow[r] & X \end{tikzcd}$$

$$\mathrm{Map}(S,X) \hookleftarrow \mathrm{Map}(S/R, X)$$

sheaf condition

③ $R \subseteq S \times S$ ⇝
closed equiv. relation

$$\begin{tikzcd} X(S/R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(R) \arrow[d] \\ X(S) \arrow[r] & X(R) \times X(R) \end{tikzcd}$$

Def^n A pyknotic set is a **sheaf**

$$X : \mathbf{Comp}^{op} \to \mathbf{Set}, \quad K \mapsto X(K)$$

for the topology given by quotients.

$X \in C$

$$よ_X : C^{op} \to \mathbf{Set}$$

$$Y \mapsto \mathrm{Map}(Y, X)$$

Helpful observation

Every compactum is a quotient of a free compactum

= compact Hausdorff space.

$$
\begin{tikzcd}
S \arrow[d] \arrow[dr, bend left] & \\
\beta(S) \arrow[r, dashed, "\exists!\text{-cts}"'] & K
\end{tikzcd}
$$

$\beta(S)$ – cptHaus

$\beta(S)$ = Stone-Čech compactification

↓

the closure of an open is still open.

$$\begin{array}{c} \mathbb{R}^\delta(K) = \mathrm{Map}(K, \mathbb{R}^\delta) = \text{locally constant fns. on } K \\ \Big\downarrow \\ \mathbb{R}(K) = \mathrm{Map}(K, \mathbb{R}) = \text{cts fns on } K \\ \Big\downarrow \\ \mathrm{Map}^{\mathrm{cts}}(K, \mathbb{R}) \big/ \mathrm{Map}^{\mathrm{lc}}(K, \mathbb{R}) \end{array}$$

Def^n A pyknotic set is a functor

$$X : \mathrm{Free}^{op} \to \mathrm{Set}$$

such that

1. $X(K \sqcup L) = X(K) \times X(L)$
2. $X(\emptyset) = *$

Defn The underlying set of X is $X(*)$

$$\mathrm{Pyk}(\mathrm{Set}) \longrightarrow \mathrm{Set}$$

Example: Every reasonable topological space.

Example:

$$0 \longrightarrow \mathbb{R}^\delta \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{R}^\delta \longrightarrow 0$$

Example: If R is a pyknotic ring, + $\mathrm{Mod}(R)$ is a pyknotic category.

$$\mathrm{Free}^{op} \longrightarrow \mathrm{Set}$$
$$\sqcup \qquad\qquad X$$

$$\mathrm{Mod}(R) : \mathrm{Free}^{op} \longrightarrow \mathrm{Cat}$$

$\mathrm{TVS}_{\mathbb{R}} \rightsquigarrow \mathrm{Pyk}(\mathrm{Vect}_{\mathbb{R}})$ $\qquad$ $\underleftarrow{\mathrm{Fin}}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$

$\mathrm{Pyk}(\mathrm{DVect}_{\mathbb{R}})$

X : scheme / Variety

$$\mathrm{Cnstr}(X, D^b(\overline{\mathbb{Q}}_\ell)) \simeq \mathrm{Fun}^{cts}(\mathrm{Exit}(X), D^b\overline{\mathbb{Q}}_\ell)$$

Spec $\mathbb{Z}$

Spec $\mathbb{F}_p$

Spec $\mathbb{Q}_p$

Y^{FF}

$Y^{FF}(R)$