Joint work with Bernd Siebert (2019).

History of mirror symmetry.


Calabi - Yau 3-folds tend to come in pairs $X, \bar{X}$.

(Calabi - Yau: compact complex manifold with a nowhere vanishing top-dimensional holomorphic form.)

$h^{1,1} (X) = h^{1,2} (\bar{X})$

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Example: The quintic 3-fold and its mirror.
\[ X = \mathbb{Z}(x_0^5 + \ldots + x_4^5) \leq \mathbb{P}^4 \]

(the quintic 3-fold)

\[ h^{1,1}(X) = 1, \quad h^{1,2}(X) = 101. \]

Greene-Plesser construction:

\[ \mathbb{Z}^5 \subset \mathbb{C} \mathbb{P}^4 \]

\[ (a_0, \ldots, a_4) \in \mathbb{Z}^5 \]

acts on \( \mathbb{P}^4 \) by

\[ (x_0, \ldots, x_4) \rightarrow (\zeta a_0 x_0, \ldots, \zeta a_4 x_4) \]

where \( \zeta = e^{2\pi i/5} \).

\[ G = \mathbb{Z}_5^4 \leq \mathbb{Z}_5^5 \]

\[ G = \zeta (a_0, \ldots, a_4) \in \mathbb{Z}_5^5 \mid \sum a_i = 0 \]

\( G \) acts on \( X \), and then \( X/G \) has a Calabi-Yau resolution \( \tilde{X} \rightarrow X/G. \)
Caldeira, de la Ossa, Green and Puc (after)

One can calculate the numbers of rational curves on \(X\) by performing certain period integrals on \(X\)

\[
\int_{\Delta_{\tilde{\mathcal{M}}}} \alpha \wedge \tilde{\omega}_{\tilde{\mathcal{M}}} \quad \alpha \in H^3(X, \mathbb{Z})
\]

\(\tilde{\mathcal{M}}\) is a nonholo-cyclic vanishing holomorphic

3-form.

A rational curve should be thought to first approximation, as a map \(f: \mathbb{P}^1 \to X\) representing some homology class \(\beta \in H_2(X, \mathbb{Z})\), i.e. \(f_* [\mathbb{P}^1] = \beta\). (Work modulo reparametrization.)
For example, the number of lines in $X$ is the number of holomorphic maps $f: \mathbb{C} \mathbb{P}^1 \to X$ representing the class of a line in $H_2(X, \mathbb{Z})$

(Number = 2875)

# cones = 609250 (S. Katz)

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Constructions of mirror pairs:

- Batyrev: Mirror pairs can be constructed as hypersurfaces in toric varieties. (1992)

Question: Is there a general mirror construction?

Yes!
Our context:

we fix a log Calabi-Yau manifold, i.e., a pair $(X, D)$ where $X$ is a non-singular variety and $D \subseteq X$ is a normal crossings divisor with $K_X + D = 0$. In other words, $X$ is a nowhere holomorphic form on $X \times D$ with simple poles along $D$.

In general, we will either consider the case that \( X \) is compact or the case that \( X \) is a family \( X_0 \in X \) with \( D = X_0 \) fibers of one-dimensional germ and \( T \) should be of a curve.

Here the fibers of \( T \) are CX manifolds.
Let \( D = D_1 + \ldots + D_5 \) be the irreducible decomposition of \( D \).

\[ \text{dual complex of } D. \]

Assume, for \( I \subseteq \{1, \ldots, 5\} \),

\[ D_I = \bigcap_{i \in I} D_i \] is connected.

\[ \text{e.g. } (1^2, A) \quad (1^2, \emptyset) \]

we will build the dual complex of \( D \) as a cone complex in the \( \mathbb{R}^3 \)-vector space with basis \( D_1, \ldots, D_5 \).

\[ \mathcal{P} = \left\{ \sum_{i \in I} \mathbb{R}_{\geq 0} D_i \mid I \subseteq \{1, \ldots, 5\}, \: D_I \neq \emptyset \right\} \]

Let \( B = \bigcup_{D \in \mathcal{P}} \).
Example: \( (\mathbb{P}^1 \times \mathbb{P}^1, \overline{D} = (303 \times \mathbb{P}^1) \cup 303 \times 101) \cup (101 \times 303) \cup (101 \times 303) \)

Blow-up a point in \( 303 \times \mathbb{P}^1 \) and let \( X \) be the blow-up and \( D \) the strict transform of \( \overline{D} \).

\[ \text{Note this is a piecewise linear object.} \]
\[ B(\mathbb{Z}) = \{ \sum a_i D_i \in B \mid a_i \in \mathbb{Z} \}. \]

Point of \( B(\mathbb{Z}) \) record tangency orders for holomorphic maps of curves
\[ f: C \longrightarrow X. \]

Typical question: consider maps from pointed curves
\[ f: (C, x_1, \ldots, x_n) \longrightarrow X \]
\[ x_1, \ldots, x_n \in C \text{ distinct points}, \]
and we wish to impose tangency of \( f \) at a point \( x_i \) with \( \operatorname{disc} \) \( D_y \).
f. Being tangent to $D_j$ at $x_i$ to order $a$ means: take a local defining equation for $D_j$ near $f(x_i)$, say $t_j = 0$. Then $t_j$ of $f$ is a function on $C$ in a neighborhood of $x_i$. It has an order of vanishing, which should be $a$.

A point $E_q$ of $B(Z)$ encodes such a tangency condition; i.e., we should be tangent to $D_j$ to order $a_j$. 
Assume: \( \dim_k \mathcal{B} = \dim_k \mathcal{X} \)

( In the case \( \mathcal{X} \to \mathcal{Y} \) with CY fibers, this is the maximally unipotent condition.)

We will construct a ring \( R \) which is either the affine or projective coordinate ring (in the lug und relative cases respectively) of the mirror.

Fix \( P \in H_2(X, \mathbb{Z}) \) with the property that

- \( P \) contains the class of every effective curve,
- \( P \) closed under addition,
- \( 0 \in P \)
- \( P \cap (-P) = H_2(X, \mathbb{Z})_{tor} \)
\[ A = k \mathbb{EPZ} = \text{monoid ring of } P = \bigoplus_{p \in P} k \cdot t^{p}, \]
\[ t^{p} \cdot t^{p'} = t^{p+p'}, \]

Fix also a monomial ideal \( \mathbb{I} \subseteq A \) such that \( A_{\mathbb{I}} := A/\mathbb{I} \) is Artinian. We construct an \( A_{\mathbb{I}} \)-algebra \( R_{\mathbb{I}} \), giving a family

\[
\begin{array}{ccc}
\text{Spec } R_{\mathbb{I}} & \xrightarrow{p} & \text{Proj } R_{\mathbb{I}} \\
\downarrow & & \downarrow \\
\text{Spec } A_{\mathbb{I}} & & \text{Spec } A_{\mathbb{I}} \\
\end{array}
\]

In fractesimal neighborhood of large Kähler limit of \( X \).
(Taking limit over all $D$ will give mirror construction in a formal neighborhood of the large Kähler limit.)

Construction of $R_D$:

$$R_D := \bigoplus_{p \in B(\mathbb{Z})} A_D \cdot \Theta_p$$

a free $A_D$-module.

Need to define

$$\Theta_p \cdot \Theta_q = \sum_{t \in B(\mathbb{Z})} \alpha_{pqt} \Theta_t$$

where $\alpha_{pqt} \in A_D = k[\mathbb{P}]/I$

$$\alpha_{pqt} = \sum_{\mathbf{r} \in \mathbb{Q}^+} N_{pqrst} \cdot t^r$$

where $N_{pqrst} \in \mathbb{Q} \leq k$. 
Define \( N_{pqr}^\beta \).

\[ r \in B(Z), \quad r = \sum_{i \in \mathbb{Z}} a_i D_i \text{ with } a_i > 0. \]

Fix a point \( z \in D^\mathbb{Z} \). (A point of the variety.)

Let \( N_{pqr}^\beta = \# \) of maps

\[ f : (C, x_1, x_2, x_{\alpha+1}) \to X \]

with

- \( C \) rational (genus 0)
- \( f|_C = \beta \)
- The tangency condition at \( x_1 \) is given by \( p \).
- The tangency condition at \( x_2 \) is given by \( z \).
- The tangency condition at \( x_{\alpha+1} \) is given by \( -r \).

and \( f(x_{\alpha+1}) = z \).
To define negative orders of tangency one uses logarithmic geometry and defines punctured invariants (Abrahamovich, Chang - , Siebert)

**Theorem (G-Siebert)** The numbers $N^B_{pqr}$ can be derived rigorously, and with this product $R_D$ is an associative, commutative algebra with $1 = \Theta$. 