RESURGENCE AND NON-PERTURBATIVE TOPOLOGICAL STRINGS

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In most quantum theories, we can use perturbation theory to calculate observables as power series in a "small" coupling z:

$$\varphi(z) = \sum_{n \geq 0} a_n z^n$$

Unfortunately, these series are typically factorially divergent

[Dyson]

$$a_n \sim n!$$

It is therefore nontrivial to extract, say, numerical predictions from these series!

String theory is no exception!

String Perturbation Theory Diverges

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(Received 11 April 1988)

It has been recognized for a long time that the factorial divergence of perturbative series is a signal of "non-perturbative effects" that have to be taken into account, of the form

$$\mathrm{e}^{-A/z}$$

This idea has been given appropriate mathematical form in the **theory of resurgence** of Jean Ecalle, who was inspired in part by the work of physicists studying non-perturbative effects in quantum theory.

In this talk, based on this theory, I will introduce the concept of a **resurgent structure** associated to a factorially divergent series. This gives a proper mathematical framework to understand "non-perturbative sectors."

I will then show how this idea can be applied to topological string theory.

From wild series to analytic functions

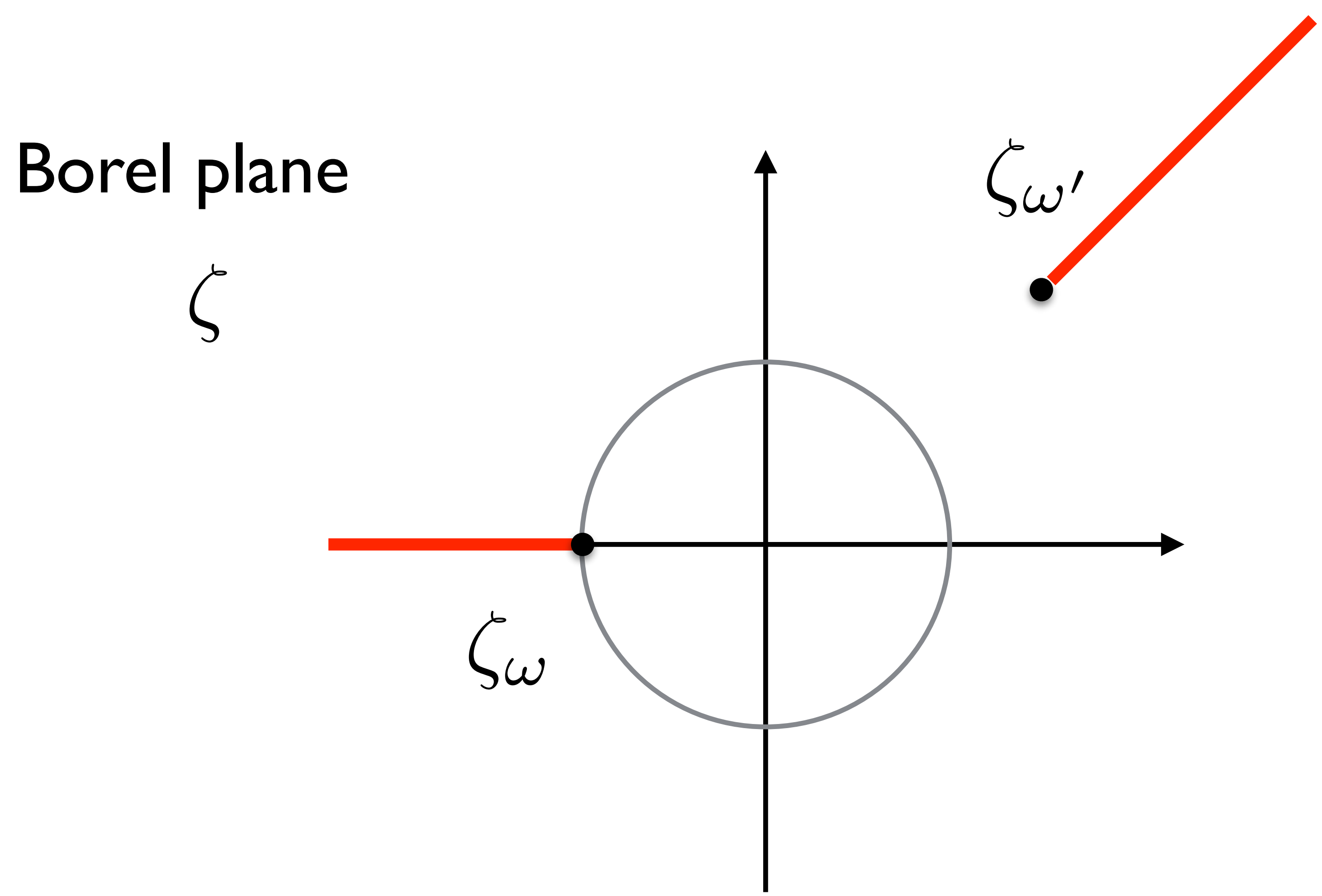
Let us consider a formal power series with factorially growing coefficients

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \qquad\qquad a_n \sim n!$$

These are sometimes called Gevrey-1 series. The first step in resurgence is the **Borel transform**, a deceptively simple way of transforming these series into "nice" functions

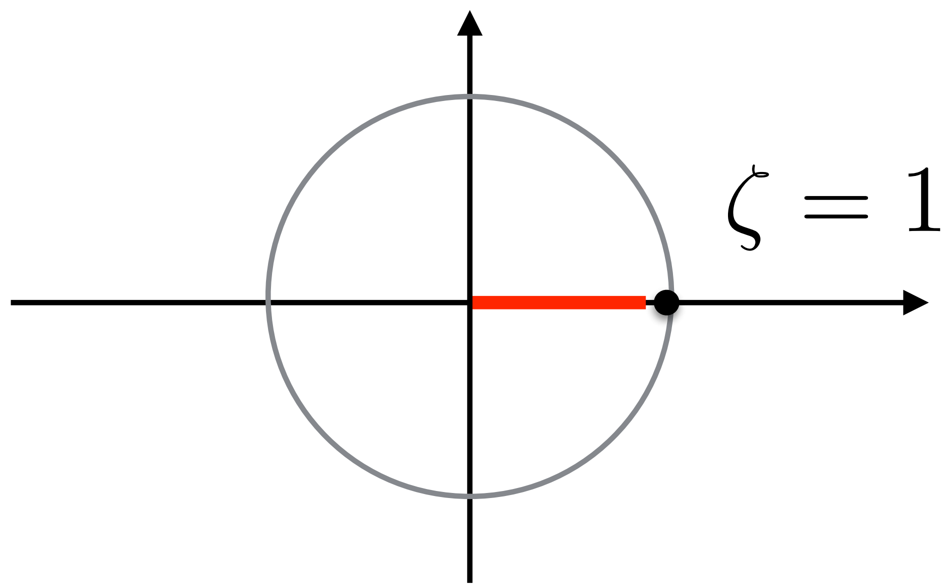
$$\varphi(z) = \sum_{n \geq 0} a_n z^n \xrightarrow{\text{Borel transform}} \widehat{\varphi}(\zeta) = \sum_{n \geq 0} \frac{a_n}{n!} \zeta^n$$

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. We now demand that it can be "endlessly analytically continued" to the complex plane, displaying a set of **singularities** (poles, branch cuts)



Example: $$\varphi(z) = \sum_{k\geq 0} k! z^k$$

$$\widehat{\varphi}(\zeta) = \sum_{k\geq 0} \zeta^k = \frac{1}{1-\zeta}$$



One of the main ideas in the theory of resurgence is that the singularities of the Borel transform contain non-perturbative information, which is hidden in the original divergent series

To extract this information, we have to consider the expansion of the Borel transform around each singularity. These leads to **new formal power series**.

Let us consider for simplicity the so-called **simple resurgent functions**, where singularities are logarithmic branch cuts. The expansion around a singularity at $\zeta = \zeta_\omega$ has the form

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega\, \widehat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi \mathrm{i}} + \text{regular}$$

The function $\widehat{\varphi}_\omega(\xi)$ is typically analytic at the origin

$$\widehat{\varphi}_\omega(\xi) = \sum_{n\geq 0} \widehat{a}_{n,\omega}\xi^n$$

but we can think about it as the Borel transform of a **new, factorially divergent power series** associated to the singularity:

$$\varphi_\omega(z) = \sum_{n\geq 0} a_{n,\omega} z^n \qquad a_{n,\omega} = n!\,\widehat{a}_{n,\omega}$$

The constant S_ω is called a **Stokes constant**. Its value depends on the normalization of $\varphi_\omega(z)$

Resurgent structures

We can repeat the same analysis for the new power series found in this way, and generate further series. At the end, we obtain **a set of formal power series** associated to the original power series, which I will call the **resurgent structure** associated to $\varphi(z)$

$$\varphi(z) \longrightarrow \mathfrak{B}_\varphi = \{\varphi_\omega(z)\}_{\omega \in \Omega}$$

Therefore, from a single factorially divergent series we obtain a very rich structure!

In actual calculations, the new series are multiplied by an exponential involving the location of the singularity

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z} \varphi_\omega(z)$$

These objects, which involve a new, exponentially small parameter, are sometimes called **trans-series**. They have many mathematical applications, from the solutions of ODEs to the saddle-point analysis of multi-dimensional integrals.

In physics, trans-series can be sometimes interpreted as **instanton corrections**, i.e. expansions around non-trivial saddle-points of the path integral.

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z)$$

instanton action

quantum fluctuations around the instanton

Resurgence and asymptotics

The "largest" exponentially small trans-series, associated to the closest Borel singularity to the origin

$$\Phi_{\mathcal{A}} = \mathrm{e}^{-\mathcal{A}/g_s}\left(c_0 + c_1 g_s + \cdots\right)$$

turns out to determine the asymptotic behavior of the perturbative series

$$a_n \sim \frac{\mathcal{A}^{-n}}{2\pi}\Gamma(n)\left(c_0 + \frac{c_1\mathcal{A}}{n-1} + \cdots\right)$$

The trans-series **"resurges"** in the perturbative series

This is the famous connection between perturbative and non-perturbative sectors predicted by resurgence, which goes back to the pioneering work by Bender and Wu

Large-Order Behavior of Perturbation Theory

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(Received 28 June 1971)

This connection will be useful to obtain experimental tests of our results.

Topological string theory

Let *X* be a Calabi-Yau (CY) threefold. At each genus *g* we can compute the topological string free energy $F_g(t)$, which depends on the Kahler moduli *t* (I will often consider one-modulus CYs for simplicity)

At large *t* this has an expansion encoding Gromov-Witten invariants of *X*, which "count" holomorphic curves of genus *g* in *X*:

$$F_g(t) = \sum_{d \geq 1} N_{g,d}\, \mathrm{e}^{-dt}$$

I will use mirror symmetry throughout. In the mirror manifold one can calculate **periods** by integrating the holomorphic 3-form over a symplectic basis of 3-cycles

$$X^I = \int_{\alpha^I} \Omega \qquad \mathcal{F}_I = \int_{\beta_I} \Omega$$

$$I = 0, 1, \cdots, n$$

The X^I are projective coordinates of the CY moduli space, and the mirror map is

$$t^a(z) = \frac{X^a(z)}{X^0(z)}$$

String perturbation theory tells us that the **total free energy** is given by a genus expansion in a small parameter, a.k.a. the string coupling constant

$$F(t, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

General arguments [Gross-Periwal, Shenker] indicate that this series grows doubly-factorially, at fixed *t*

$$F_g(t) \sim (2g)!, \qquad g \gg 1$$

What is the resurgent structure associated to this series?

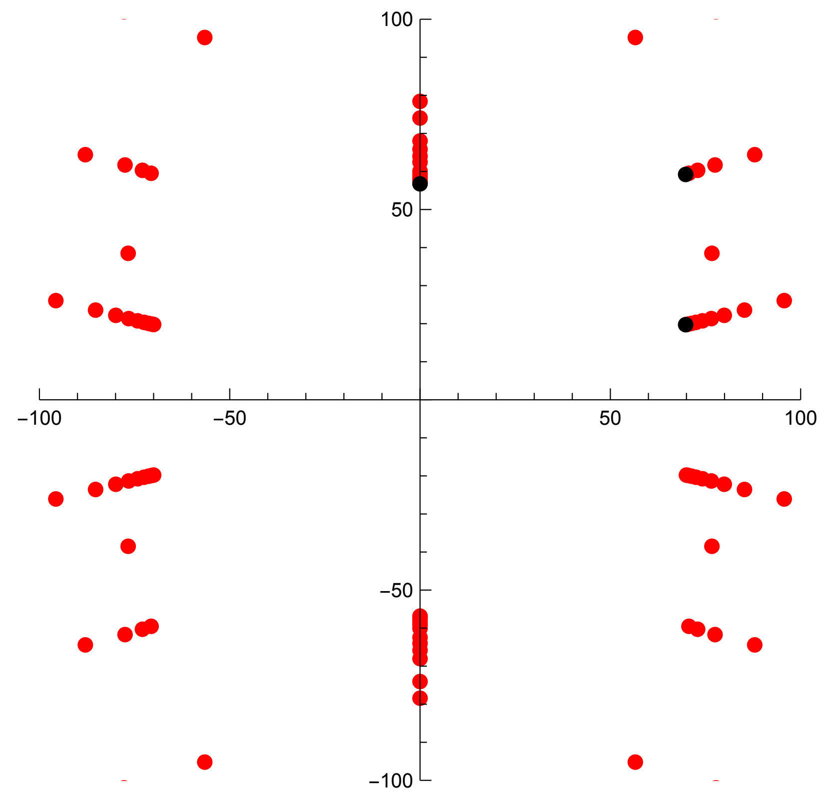
This is a difficult problem. Note that in this case the resurgent structure depends on the moduli of the CY manifold, parametrized by t.

I will present a conjecture on the possible location of Borel singularities, and an exact description of the trans-series associated to singularities.

Borel plane and CY periods

Conjecture: the Borel singularities for the series of free energies are **integral periods**

Determining which periods are actual singularities is much harder and only partial information is available (often based on numerical calculations)



Trans-series for topological strings

How do we determine the trans-series associated to the singularities?

In the case of ODEs, a simple way to obtain trans-series is to use an ansatz involving exponentially small terms [Ecalle, Costin, …]

Euler equation: $$x^2 y'(x) - y(x) = -x$$

perturbative solution: $$y_{\rm p}(x) = \sum_{n\geq 0} n! x^{n+1}$$

trans-series solution: $$y(x) = y_{\rm p}(x) + \boxed{C{\rm e}^{-1/x}}$$

In the case of the topological string, and in contrast to e.g. non-critical strings, there is no ODE in the string coupling constant.

However, we have a PDE governing the total free energy: the famous **holomorphic anomaly equations** (HAE) of BCOV

I will write a simplified version of the HAE, involving a single **propagator** S and the complex modulus z of the CY

$$\frac{\partial F}{\partial S} = \frac{g_s^2}{2} D_z^2 F + \frac{1}{2}(D_z F)^2$$

One can solve this equation perturbatively, to obtain the free energies as polynomials in the propagator, and involving known functions of the modulus *z* [BCOV, Yamaguchi-Yau, Grimm-Klemm-M.M.-Weiss, Alim-Lange, Klemm et al., …]

$$F_2(S, z) = \frac{\partial_z^3 F_0}{24} S^3 + \cdots$$

The conventional topological string free energies are recovered in the so-called **holomorphic limit**, where *S* becomes a (known) function of *z*.

In this way one can obtain the perturbative series to high order in the genus [Klemm et al.]

The CESV trans-series ansatz

CESV [Couso-Edelstein-Schiappa-Vonk] proposed to solve the HAE with a **trans-series ansatz** as in Ecalle's theory of ODEs

$$F = \sum_{g \geq 0} F_g(S, z) g_s^{2g-2} + \mathrm{e}^{-\mathcal{A}/g_s} \sum_{n \geq 0} F_n^{(1)}(S, z) g_s^{n-1} + \cdots$$

perturbative series instanton correction

In the case of one-modulus, toric CYs they obtained solutions at low orders in the string coupling constant

In recent work [Gu-M.M, Gu-Kashani-Poor-Klemm-M.M.] we obtained an all-orders, **exact solution** for the trans-series, for **any** CY (compact or not)

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I = c^I \frac{\partial F_0}{\partial X^I}$$

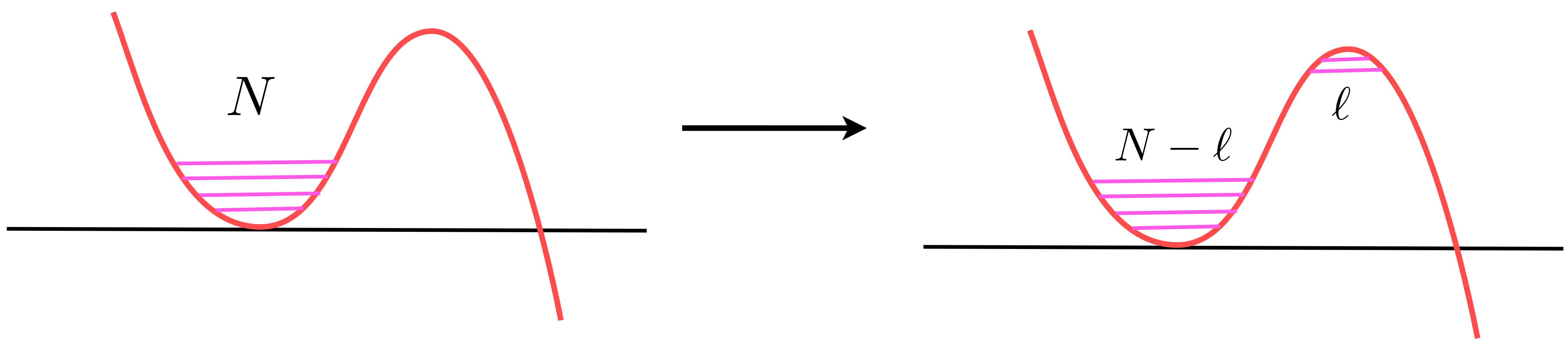
$$\Phi_{\mathcal{A}} = \left(1 + g_s c^J \frac{\partial F}{\partial X^J}\left(X^I - g_s c^I\right)\right) \mathrm{e}^{F\left(X^I - g_s c^I\right) - F\left(X^I\right)}$$

This is a universal formula for the "one-instanton amplitude" associated to $\mathcal{A}$

Remarkably, it can be written in terms of perturbative data only. Multi-instanton solutions are available, too.

This formula suggests that the the periods X^I are **quantized** in units of the string coupling constant, as postulated in large *N* dualities.

In fact, the form of the one-instanton amplitude is reminiscent of "eigenvalue tunneling" in matrix models and non-critical strings

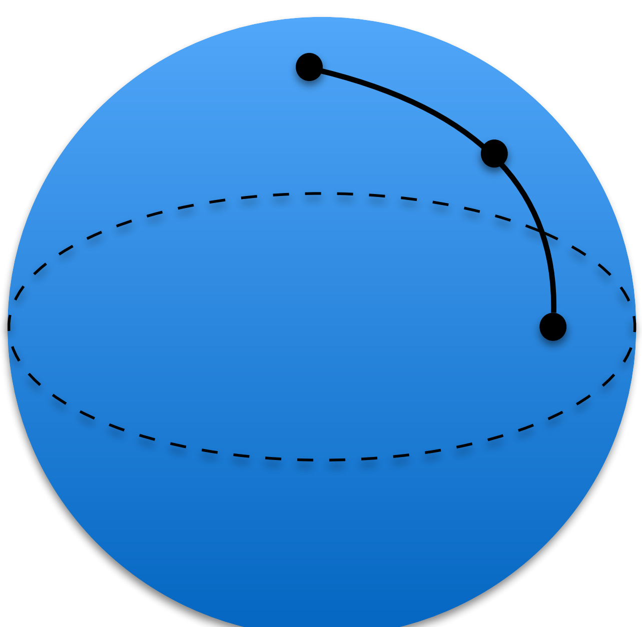


Experimental evidence: asymptotics in the quintic CY

We can test the one-instanton amplitude against the large genus asymptotics of the free energies in e.g. the famous quintic CY

large radius

$$z = 0$$

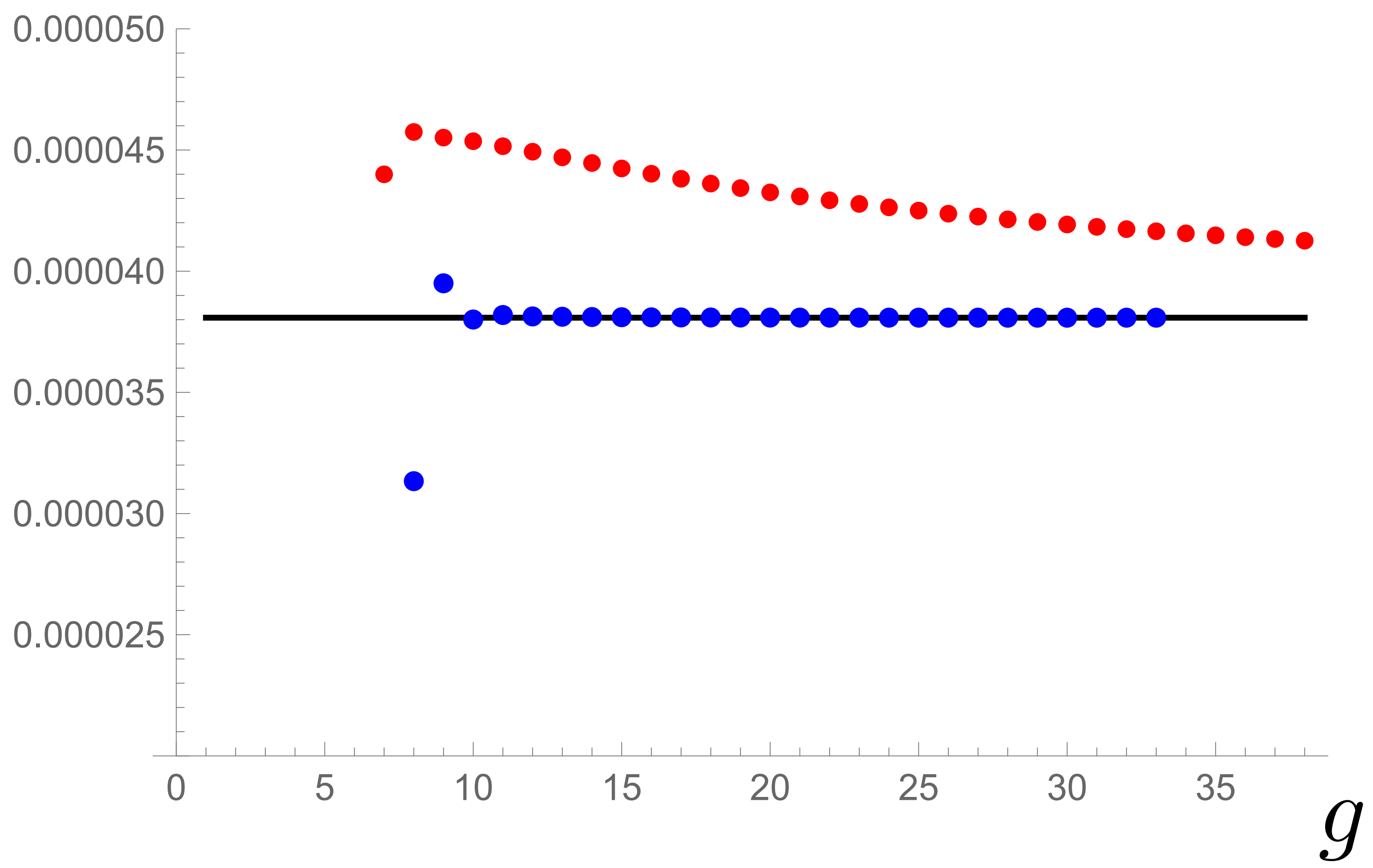


$$z = \frac{5^{-5}}{2} \longrightarrow \mathcal{A} = \mathcal{F}_0$$

vanishing period at the conifold

$$z = 5^{-5}$$

conifold point



red dots: sequence $\frac{\mathcal{A}^{2g-1}}{\Gamma(2g-1)} F_g$

blue dots: Richardson acceleration

black line: prediction from one-instanton formula

Stokes constants

An important piece of the resurgent structure are the Stokes constants. In some interesting examples (e.g. WKB, complex Chern-Simons theory) they turn out to be **integers** related to BPS invariants [Gaiotto-Moore-Neitzke, Garoufalidis-Gu-M.M].

We expect a similar picture here. We can show e.g. that the genus zero Gopakumar-Vafa invariants $n_{0,d}$ arise as Stokes constants associated to towers of Borel singularities at

$$dX^1 + mX^0 \qquad m \in \mathbb{Z}$$

Relation to other work

Some recent works have explored the relation between susy gauge theories, topological strings, Riemann-Hilbert problems, and BPS counting [Bridgeland, Neitzke, Hollands, Grassi, Longhi, Alim, Teschner, …]. Many of these works assume the Delabaere-Pham (or Kontsevich-Soibelman) form for Stokes automorphisms.

Our approach is based on a more universal tool, the theory of resurgence, where Stokes automorphisms are more general. It follows from our results that Stokes automorphisms for topological strings do not seem to have the Delabaere-Pham form.

Conclusions and outlook

The theory of resurgence gives a precise mathematical framework to understand non-perturbative sectors, which can be applied successfully to (topological) string theory.

To do this, we have developed instanton calculus in the Kodaira-Spencer theory of BCOV, and managed to find **exact** solutions for instanton amplitudes. They lead to precise formulae for large genus asymptotics.

Our results apply as well to other systems governed by the HAE, like large N matrix models

The full resurgent structure requires determining the possible Borel singularities and their Stokes constants, and we expect a very rich mathematics and physics related to BPS invariants and Riemann-Hilbert problems.

What is the meaning of the "instanton" amplitudes we obtained? They look like D-branes of the "wrong" type. Are they rather "renormalons" of the topological string?

Relation to proposals for non-perturbative definitions of the topological string, like the one in [Grassi-Hatsuda-M.M.]?

Thank you for your attention!