LORENTZIAN

FLAT

Haag-Kastler

$\mathcal{U} \subset \mathbb{R}^{d-1,1}$
$\rightsquigarrow$ $*$-algebra
$A_{\mathcal{U}} \subset \operatorname{End} \mathcal{H}$

$\mathcal{H}$: Hilbert space
Action of $\mathrm{Poincaré}(\mathbb{R}^{d-1,1})$, Energy ≥ 0

Wightman

$\phi(x) \in S'(\mathbb{R}^{d-1,1}) \otimes \operatorname{End} \mathcal{H}$

v.e.v. $\langle vac |\, \phi(x_1) \cdots \phi(x_n)\, | vac \rangle$

Depends on order

commute if $x_i - x_{i+1}$ is space-like

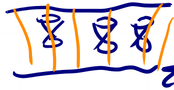
CURVED

Brunetti-Fredenhagen-Verch-Fewster

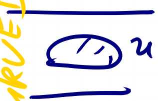
$\forall$ space-like hypersurface $\Sigma \rightsquigarrow \mathcal{H}_\Sigma$

Hilbert space

Globally hyperbolic



$\rightsquigarrow$ Unitary $\approx$ $\mathcal{H}_{\Sigma_-} = \mathcal{H}_{\Sigma_+}$



$A_{\mathcal{U}} \otimes \mathcal{H}_{\Sigma_-} \to \mathcal{H}_{\Sigma_+}$

EUCLIDEAN

Wick rotation

Osterwalder-Schrader

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \text{"} \int e^{-S(\phi)} \phi(x_1) \cdots \phi(x_n)\, \mathcal{D}\phi \text{"}$$

symmetric

$x_i \neq x_j$

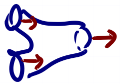
Reflection positivity $\langle$  $\rangle \geq 0$

Nobody

("Atiyah-like", for 2d CFT: G. Segal M, K. '96)

$$\mathrm{Bordisms}_{\mathrm{met}} \xrightarrow{\otimes} \text{vector spaces}$$

$\sqcup$ $\qquad\qquad$ $\otimes$



not Hilbert spaces in general

Linear algebra

$V/\mathbb{R}$ $\quad \dim V = d < +\infty$

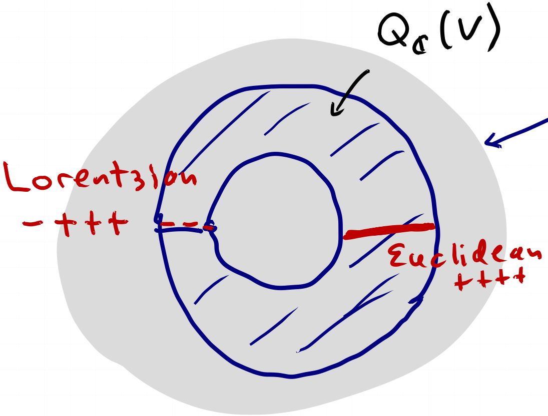
subset of $\mathrm{Hom}_{\mathbb{R}}(\mathrm{Sym}^2 V, \mathbb{C})$

Def. $Q_{\mathbb{C}}(V) := \{\mathbb{C}$-valued quadr. forms on V such that $\exists$ real basis,

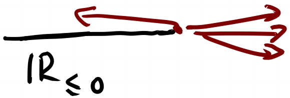
$$\text{form} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} \qquad \lambda_i \in \mathbb{C} - \mathbb{R}_{\leq 0}$$

$$\boxed{\sum_i |\mathrm{Arg}\,\lambda_i| < \pi}$$

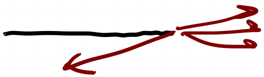
$$Q_{\mathbb{C}}(V) \underset{\text{open}}{\subset} GL(d,\mathbb{C})/O(d,\mathbb{C}) \underset{\text{open}}{\subset} \mathbb{C}^{d(d+1)/2}$$



$GL(d,\mathbb{C})/O(d,\mathbb{C})$



or



Facts:

① $Q_{\mathbb{C}}(V)$ is contractible

closure $\overline{Q_{\mathbb{C}}(V)}$ in $GL(d,\mathbb{C})/O(d,\mathbb{C})$ $\sim S^1$

② $Q_{\mathbb{C}}(V)$ is domain of holomorphy

③ $\forall$ $W \subset V$ get restriction

$$Q_{\mathbb{C}}(V) \xrightarrow[|_W]{} Q_{\mathbb{C}}(W)$$

Def Unitary QFT :

$$\text{Bordisms}_{\mathbb{C}\text{-met}} \xrightarrow{\otimes\text{-functor}} \text{nuclear Fréchet spaces}$$

(non-unital category) $\quad \sqcup \mapsto \otimes \quad$ and nuclear morphisms (non-unital category)

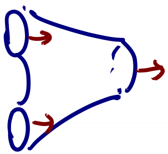
$\varprojlim (H_1 \leftarrow H_2 \leftarrow H_3 \leftarrow \cdots)$

Hilbert spaces
Hilbert-Schmidt maps.

Morphisms : $(M, g_{\mathbb{C}})$ C^ω

compact with cooriented ∂M

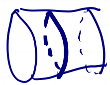
$= \partial M_- \sqcup \partial M_+$



$$g_{\mathbb{C}} \in \Gamma \left(\begin{array}{ccc} \text{Bundle} & & Q_{\mathbb{C}}(T_x M) \\ \downarrow & & \downarrow \\ M & \ni & x \end{array} \right)$$

Objects : Σ^{d-1} + collar + $g_{\mathbb{C}}$ on collar

$\partial \Sigma^{d-1} = \phi$



Notation :

Σ, collar, $g_\Sigma \rightsquigarrow \mathcal{H}_\Sigma$

$M, g_M \rightsquigarrow \psi_M$

$\psi_M : \mathcal{H}_{\partial M_-} \to \mathcal{H}_{\partial M_+}$

Axioms:

1. Holomorphicity

2. Semi-continuity

3. Positivity (= unitarity)

① Holomorphicity.

$\mathcal{H}_\Sigma$: holomorphic bundle on $\mathrm{Met}_{\mathbb{C}}\left(\begin{smallmatrix}\text{collar}\\ \text{of } \Sigma\end{smallmatrix}\right)$

∞-dim $\mathbb{C}$-manifold

Ψ_M : holomorphic on $\mathrm{Met}_{\mathbb{C}}(M)$

for finite-dim. families.

Covariance : $\mathcal{H}_\Sigma \circlearrowleft \mathrm{Diff}^\omega(\mathrm{Collar}_\Sigma)$

Ψ_M equiv. $\circlearrowleft \mathrm{Diff}^\omega(M)$

$\partial M = \emptyset$

$\mathcal{H}_{\partial M} = \mathbb{C}$ $\Rightarrow$ $\Psi_M =: Z_M \in \mathbb{C}$

partition function

: speial case

Function $g_\mathbb{C} \longmapsto Z_{M,g_\mathbb{C}}$ is holomorphic on ∞-dim $\mathbb{C}$-mfd $\mathrm{Met}_\mathbb{C}(M)$

invariant under ∞-dim $\mathbb{R}$-Lie group $\mathrm{Diff}^\omega(M)$

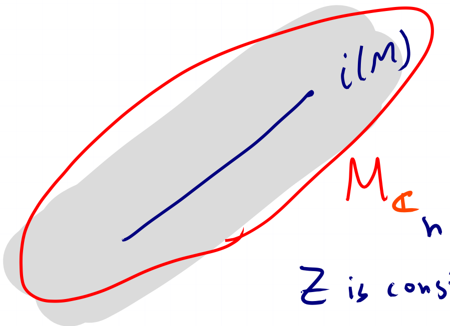
$\rightsquigarrow$ is invariant under "action of complexification")

$(M, g_\mathbb{C})$ is C^ω ← real-analytic $\rightsquigarrow$ $i : M \hookrightarrow (M_\mathbb{C}, g_\mathbb{C}^{hol})$

complex d-dim mfd

holom. section of $\mathrm{Sym}^2 T^*_{M_\mathbb{C}}$

$$g_\mathbb{C} = i^* g_\mathbb{C}^{hol}$$

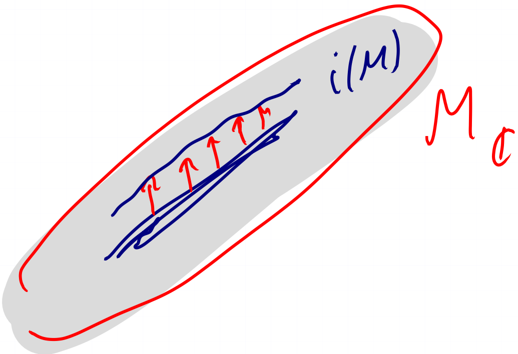


hence on $\{ \tilde{i} : M \to M_\mathbb{C} \mid \tilde{i} \text{ close to } i \}$ ∞-dim $\mathbb{C}$-mfd

$\cup$ totally real

Z is constant on $\{ M \xrightarrow{f} M \overset{i}{\hookrightarrow} M_\mathbb{C} \mid f \in \mathrm{Diff}^\omega(M) \text{ close to } id_M \}$

Conclusion:

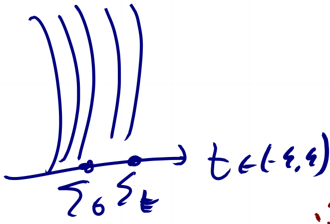


instead of allowable $\mathbb{C}$-metrics $/ \mathrm{Diff}^{\omega}$

$\rightsquigarrow$ allowable totally real submfds in $(M_{\mathbb{C}}, g_{\mathbb{C}}) /$ isotopy among allowable submfds.

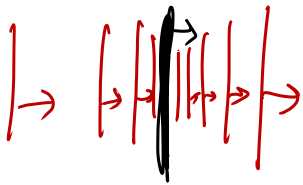
② Semicontinuity (in purely Euclidean setup)

$\Sigma = \Sigma_0$



$$\underset{\xrightarrow[t<0]{}}{\lim} \mathcal{H}_t \to \mathcal{H}_\Sigma \to \underset{\xleftarrow[t>0]{}}{\lim} \mathcal{H}_{\Sigma_t}$$

$\boxed{\text{Axiom: it is } =}$



$\mathcal{H}^{small}_\Sigma$

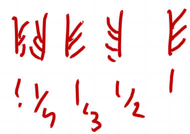
continuous dual

Lemma: perfect duality

$$\begin{cases} \mathcal{H}_\Sigma = \left(\mathcal{H}^{small}_{\Sigma^*}\right)^* \\ \mathcal{H}^*_\Sigma = \mathcal{H}^{small}_{\Sigma^*} \end{cases}$$

opposite coorientation

Proof: $\mathcal{H}_\Sigma = \underset{\xleftarrow[n]{}}{\lim} \mathcal{H}_{\Sigma_{\frac{1}{n}}} = \underset{\longleftarrow}{\lim} \left(H_{\Sigma^*_{\frac{1}{2n}}}\right)^*$



$$\cdots \to \left(H_{\Sigma^*_{\frac{1}{4}}}\right)^* \to H_{\Sigma_{1/3}} \to \left(H_{\Sigma^*_{1/2}}\right)^* \to H_\Sigma.$$

$A \otimes B \to C \rightsquigarrow A \to B^*$

$C \to A \otimes B \rightsquigarrow A^* \to B$

3 Positivity

3.1 Reality

$$Z(M, \overline{g_{\mathbb{C}}}) = \overline{Z(M, g_{\mathbb{C}})}$$

+ change global orientation

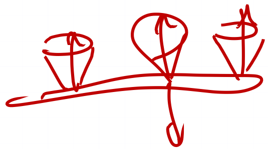
same for Σ, $\quad \underset{\text{opp. orientation}}{\mathcal{H}_{\Sigma, \overline{g_{\mathbb{C}}}}} = \overline{\mathcal{H}_{\Sigma, g_{\mathbb{C}}}}$

Axiom: $\quad (\Sigma, \overline{g_{\mathbb{C}}}) = (\Sigma^*, g_{\mathbb{C}}) \overset{1:1}{\Longleftrightarrow}$ space like hypersurface in Lorentzian signature

$$\mathcal{H}^{\text{smal}}_{\Sigma, g_{\mathbb{C}}} \otimes \overline{\mathcal{H}}^{\text{small}}_{\Sigma, g_{\mathbb{C}}} \longrightarrow \mathbb{C}$$

(pseudo)-hermitean pairing

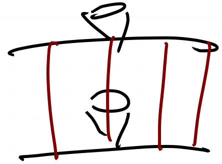
is ≥ 0



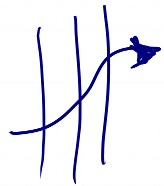
it

Conclusions

globally hyperbolic space-time

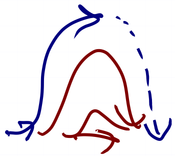


$$\mathcal{H}_{\Sigma_-} \xrightarrow[\approx]{\text{unitary}} \mathcal{H}_{\Sigma_+}$$

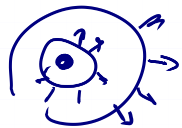


$d = 1$ $\quad M_{\mathbb{C}} = \mathbb{C}, \; g^{hol} = (dz)^2$

$i\mathbb{R} \uparrow$

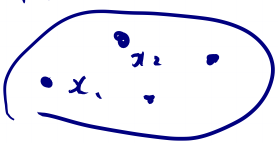


Local observables



$\mathcal{H}_{pt} := \lim\limits_{\text{radius} \to 0} \mathcal{H}$

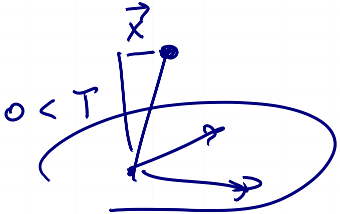
"$\varphi_i(x_i) \in \mathcal{H}_{x_i}$"



$\rightsquigarrow \langle \prod \varphi_i(x_i) \rangle$

In the Euclidean setup ("Atiyah axioms")

$$Z(\text{flat torus}) : SO(d,\mathbb{R})\backslash GL_+(d,\mathbb{R})/SL(d,\mathbb{Z}) \to \mathbb{C}$$

Property: 

$\Gamma_{d-1} \subset \mathbb{R}^{d-1}$ lattice, $\simeq \mathbb{Z}^{d-1}$. Fix Γ_{d-1}

$0 < T$, $\vec{x} \in \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$,

$$Z(\vec{x}, T, \Gamma_{d-1}) = \sum e^{-(E_\alpha T + 2\pi\sqrt{-1}(\vec{x}, \mu_\alpha))}$$

$\mu_\alpha \in \mathbb{Z}^{d-1}$

$E_\alpha \in \mathbb{C}$ depend on $[\Gamma] \in SO(d-1,\mathbb{R})\backslash GL_+(d-1,\mathbb{R})/SL(d-1,\mathbb{Z})$

$\mathrm{Re}\, E_\alpha \to +\infty$

$\rightsquigarrow$ Analytic bootstrap

$$Z \in \mathcal{O}_{hol}\left(Q_{\mathbb{C}}(\mathbb{R}^d)/SL(d,\mathbb{Z})\right)$$

Why $\sum |\operatorname{Arg} \lambda_j| < \pi$?

$$\Updownarrow$$

Necessary :

$\forall \quad 1 \leq p \leq d-1$

$\forall$ real p. form $\alpha \neq 0$

$$\boxed{\operatorname{Re} \int \alpha \wedge * \alpha > 0}$$

Higher abelian gauge theories

$$\int \exp\left(-C\int(\alpha\wedge *\alpha)\right)$$

$C>0$
coupling constant.

$\alpha\in\Omega^p : d\alpha=0$

$[\alpha]\in H^p(M;\mathbb{Z})$

coupling constant >0

$$Z\left(\begin{array}{l}\text{torus } \mathbb{R}^d/\mathbb{Z}^d\\ \cdot\ \text{constant}\\ \quad \text{ } - \text{metric}\end{array}\right)=(\det{}'\Delta_g)^{?}(\det g)^{?}\cdot\sum_{\substack{\alpha\in\Omega^p \text{ constant coeff.}\\ [\alpha]\in H^p(M,\mathbb{Z})}}\exp\left(-C\int\alpha\wedge*\alpha\right)$$

$p=1$ Maps to S^1

$p=2$ $U(1)$ - gauge theory

Why it is sufficient?

Use Wightman axioms : Wick rotation is based on

Extended tube domain

$$\mathbb{D}_n := \left\{ (\vec{x}_1 \ldots \vec{x}_n) \;\middle|\; \begin{array}{l} \vec{x}_i \in \mathbb{C}^d \\ \vec{x}_i \neq \vec{x}_j \end{array} : \begin{array}{l} \exists\, g \in SO(d-1,1;\mathbb{C}) \\ \exists\, \sigma \in \mathrm{Sym}_n \end{array} \right.$$

$$\mathrm{Im}\left(g(\vec{x}_{\sigma(i+1)}) - g(\vec{x}_{\sigma(i)}) \right) \in \underset{\text{open Light cone}}{V_+}$$

$$\mathbb{D}_n \underset{\text{open}}{\subset} \mathrm{Conf}_n(\mathbb{C}^d)$$

$$\mathbb{D}_n \supset \mathrm{Conf}(\underset{\text{Euclidean}}{\mathbb{R}^d})$$

Claim: $\forall\ V \subset \mathbb{C}^d$

totally real d-dim. subspace.

such that $-dz_1^2 + dz_2^2 + \cdots + dz_d^2 \,|\, V$

is allowable

$\rightsquigarrow \forall\ n \quad \mathrm{Conf}_n(V) \subset \mathbb{D}_n$

Fix n distinct points $\vec{x}_a$

$\rightsquigarrow$ we have well-defined v.e.v.

$V \longrightarrow \mathbb{R}^d_{\text{Euclidean}}$

1 : 1

Proof: Bochner type argument:

$i = 1 \ldots d$

$y_i \longrightarrow y_i \cdot \sqrt{\lambda_i}$ in some orthonormal basis in $\mathbb{R}^d_{\text{Euclidean}}$

$\mathbb{R}^d \longrightarrow V$

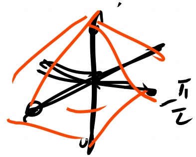
$$y_i \longrightarrow y_i\, e^{\alpha_i}$$

If $\left(\mathrm{Im}\,\alpha_i\right) = \Big(0, 0, \dots, \underbrace{\theta}_{i\text{-th place}}, \dots, 0\Big)$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$\rightsquigarrow$ v.e.v. is well-defined

No restrictions on $\mathrm{Re}\,\alpha_i$!

$\Big[$ "Tube domain": $\left(\mathrm{Im}\,\alpha_i\right) \in$ 

$\Big] \subset \mathbb{C}^1$

convex hull

$$\boxed{\rightsquigarrow \quad \sum \left|\tfrac{1}{2}\,\mathrm{Arg}\,\lambda_i\right| < \frac{\pi}{2}}$$