The asymptotic geometry of the Hitchin moduli space

Laura Fredrickson

University of Oregon

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Abstract: Hitchin's equations are a system of gauge theoretic equations on a Riemann surface that are of interest in many areas including representation theory, Teichmüller theory, and the geometric Langlands correspondence. The Hitchin moduli space carries a natural hyperkähler metric. An intricate conjectural description of its asymptotic structure appears in the work of physicists Gaiotto-Moore-Neitzke and there has been a lot of progress on this recently. I will discuss some recent results using tools coming out of geometric analysis which are well-suited for verifying these extremely delicate conjectures. This strategy often stretches the limits of what can currently be done via geometric analysis, and simultaneously leads to new insights into these conjectures.

The Hitchin moduli space

Fixed data:

- C, a compact Riemann surface (possibly with punctures D)
- $G = SU(n)$, $G_{\mathbb{C}} = SL(n, \mathbb{C})$
- $E \to C$, a complex vector bundle of rank n with $\mathrm{Aut}(E) = SL(E)$

$\rightsquigarrow$ Hitchin moduli space, $\mathcal{M}$.

Fact #1: $\mathcal{M}$ is a noncompact hyperkähler manifold with metric g_{L^2}
$\Rightarrow$ have a $\mathbb{CP}^1$-family of Kähler manifolds $\mathcal{M}_\zeta = (\mathcal{M}, g_{L^2}, I_\zeta, \omega_\zeta)$.

- $\mathcal{M}_{\zeta=0}$ is $G_{\mathbb{C}}$-Higgs bundle moduli space
- $\mathcal{M}_{\zeta\in\mathbb{C}^\times}$ is moduli space of flat $G_{\mathbb{C}}$-connections

The Higgs bundle moduli space

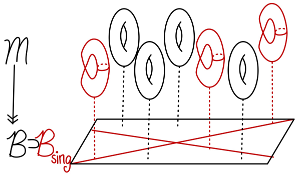
Definition

A **Higgs bundle** is a pair $(\bar\partial_E, \varphi)$ consisting of a holomorphic structure $\bar\partial_E$ on E and a "Higgs field" $\varphi \in \Omega^{1,0}(C, \mathrm{End}_0 E)$ such that $\bar\partial_E \varphi = 0$.

(Locally, $\bar\partial_E = \bar\partial$ and $\varphi = P\mathrm{d}z$, where P is a tracefree $n \times n$ matrix with holomorphic entries.)

Ex: The $GL(1)$-Higgs bundle moduli space is $\mathcal{M} = \underbrace{\mathrm{Jac}(C)}_{\bar\partial_E} \times \underbrace{H^0(K_C)}_{\varphi}$.
For $C = T^2_\tau$, $\mathcal{M} = T^2_\tau \times \mathbb{C}$.

Fact #2: In its avatar as the Higgs bundle moduli space, $\mathcal{M}$ is an algebraic completely integrable system.



Hitchin's equations

Hitchin's equations are equations for a hermitian metric h on E.

Definition

A Higgs bundle $(\bar\partial_E, \varphi)$, together with a Hermitian metric h on E, is a **solution of Hitchin's equations** if

$$F_D^\perp + [\varphi, \varphi^{*_h}] = 0.$$

(Here, D is the Chern connection for $(\bar\partial_E, h)$.)

There is a correspondence between stable Higgs bundles and solutions of Hitchin's equations. [Hitchin, Simpson]

$$\left\{ \begin{matrix}\text{stable Higgs bundles} \\ (\bar\partial_E, \varphi)\end{matrix} \right\} \Big/ \mathcal{SL}(E) \xleftrightarrow{\cong} \left\{ \begin{matrix}\text{soln of Hitchin's eqn} \\ (\bar\partial_E, \varphi, h)\end{matrix} \right\} \Big/ \mathcal{SU}(E) =: \mathcal{M}$$

Conjecture of Gaiotto, Moore, and Neitzke

The Hitchin moduli space (with parameter $t > 0$)

$$\mathcal{M}_t = \{\text{solutions of } F_D^\perp + t^2 [\varphi, \varphi^{*_h}] = 0\}/\sim$$

arises as the moduli space of certain $\mathcal{N} = 2$, $4d$ SUSY theories (namely "theories of class S", $S[\mathfrak{g}, C, D]$) compactified on a circle S^1_t.

Gaiotto-Moore-Neitzke:

- The BPS spectrum

$$\left\{ \Omega(\gamma; u) \;\middle|\; u \in \mathcal{B}, \gamma \in H_1(\Sigma_u; \mathbb{Z})_\sigma \right\}$$

  of the $\mathcal{N} = 2$ $4d$ theory $S[\mathfrak{g}, C, D]$ can be recovered from the geometry of the family $\mathcal{M}_t$ as $t \to \infty$. Satisfies Kontsevich-Soibelman wall-crossing.

  Schematically, the length scale of Lagrangian fibers is $\frac{1}{t}$ and

$$g_{\mathcal{M}_t} - g_{\mathrm{sf},t} = t^2 \sum_{\gamma \in H_1(\Sigma_u;\mathbb{Z})_\sigma} \Omega(\gamma; u) \mathrm{e}^{-\ell(\gamma;u)t}.$$

- GMN also give a recipe for constructing hyperkähler metrics from integrable system data and BPS indices $\Omega(\gamma; u)$

Note: If $\mathcal{M}$ admits a $\mathbb{C}^\times_\zeta$-action $(\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \zeta\varphi)$, then conjecture is about the asymptotic geometry of a *single* Hitchin moduli space, $\mathcal{M}$.

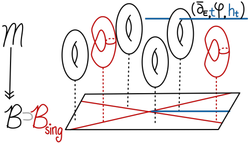
Two hyperkähler metrics on the regular locus $\mathcal{M}'$

- g_{L^2} Hitchin's L^2 hyperkähler metric—uses h
- g_{sf} semiflat metric—from integrable system structure

Gaiotto-Moore-Neitzke's Conjecture

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$,

$$g_{L^2} - g_{\mathrm{sf}} = \Omega \mathrm{e}^{-\ell t} + \text{faster decaying}$$



Progress:

- Mazzeo-Swoboda-Weiss-Witt proved polynomial decay for $SU(2)$-Hitchin moduli space. ['17]
- Dumas-Neitzke proved exponential* decay in $SU(2)$-Hitchin section with its tangent space. ['18]
- **F** proved exponential* decay for $SU(n)$-Hitchin moduli space. ['18]
- **F**-Mazzeo-Swoboda-Weiss proved exponential* decay for $SU(2)$ parabolic Hitchin moduli space. (Higgs field has simple poles along divisor $D \subset C$.) ['20]

*: Rate of exponential decay is not optimal.

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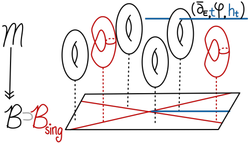


Plan:

(1) Describe important elements of general proof.

- We can gain insight into physics conjecture from geometric analysis.
- Trying to prove intricate conjectures of physics stretches limits of geometric analysis.

(2) Specialize to $4d$ Hitchin moduli spaces, since $4d$ noncompact hyperkähler spaces are well-studied. In particular, I'll describe progress for $SU(2)$-Hitchin moduli space on the four-punctured sphere. (Here, we get optimal rate of exponential decay.)

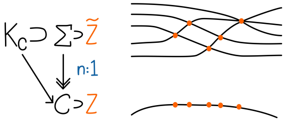
Main Theorem

Theorem [F, F-Mazzeo-Swoboda-Weiss]

Fix $(\bar\partial_E, \varphi) \in \mathcal{M}'$ and a Higgs bundle variation $(\dot\eta, \dot\varphi) \in T_{(\bar\partial_E, \varphi)}\mathcal{M}$.
Along the ray $T_{(\bar\partial_E, t\varphi, h_t)}\mathcal{M}'$, as $t \to \infty$,

$$\|(\dot\eta, t\dot\varphi)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

As $t \to \infty$, $F_{D(\bar\partial_E, h_t)}$ concentrates along branch divisor $Z \subset C$.
The limiting metric h_∞ is flat with singularities along Z.



The main difficulty is dealing with the contributions to the integral $\|\cdot\|_{g_{L^2}} = \int_C \cdots$ from infinitesimal neighborhoods around Z.

Idea #1: Semiflat metric is an L^2-metric

Hitchin's hyperkähler metric g_{L^2} on $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$ is

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} = 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t\right|^2_{h_t} + t^2\left|\dot\varphi + [\dot\nu_t, \varphi]\right|^2_{h_t}$$

where the metric variation $\dot\nu_t$ of h_t is the unique solution of

$$\partial_E^{h_t}\bar\partial_E \dot\nu_t - \partial_E^h \dot\eta - t^2\left[\varphi^{*_{h_t}}, \dot\varphi + [\dot\nu_t, \varphi]\right] = 0.$$

The semiflat metric, from the integrable system structure, on $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$ is an L^2-metric defined using h_∞.

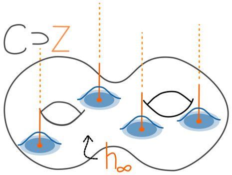
$$\|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_\infty\right|^2_{h_\infty} + t^2\left|\dot\varphi + [\dot\nu_\infty, \varphi]\right|^2_{h_\infty},$$

where the metric variation $\dot\nu_\infty$ of h_∞ is independent of t and solves

$$\partial_E^{h_t}\bar\partial_E \dot\nu_\infty - \partial_E^h \dot\eta = 0 \qquad \left[\varphi^{*_{h_\infty}}, \dot\varphi + [\dot\nu_\infty, \varphi]\right] = 0.$$

Idea #2: Approximate solutions

Desingularize h_∞ (singular at Z) by gluing in solutions h_t^{model} of Hitchin's equations on neighborhoods of $p \in Z$. $\rightsquigarrow$ h_t^{approx}.



Perturb h_t^{approx} to an actual solution h_t using a contracting mapping argument.

(Difficulty: Showing the first eigenvalue of $L_t : H^2 \to L^2$ is $\geq Ct^{-2}$)

Theorem

$$h_t(v, w) = h_t^{\mathrm{app}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w) \qquad \text{for } \|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}.$$

Idea #2: Approximate solutions

Define an *non-hyperkähler L^2-metric* g_{app} on $\mathcal{M}'$ using variations of the metric h_t^{app}.

$$\begin{aligned}
\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t\right|^2_{h_t} + t^2\left|\dot\varphi + [\dot\nu_t, \varphi]\right|^2_{h_t} \\
\|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_\infty\right|^2_{h_\infty} + t^2\left|\dot\varphi + [\dot\nu_\infty, \varphi]\right|^2_{h_\infty} \\
\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t^{\mathrm{app}}\right|^2_{h_t^{\mathrm{app}}} + t^2\left|\dot\varphi + [\dot\nu_t^{\mathrm{app}}, \varphi]\right|^2_{h_t^{\mathrm{app}}}.
\end{aligned}$$

Then, break the $g_{L^2} - g_{\mathrm{sf}}$ into two pieces:

$$\left(\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}}\right) + \left(\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}}\right)$$

Corollary

Since $h_t(v, w) = h_t^{\mathrm{app}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w)$ for $\|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}$, as $t \to \infty$ along the ray $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$,

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} = O(\mathrm{e}^{-\varepsilon t}).$$

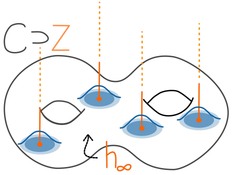
Idea #2: Approximate solutions

Our goal is to show that the following sum is $O(\mathrm{e}^{-\varepsilon t})$:

$$\underbrace{\left(\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}}\right)}_{O(\mathrm{e}^{-\varepsilon t})} + \left(\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}}\right)$$

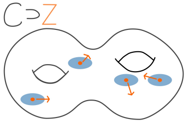
It remains to show that $\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$.

Since h_t^{app} differs from h_∞ only on disks around $p \in Z$, the difference $g_{\mathrm{app}} - g_{\mathrm{sf}}$ localizes (up to exponentially-decaying errors) to disks around $p \in Z$.



Idea #3: Holomorphic variations

When Mazzeo-Swoboda-Weiss-Witt proved that $g_{L^2} - g_{\mathrm{sf}}$ was at least polynomially-decaying in t, all of their possible polynomial terms came from infinitesimal variations in which the branch points move.



Dumas-Neitzke used a family of biholomorphic maps on local disks (originally defined by Hubbard-Masur) to match the changing location of the branch points. This uses subtle geometry of Hitchin moduli space. E.g. for $SU(2)$, conformal invariance.

Remarkably, this can be generalized off of the Hitchin section and from $SU(2)$ to $SU(n)$.

Theorem [F, F-Mazzeo-Swoboda-Weiss]

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

Main Theorem

Gaiotto-Moore-Neitzke's Conjecture

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$,

$$\|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{L^2}} - \|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{\mathrm{sf}}} = \Omega \mathrm{e}^{-\ell t} + \text{faster decaying.}$$

Theorem [**F**, **F**-Mazzeo-Swoboda-Weiss]

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$ and a Higgs bundle variation $(\dot{\eta}, \dot{\varphi}) \in T_{(\bar{\partial}_E, \varphi)}\mathcal{M}$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$, as $t \to \infty$,

$$\|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{L^2}} - \|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t}).$$

- **F** proved exponential* decay for $SU(n)$-Hitchin moduli space. ['18]
- **F**-Mazzeo-Swoboda-Weiss proved exponential* decay for $SU(2)$ parabolic Hitchin moduli space. (Higgs field has simple poles along divisor $D \subset C$.) ['19]

*: Rate of exponential decay is not optimal.

Here, $\varepsilon = \frac{\ell}{2} - \delta$ for δ arbitrarily small.

4d Hitchin moduli spaces

Noncompact hyperkähler four-manifolds X

There are several known families: the 'classical' spaces of types ALE, ALF, ALG, ALH, as well as two more recently discovered types, now frequently called ALG* ALH*.

Theorem [Chen-Chen]

If (X, g) is a noncompact complete connected hyperkähler manifold of real dimension 4 (i.e. if (X, g) is a "gravitational instanton") whose Riemannian curvature tensor decays faster than $1/r^2$, i.e., $|\mathrm{Riem}_g(q)| \leq C\mathrm{dist}(p, q)^{-2-\epsilon}$ as $q \to \infty$, where p is a fixed point in X, then (M, g) necessarily belongs to one of the families ALE, ALF, ALG and ALH.

Categories based on asymptotic volume growth: ALE/ALF/ALG/ALH

ALE: $[\mathrm{Vol} \sim r^4]$ Any X is asymptotic to some standard model $X_\Gamma^\circ = \mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $SU(2)$. [Kronheimer]

ALF: $[\mathrm{Vol} \sim r^3]$ S^1-fibrations over $3d$ spaces, described by Gibbons-Hawking Ansatz.

ALG: $[\mathrm{Vol} \sim r^2]$ T^2 fibrations over cone

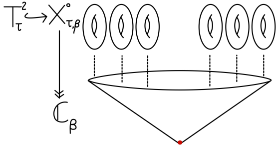
ALH: $[\mathrm{Vol} \sim r^1]$ T^3 fibrations over $\mathbb{R}$ or $\mathbb{R}^+$.

Curvature decay condition $\Rightarrow$ volume growth is r^4, r^3, r^2 or r^1. Rigidity!

Noncompact hyperkähler four-manifolds X

More recently, Chen-Chen-Chen classified *ALG* and *ALH* spaces.

ALG: Any X (with faster than quadratic curvature decay) is asymptotic to some standard model $X^\circ_{\tau,\beta}$ fibered over $\mathbb{C}_\beta$ of angle $2\pi\beta$ with fiber T^2_τ. [Chen-Chen]



What about these other types ALG* and ALH*?

- ALG*: There are examples with $\mathrm{Vol} \sim r^2$ but $\mathrm{Riem} \sim r^{-2}(\log r)^{-1}$ [Hein].
- ALH*: There are examples with $\mathrm{Vol} \sim r^{\frac{4}{3}}$ but $\mathrm{Riem} \sim r^{-2}$ [Hein].

Both of these examples are given by rational elliptic surfaces.

Not sure if there are other exceptional examples.

Question

Where do $4d$ Hitchin moduli spaces fit in to classification of gravitational instantons?

Modularity Conjecture [Boalch]

Modularity Conjecture

Every $4d$ Hitchin moduli space is of type ALG or ALG*. Conversely, every ALG & ALG* hyperkähler metric with $\mathrm{Vol} \sim r^2$ can be realized as the hyperkähler metric on a Hitchin moduli space.

Here are the types of ALG metrics, and the conjectural associated families of 4d Hitchin moduli spaces.

	Regular	I_0^*	II	III	IV	II^*	III^*	IV^*
β	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$\in \mathbb{H}$	$\mathrm{e}^{2\pi\mathrm{i}/3}$	i	$\mathrm{e}^{2\pi\mathrm{i}/3}$	$\mathrm{e}^{2\pi\mathrm{i}/3}$	I	$\mathrm{e}^{2\pi\mathrm{i}/3}$
C	T^2_τ	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$	$\mathbb{CP}^1$
D		$\{0, 1, \infty, p_0\}$	$\{0, 1, \infty\}$	$\{0, 1, \infty\}$	$\{0, 1, \infty\}$	$\{\infty\}$	$\{\infty\}$	$\{0, \infty\}$
G	$U(1)$	$SU(2)$	$SU(3)$	$SU(4)$	$SU(6)$	$SU(2)$	$SU(2)$	$SU(2)$

The Hitchin moduli spaces associated to the $SU(2)$ theories with $N_f = 0, 1, 2, 3$ are conjecturally of type ALG* with fiber $I^*_{4-N_f}$. (They are not ALG since the modulus, τ, of torus does not converge at ends.)

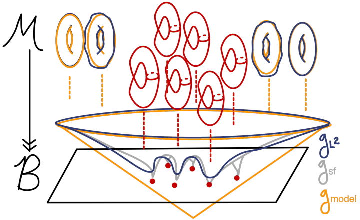
Example: The four-punctured sphere

There is a 12-parameter family of $SU(2)$-Hitchin moduli spaces on $\mathbb{CP}^1 - \{0, 1, \infty, p_0\}$.

Data: At each $p \in \{0, 1, \infty, p_0\}$,

- fix a "complex mass" $m_p \in \mathbb{C}$ (eigenvalue of $\mathrm{Res}\,\varphi$)
- fix a "real mass" $\alpha_p \in (0, \frac{1}{2})$ (parabolic weights at p are α_p and $1 - \alpha_p$)

$\rightsquigarrow$ Hitchin moduli space, $\mathcal{M} = \mathcal{M}(\boldsymbol{m_\bullet}, \boldsymbol{\alpha_\bullet})$.



In this case, the model metric metric has cone angle π and fibers are T^2_τ where $\tau = \lambda(p_0)$. The volume of each fiber is $4\pi^2$.

Generically, $\mathcal{M} \to \mathcal{B}$ has 6 singular fibers. The semiflat metric g_{sf} coincides with g_{model} iff complex masses are all zero. Call such spaces "strongly parabolic" (as opposed to "weakly parabolic").

Example: The four-punctured sphere

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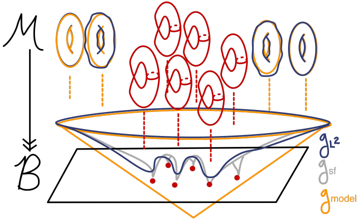


How fast does curvature decay for each of these Hitchin moduli spaces?

Use our results about the asymptotic geometry to compare g_{L^2} versus g_{sf}.

- $\mathrm{Riem} \sim \frac{1}{r^2}$ if $\mathbf{m} \neq 0$ since $g_{\mathrm{sf}} \neq g_{\mathrm{model}}$
- $\mathrm{Riem} \sim \mathrm{e}^{-\varepsilon r}$ if $\mathbf{m} \equiv 0$ since $g_{\mathrm{sf}} = g_{\mathrm{model}}$.

Corollary [F-Mazzeo-Swoboda-Weiss]

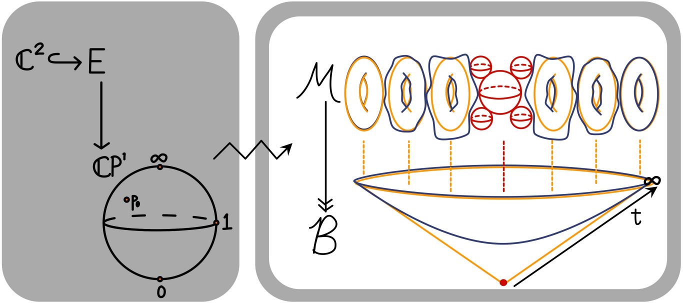
Each of these Hitchin moduli spaces is ALG.

Example: The four-punctured sphere

Now restrict our attention to **strongly parabolic** Hitchin moduli moduli spaces (i.e. $\mathbf{m} \equiv 0$) where $g_{\mathrm{sf}} = g_{\mathrm{model}}$.

Question

Is it possible to prove Gaiotto-Moore-Neitzke's conjectured rate of exponential decay for in the case of the strongly parabolic $SL(2, \mathbb{C})$-Higgs bundles on the four-punctured sphere?



Bootstrapping to optimal exponential decay

LeBrun gave a framework to describe all Ricci-flat Kähler metrics of complex-dimension two with a holomorphic circle action in terms of two functions u, w.

Generalized Gibbons-Hawking Ansatz specialized to our case:

Consider a hyperkähler metric on $T^2_{x,y} \times \mathbb{R}^+_r \times S^1_\theta$ with holomorphic circle action. The hyperkähler metric is

$$g_{L^2} = \mathrm{e}^u u_r(\mathrm{d}x^2 + \mathrm{d}y^2) + u_r \mathrm{d}r^2 + u_r^{-1}\mathrm{d}\theta^2$$

where $u : T^2_{x,y} \times \mathbb{R}^+_r \to \mathbb{R}$ solves

$$\Delta_{T^2} u + \partial_r^2 \mathrm{e}^u = 0.$$

The semiflat metric g_{sf} corresponds to $u_{\mathrm{sf}} = \log r$.

Goal

Show that $u - u_{\mathrm{sf}}$ has conjectured rate of exponential decay.

Bootstrapping to optimal exponential decay

Let $v = u - u_{\mathrm{sf}}$. Then,

$$\underbrace{\Delta_T v + r\partial_r^2 v + 2\partial_r v}_{Lv} = \underbrace{-e^v r(\partial_r v)^2 - (e^v - 1)\left(r\partial_r^2 v + 2\partial_r v\right)}_{Q(v,\partial_r v,\partial_{rr} v)},$$

Observation #1: The first exponentially-decaying function in $\ker L$ decays like $\mathrm{e}^{-2\lambda_T\sqrt{r}}$, where λ_T^2 is the first positive eigenvalue of $-\Delta_{T^2}$. In the torus T_τ^2 with its semiflat metric $\lambda_T^2 = \frac{2}{\mathrm{Im}\ \tau}$.

Observation #2: If $v \sim \mathrm{e}^{-\varepsilon\sqrt{r}}$, then $Q(v, \partial_r v, \partial_{rr} v) \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$.

Solving the non-homogeneous problem $Lv = f$ for $f \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$, we find

$$v \sim \mathrm{e}^{-2\min(\varepsilon,\lambda_T)\sqrt{r}}.$$

Conclusion: $v \sim \mathrm{e}^{-2\lambda_T\sqrt{r}}$ where $\lambda_T = \sqrt{\frac{2}{\mathrm{Im}\ \tau}}$

Bootstrapping to optimal exponential decay

Gaiotto-Moore-Neitzke's Conjecture (Schematically)

Pick a point $u \in \mathcal{B} \simeq \mathbb{C}$ and let $|u| = r$.

$$g_{L^2} - g_{\mathrm{sf}} = \sum_{\gamma \in H_1(\Sigma_u;\mathbb{Z})} \Omega(\gamma, u) \mathrm{e}^{-\ell(\gamma,u)\sqrt{r}}.$$

The first correction is $\Omega(\gamma_0, u) = 8$ and $\ell(\gamma_0, u) = 2\sqrt{\frac{2}{\mathrm{Im}\tau}}$.

Theorem [F-Mazzeo-Swoboda-Weiss]

Let $\mathcal{M}$ be a (strongly-parabolic) $SU(2)$ Hitchin moduli space for the four-punctured sphere. The rate of exponential decay for the Hitchin moduli space is as Gaiotto-Moore-Neitzke conjecture:

$$g_{L^2} - g_{\mathrm{sf}} = O(\mathrm{e}^{-2\sqrt{\frac{2}{\mathrm{Im}\tau}}\sqrt{r}}).$$

$(\mathcal{M}, g_{L^2})$ is an ALG metric asymptotic to the model metric g_{sf}.

Thank you!