Topological Strings
Twistors
Skyrmions
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Background on Twisters

- \( \Pi = O(-1) \oplus O(-1) \rightarrow \mathbb{CP}^1 \)

- \( \Pi \rightarrow \mathbb{R}^4 \), Twistor fibration: fibres are \( \mathbb{CP}^1 \)

- \( \bar{\partial} \) equation on \( \Pi \) \( \leftrightarrow \) ordinary PDE on \( \mathbb{R}^4 \)

- e.g., \( H^1(\Pi, O(-2)) = \{ \phi \in C^\infty(\mathbb{R}^4), \Delta \phi = 0 \} \)
More generally,

Holomorphic field theory on $\mathbb{P}^1$ $\Rightarrow$ Ordinary field theory on $\mathbb{R}^4$

Examples:

- $A \in \mathfrak{so}_0(1|\mathbb{P}^1, \mathfrak{o}(-2)) = \mathfrak{so}_0^0(1|\mathbb{P}^1, K^{1/2})$

  gauge field, $S(A) = \int_{1|\mathbb{P}^1} A \delta A$

$\Rightarrow$ Free massless scalar $\int_{\mathbb{R}^4} \phi \Delta \phi$
Many more examples:

- Holomorphic BF theory
  \(\leftrightarrow\) Self-Dual Yang Mills theory
  (Penrose-Ward, Mousheg, Mason et al.)

- Holomorphic Chern-Simons on Super-Twistor space
  \(\leftrightarrow\) Self dual limit of
  \(N = 4\) Yang-Mills
  (Witten, Berkovits, Mason et al., ... )
All these theories are free or almost free (one loop exact).

E.g. Full $N=4$ SYM requires adding non-local terms to the Lagrangian on twistor space

($D1$-instantons: Witten, Berkovits, Boels, Mason, Skinner)
Goal Today:

- Study a new twistor string theory
- Local field theory on twistor space (holomorphic Chern-Simons)
- Corresponding field theory on $\mathbb{R}^4$
- $\sigma$-model with target $SO(8)$
- No supersymmetry
Holomorphic Chern-Simons

\[ \Omega \]

Calabi-Yau 3-fold

\[ \Omega \]

holomorphic volume form

Gauge field \( A \in \Omega^{0,1}(X, g) \)

\[ S(A) = \int X \wedge CS(A) \wedge \Omega \]
hCS on Twistor Space

PT is not Calabi-Yau!

\[ K_{PT} = \Theta (-4) \]

Take \( \Omega \) to have second order poles at \( z = 0 \) and \( z = \infty \)

Locally \( \Omega = \prod v_1 \prod v_2 dz/z^2 \)

\( z \): coordinate on \( \mathbb{P}^1 \), \( v_i \) on \( \Omega(-1)^2 \) fibres
$A \in SL^0(\mathbb{H}, g)$ hCS gauge field

$\int_{\mathbb{H}} CS(A)\gamma^2$ not gauge invariant

because $\gamma^2$ has poles

Fix:

- $z = 0, z = \infty$ set $A = 0$

(and gauge transformations = 1)

Dirichlet boundary conditions
Field theory on $R^4$

$x \in R^4$, $P^1_x \subset I\Pi T$ twistor fibre

$A|_{P^1_x}$ is a holomorphic $G$-bundle

trivialized at $z = 0$, $z = \infty$

$\iff$ an element $\sigma(x) \in G$

(compare trivializations)

4-d field: $\sigma: R^4 \rightarrow G$. 
\[ \sigma \text{-model Lagrangian} \]

A short calculation shows hCS Lagrangian on IP\(\mathbb{T}\) gives

\[ \int_{\mathbb{R}^4} \text{Tr}(J \wedge * J) - \frac{1}{3} \int_{\mathbb{R}^4 \times \mathbb{R}_{\geq 0}} \text{Tr}(\hat{J} \wedge \hat{J} \wedge \hat{J}) \omega \]

\(J = \sigma^{-1} d\sigma\)

\(\hat{\sigma}\) extension to \(\mathbb{R}^4 \times \mathbb{R}_{\geq 0}\), \(\hat{J} = \hat{\sigma}^{-1} d\hat{\sigma}\)

\(\omega \in \Omega^2(\mathbb{R}^4)\) Kähler form
\[ \int_{\mathbb{R}^4} J \wedge \ast J \] usual 6-model Lagrangian

\[ \int_{\mathbb{R}^4 \times \mathbb{R}^2} \text{Tr} (\hat{J} \wedge \hat{J} \wedge \hat{J}) \] analog of WZ term

Lorentz invariance broken
Interpretation of WZ term

4d 5-model with target G has a topological U(1) symmetry

Current is $\text{Tr}(J \wedge J \wedge J)$

WZ term: couple to background U(1) gauge field for this symmetry:

$\int A \text{Tr} J^3 \quad F(A) = \omega$, Kähler form
Skyrme Model

Low energy EFT for QCD

\[ G = SU(N_F) \quad \sigma : \mathbb{R}^4 \rightarrow G, \]
\[ \sigma = \exp(q \bar{q}) \text{ encodes mesons} \]

Topological $U(1)$ charge

= Baryon number
Anomalies for holomorphic CS

So far: hCS on twistor space $\to$ 5-model with target $G$, \textit{classically}.

At the quantum level, hCS has a one-loop anomaly: $(KC, Si Hi)$

Gauge variation of diagram

\[
\text{is} \quad \int_\Gamma \text{Tr}_g (c(\partial A)^3)
\]

(Gauge variation is $\delta A = \delta c + [c, A]$)
Green-Schwarz mechanism

Introduce closed string fields

\( \alpha \in \Omega^{2,1}(\mathbb{IP}^1, \log D), \quad \partial \alpha = 0 \)

(Type I Kachira- Spencer Theory)

Lagrangian

\[ \int \frac{1}{2} \text{Tr} \left( \alpha A A A \right) + \int \partial \alpha \tilde{\partial} \alpha + \ldots \]

These cancel the anomaly of hCS for

\[ G = \text{SO}(8) \quad (C, S; Li) \]
\[ \sim \text{Tr}_g (c (\partial A)^3) \]

closed string propagator

\[ \sim \text{Tr}_g (c \partial A) \text{Tr}_g ((\partial A)^2) \]

These cancel for a lie algebra \( g \) where

\[ \text{Tr}_g \left( X^4 \right) \propto \left( \text{Tr}_g X^2 \right)^2 \]

(trace in adjoint representation)

This happens for \( g = \text{SO}(8) \) (and \( E_8 \) ...)
\[ g = \text{so}(8) \]: All higher-loop anomalies for \( hCS + CS \) cancel, and counterterms fixed uniquely \((KC, S; Li)\)

**Worldsheet Perspective:**

Unoriented topological \( B \)-model with target \( \mathbb{PT} \)

\[ \mathbb{PT} \rightarrow \mathbb{PT} \]

Type I top \( \mathbb{L} \) string.
Compare to Witten, Berkovits: ordinary B-model on super-twistor space $\mathbb{C}P^{3}\mathbb{F}_4$

$\longrightarrow N = 4$ SYM on $\mathbb{R}^4$

Here: Type I B-model on ordinary twistor space $\mathbb{C}P^{11}$

$\longrightarrow$ 4d $\sigma$-model with target $SO(8)$
Implications of the twistorial origin

- Control over the R6 flow and counterterms

- Integrable properties and a lax matrix

- Good analytic behaviour of correlation functions
RG flow

$R_+ \to R_+^4$ by scaling.

On $\mathbb{P}^2$ comes from scaling $O(-1)^2$

fibres of projection $\mathbb{P}^2 \to \mathbb{C}P^1$

Extends to an action of $C^*$

**Corollary** For any theory of twistorial origin the RG flow is periodic with period $i$
If \( \lambda = \log \text{ scale} \), then periodicity says

\[
T(\lambda + i) = T(\lambda)
\]

Periodicity takes place in the world of analytically continued theories. Strongly constrains divergences in Feynman diagrams. No divergences like \( \log \varepsilon \) or \( e^{-\varepsilon \log \varepsilon} \).
Very unusual behaviour!
- Interacting scalar field theory cannot come from twistor space (because of log divergences)
- 4d YM cannot come from twistor space

How can σ-model with SO(8) target have periodic RG trajectory?

Answer: 4d version of Green-Schwarz mechanism that cancels log divergence instead of anomalies.
$\sigma : \mathbb{R}^4 \to G \quad \psi = \log \sigma : \mathbb{R}^4 \to \mathfrak{g}$

$\sum \omega J^n = 3 - \frac{1}{3} \sum \omega J^3$

$\implies \int_{\mathfrak{g}} \text{Tr} \Delta \psi + \int_{\mathfrak{g}} \left( \psi \{ \partial \psi, \bar{\partial} \psi \} \right) \omega$

+ higher order terms

One loop log divergence

$\square \sim (\log \epsilon) \text{Tr} \left( \partial \psi \bar{\partial} \psi \partial \psi \bar{\partial} \psi \right)$

(*) Lagrangian studied by Goncharov, "Hodge Field Theory"
Log divergence is cancelled by closed string fields. On \( \mathbb{IP}^1 \), have \( \omega \in \Omega^{2,1}(\mathbb{IP}^1, \log D) \)
\[ \delta \alpha = 0, \quad \int \delta \alpha \, \alpha^{-1} \alpha \]
On \( \mathbb{IR}^4 \), \( \alpha \mapsto B \in \Omega^2_- \), \( \gamma \) a scalar,
\[ d\alpha + \ast d\gamma = 0, \quad \text{two point function} \]
\[ \langle B(0), B(x) \rangle = 1/|x|^4, \quad \langle \gamma, \gamma \rangle = 0, \quad \langle \gamma, B \rangle = \ldots \]
\( B, \gamma \) are coupled by varying \( \text{Kähler form} \)
\[ \omega \mapsto \omega + B + \omega \cdot \gamma \]
\( (B, \gamma \) are gravitational fields)
In terms of \( \varphi = \log s \), fields couple by \[
\int \text{Tr}_g \left( \partial \varphi \bar{\partial} \varphi \right) B
\]

Since \[
\langle B(0), B(\omega) \rangle = \frac{1}{11} x^4
\]

\[
\sim \log \epsilon \text{Tr}_g \left( \partial \varphi \bar{\partial} \varphi \right)^2
\]

\[
\sim \log \epsilon \text{Tr}_g \left( \partial \varphi \bar{\partial} \varphi \right) \varphi \bar{\varphi}
\]

\[
\text{Cancel for } G = \text{SO}(8)!
\]
Fixing higher loop counter-terms

- \( \sigma \)-model is non-renormalizable

- 4d perspective: \( \infty \) many possible choices of counter-terms

\[ \{ \text{4d local counter-terms} \} \supset \{ \text{6d local counter-terms} \} \]

6d: many fewer possibilities

- (KC, Si, Li) All 6d counter-terms fixed by anomaly cancellation
Towards integrability of the \( SO(8) \) 5-model

There are many 2-dimensional integrable field theories.

- Classically characterized by a Lax matrix \( L(z) \), a \( \Omega \)-form so that

\[
\text{d}L(z) + \frac{1}{2} [L(z), L(z)] = 0 \Longleftrightarrow \text{Equations of motion}
\]

- Quantum level:

\[
P \exp \oint L(z) \text{ is a conserved quantity}
\]

(Better: a topological line defect)
4d: It is believed Lorenz invariant theories cannot be integrable.

SO(8) \sigma-model has a Lax matrix $L(z) \in \mathbb{C}^\times$ so that $L(z)$ is a $(0,1)$ form on $\mathbb{R}^4$ in complex structure $\mathbb{Z}$:

$L(z) \in \mathcal{S}_2^\mathbb{C}^\times(\mathbb{R}^4) \otimes \text{SO}(8)$

Satisfying the Lax equation

$$\frac{\partial}{\partial z} L(z) + \frac{1}{2} [L(z), L(z)] = 0$$

$\in \mathcal{S}_2^\mathbb{C}^\times(\mathbb{R}^4) \otimes \text{SO}(8)$
At the classical level this is very simple:
\[ \Pi_2 := \text{projection onto } (0,1) \text{ forms in complex structure } z \]
\[ L(z) = \Pi_2 \Pi_0 J \]
Lax equation follows from the equations of motion: \( u_i, \bar{u}_i; \) holomorphic coordinates in complex structure \( z = 0 \)
\[ \text{Eom are} \quad \partial u_i, J_{u_i} = 0 \]
\[ A \in S^0, (\Pi^T) \otimes \text{so}(8), \quad L(z) = \langle A(z) \rangle \]
A CS gauge field
At the quantum level, \( L(z) \) can couple to chiral fields living on a surface \( S \subseteq \mathbb{R}^4 \) which is holomorphic in complex structure \( z \). This gives a surface defect on \( S \) which depends holomorphically (in complex structure \( z \)) on the position of \( S \).
Infinitely many conserved charges

Take 4d spacetime to be \( T^3 \times \mathbb{R} \).

For countably many complex structures \( z \)
this is \( E \times \mathbb{C}^x \), \( E \) an elliptic curve,
coordinates \( u, v \).

Let \( T(z, v) \) be surface defect on \( E \times v \).
Then \( \frac{\partial}{\partial \bar{v}} T(z, v) = 0 \).
As in a 2d CFT a homomorphic operator gives infinitely many conserved charges:

$$\sum_{|v|=1} v^k T(z, v)$$

On $\mathbb{PT}$, surface defects come from $D1$ branes in the fibre over $z \in \mathbb{C} \cup \mathbb{P}^1$. 
D1's at different values of $z$ are disjoint in $\overline{\mathbb{P}}^1$

$$\Rightarrow \left[ T(z,v), T(z',v') \right] = 0$$

$$\Rightarrow \left[ \oint_{|\nu|=1} T(z,v), \oint T(z',v') \right] = 0$$

Infinitely many conserved (non-local) charges!

(Those $z$ for which $T^3 \times \mathbb{R}$ contains elliptic curve $E$ includes $z = e^{2\pi i (n/m)}$)
So far:

4d SO(8) $\sigma$-model, with WZW term and additional fields, has

- Periodic RG trajectory, period $i$
- A well-defined quantization, despite being non-renormalizable by power-counting
- Infinitely many commuting non-local charges
"Standard" twistor string theory:

Scattering amplitudes

\[ \leftrightarrow \] Geometry of algebraic curves in \( \mathbb{CP}^n \)

Here: Correlation functions of special local operators are related to algebraic curves in \( \mathbb{CP}^n \).
Vertical D1 branes: wrap $\mathbb{CP}^1$ in $\mathbb{P}^1 = \mathcal{O}(-1)^2 \rightarrow \mathbb{CP}^1$

Theory on D1 brane is symplectic bosons on moduli space $M$ of charge 1 instantons for $SO(8)$

$z = 0, \infty$: need to choose boundary conditions.

Lagrangians $L_0, L_\infty \subset M$
Vertical D1 brane $\Rightarrow$ local operator

Assume $L_0 \cap L_\infty = \text{a point } p \in M$

$SO(8)$ acts on $M$ by Hamiltonian functions $H_a$

If $\phi = \log \sigma$, then the local operator is

$$\exp \left( \phi^a H_a(\rho) \right) + ...$$

These local operators have special analytic properties: consider correlation functions

$$\langle O(x_1), ..., O(x_n) \rangle \quad x_i \in \mathbb{R}^4$$
$G(2, 4) = \left\{ \Omega | \Omega' \subseteq \Omega' \cap \Omega^3 \right\}$

$G(2, 4) \cap \mathbb{C}^4 = \left\{ \Omega | \Omega' \subseteq \Omega' \right\}$

Correlation functions analytically extend to $x_i \in \mathbb{C}^4$.

As on $\Omega'$ the D1 branes are defined on $\Omega' \subseteq \Omega'$.

$\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ has poles when $\Omega' \cap \Omega' \Rightarrow x_i, x_j \in \mathbb{C}^4$ are such that $x_i - x_j$ is null.
This implies that there is a well-behaved OPE:

\[ \Theta(0) \cdot \Theta(x) \sim \frac{1}{\|x\|^{2D}} \sum x^T \]

\(I\) is a multi-index

Suggests a bootstrap-like way to understand correlation functions