QUANTUM FIELD THEORY

Notes by Graeme Segal

For the purposes of these lectures, quantum field theory is the study of the following situation.

We have an n-dimensional space-time manifold X.

We have a space $\Phi(X)$ of "fields" on X. A crucial property of "fields" is that they are <u>locally</u> defined.

<u>Examples</u>
(a) $\Phi(X)$ = all smooth maps $\varphi : X \to \mathbb{R}$

(a′) . . . . $\varphi : X \to M$, where M is another manifold, usually with a Riemannian structure.

(b) $\Phi(X)$ = {Riemannian metrics φ on X}

(c) . . . . {principal G-bundles on X}, where G is a Lie group

(d) . . . . . { . . . . . , equipped with a connection}

(d′) . . . . { . . . . , ... and a section}.

Notice that in (c) and (d) the fields are naturally a <u>category</u>. $\Phi(X)$ means, at least roughly, the set of isomorphism classes.

We want to make "local observations" of fields $\varphi \in \Phi(X)$. Thus for $x \in X$ we consider maps $F_x : \Phi(X) \to \mathbb{C}$ such that $F_x(\varphi)$ depends only on $\varphi \mid$ (infinitesimal neighbourhood of x).

<u>Examples</u>
In (a) above : $F_x(\varphi) = \varphi(x)$

(a′) . . . : $F_x(\varphi) = f(\varphi(x))$ for some $f : M \to \mathbb{C}$

(b) . . . : $F_x(\varphi)$ = scalar curvature of φ at x

(c) : There are no local observables, although there are global ones, e.g.

$$F(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is trivial} \\ 1 & \text{if } \varphi \text{ is not trivial} \end{cases}$$

We want to put a probability measure on $\Phi(X)$, so that we can calculate "expectation values" $\langle F_x \rangle$, and "correlations" $\langle F^{(1)}_{x_1} F^{(2)}_{x_2} \dots F^{(k)}_{x_k} \rangle$ of local observables.

The measure contains all the physics of the situation. It expresses the fact that the values of φ at different points of X are correlated only by the interaction of (points) which are infinitely close to each other.

To be more precise, consider $\Phi(X) = \mathrm{Map}(X; M)$, and let $d\mu(x)$ be a probability measure on X. Then

$$\mathcal{D}\varphi = \prod_{x \in X} d\mu(\varphi(x))$$

is a measure on $\Phi(X)$ with no correlation at all between different points. The same would apply to

$$e^{-S(\varphi)} \mathcal{D}\varphi = \prod_{x \in X} e^{-L(\varphi(x))\,dx} \, d\mu(\varphi(x))$$

if $S : \Phi(X) \to \mathbb{R}$ is of the form $S(\varphi) = \int_X L(\varphi(x))\,dx$ for some $L : M \to \mathbb{R}$.

The measures we want are of the form $e^{-S(\varphi)} \mathcal{D}\varphi$ with the crucial difference that $S : \Phi(X) \to \mathbb{R}$, which is called the action, is of the form

$$S(\varphi) = \int_X L(\varphi(x), D\varphi(x))\,dx .$$

Here $L : \mathcal{J}^1(M) \to \mathbb{R}$ is a function on the space of 1-jets of maps $X \to M$. The dependence of L on the derivative $D\varphi(x)$ determines how the points of X interact with each other.

It is very hard to construct measures of the type we want. The only easy case is when $\dim(X) = 1$. Then we have

Wiener measure (or "Brownian motion" or "random walk")

Let $X = [a,b] \subset \mathbb{R}$

$\Phi(X) = \mathrm{Maps}(X; M)$, M compact Riemannian

$S : \Phi(X) \to \mathbb{R}$ is $S(\varphi) = \frac{1}{2}\int_a^b \|\dot\varphi(t)\|^2\,dt.$

Then the measure can be described as follows.

Let $\mathcal{H} = L^2(M)$, a Hilbert space.

$\Omega \in \mathcal{H}$ is the constant function 1

$U_t : \mathcal{H} \to \mathcal{H}$ for $t > 0$ is the contraction semigroup expressing the diffusion of heat on X, i.e. $U_t = e^{-t\Delta}$, where Δ is the Laplacian.

Let $m_f : \mathcal{H} \to \mathcal{H}$ be multiplication by $f : M \to \mathbb{C}$.

Notice that U_t is self-adjoint, and $U_s U_t = U_{s+t}$.

Also $U_t \Omega = \Omega$.

<u>Theorem</u> If $a \le t_1 < t_2 < \dots < t_k \le b$, and $f_1, \dots, f_k : M \to \mathbb{C}$ then

$$\langle f_k(\varphi(t_k)) \dots f_2(\varphi(t_2)) f_1(\varphi(t_1)) \rangle_{\text{Wiener}}$$

$$= \langle \Omega, U_{b-t_k} m_{f_k} \dots\dots m_{f_2} U_{t_2-t_1} m_{f_1} U_{t_1-a} \Omega \rangle_{\mathcal{H}}$$

The form of this result is basic for us. We want to understand how X_1 interacts with X_2 when we write $X = X_1 \cup X_2$.

Suppose $[a,b] = [a,c] \cup [c,b]$ and $\begin{cases} t_i \in [a,c] & \text{for } i \le r \\ t_i \in [c,b] & \text{for } i > r \end{cases}$

The right-hand side of the formula above can be written

$$\langle \Psi_2, \Psi_1 \rangle_{\mathcal{H}},$$

where $\Psi_1 = U_{c-t_r} m_{f_r} U_{t_r - t_{r-1}} \dots m_{f_1} U_{t_1 - a} \Omega \in \mathcal{H}$

Ψ_2 = similar

In terms of measures on $X = X_1 \cup X_2$, with $X_1 \cap X_2 = Y$, this tells us that

$$e^{-S(\varphi)} \mathcal{D}\varphi \neq e^{-S(\varphi_1)} \mathcal{D}\varphi_1 \,.\, e^{-S(\varphi_2)} \mathcal{D}\varphi_2$$

(where $\varphi_i = \varphi | X_i$), which would say that the points of X_1 did not interact with the points of X_2, but that we <u>can</u> define measures $e^{-S(\varphi_i)} \mathcal{D}\varphi_i$ <u>with values in $\mathcal{H}$</u> such that

$$e^{-S(\varphi)}\,\mathcal{D}\varphi \quad = \quad \langle e^{-S(\varphi_1)}\,\mathcal{D}\varphi_1 \,,\; e^{-S(\varphi_2)}\,\mathcal{D}\varphi_2 \rangle .$$

The size of the vector space $\mathcal{H}$ precisely expresses the extent of the interaction between X_1 and X_2, and it is canonically associated to the interface $Y = X_1 \cap X_2$. (More accurately, to the infinitesimal neighbourhood of Y in X.)

The construction of Wiener measure from

(i) the Hilbert space $\mathcal{H}$

(ii) the semigroup $U_t : \mathcal{H} \to \mathcal{H}$ and the vector $\Omega \in \mathcal{H}$, and

(iii) the ring of operators $\{m_f\}$

can easily be axiomatized. For many purposes most, and sometimes all, of the information is already encoded in (i) and (ii), and the ring of operators and the space $\Phi(X)$ of fields can be completely forgotten. This point of view leads us to the axiom system described in Lecture 1 of the accompanying notes.

One of the few situations when one can construct a measure like Wiener measure quite explicitly, but where $\dim(X) = 2$, is the case of Yang-Mills theory on a surface, which is discussed in § 1.4 and § 1.5 of the notes.

<u>Geometric quantization</u>

The measure on $\Phi(X)$ is concentrated very sharply around the critical set of $S : \Phi(X) \to \mathbb{R}$. Call this $\Phi_0(X)$ – the space of "classical solutions". But let us be more precise.

Let ψ be an infinitesimal variation of φ, i.e. a tangent vector to $\Phi(X)$ at φ. Usually we have a formula like

$$dS(\varphi;\psi) = \int_X A(\varphi(x))\,\psi(x)\,dx + \int_{\partial X} B(\varphi(x))\,\psi(x)\,dx \qquad (*)$$

for the derivative of S. (Here A and B depend also on $D\varphi(x)$.)

The two terms on the RHS are 1-forms on $\Phi(X)$. Let us call them α and β. The zero-set of α is the space $\Phi_0(X)$ of classical solutions.

Because of the local nature of S, A, B it makes sense to speak of the space $\Phi_0(Y)$ of germs of classical solutions along any codimension 1 submanifold Y of X, and β defines a 1-form on $\Phi_0(Y)$

In fact the closed 2-form $d\beta$ defines a symplectic structure on the manifold $\Phi_0(Y)$, and the Hilbert space $\mathcal{H}_Y$ associated to Y is obtained by "geometric quantization" from $\Phi_0(Y)$.

If $Y = \partial X$ then $\Phi_0(X) \subset \Phi_0(Y)$, and $\beta \,|\, \Phi_0(X) = dS$. Thus $d\beta = 0$ on $\Phi_0(X)$. In fact $\Phi_0(X)$ is in good cases a Lagrangian submanifold of $\Phi_0(Y)$, and when $\partial X = Y_0 \amalg Y_1$ it is the graph of a symplectic transformation

$$\Phi_0(Y_0) \longrightarrow \Phi_0(Y_1).$$

§2 The formalism of Gaussian measures

If A is a real $n \times n$ positive-definite matrix then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x, Ax\rangle}\, dx = \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} \qquad (2.1)$$

More abstractly, if V is an n dimensional real vector space, with no given measure, and $A : V \to V^*$ is positive-definite symmetric, we should think of the two sides of (2.1) as elements of $|\Lambda^n V|$, for $A : V \to V^*$ induces $\det(A) : \Lambda^n V \to \Lambda^n V^*$ so that

$$\det(A) \in (\Lambda^n V^*)^{\otimes 2} \qquad \text{and} \qquad (\det A)^{-1/2} \in |\Lambda^n V|.$$

Better, we should allow complex symmetric A with $\mathrm{im}(A)$ positive-definite. Then

$$\int_V e^{\frac{i}{2}\langle x, Ax\rangle}\, dx = \left(\det \frac{A}{2\pi i}\right)^{-1/2},$$

where now both sides belong to $|\Lambda^n V|_{\mathbb{C}}$.

The function $e^{\frac{i}{2}\langle x, Ax\rangle}$ on V defines a ray L_A in the Hilbert space $\mathcal{H} = L^2(V)$ of $\frac{1}{2}$-densities on V. The ray consists of all elements $e^{\frac{i}{2}\langle x, Ax\rangle}\, d\mu(x)^{1/2}$, where $d\mu$ is a translation-invariant measure on V. Thus L_A is canonically isomorphic to $|\Lambda^n V|_{\mathbb{C}}^{-1/2}$.

To generalize all this to infinite dimensional V the first step is to see that $\mathcal{H} = L^2(V)$ can be characterized as a group representation.

For each $\xi \in V$ we have a translation operator $T_\xi : \mathcal{H} \to \mathcal{H}$.
For each $\alpha \in V^*$ . . . . multiplication operator $M_\alpha : \mathcal{H} \to \mathcal{H}$.
(Thus $(T_\xi \varphi)(x) = \varphi(x - \xi)$, and $(M_\alpha \varphi)(x) = e^{i\alpha(x)} \varphi(x)$.)

These operators nearly commute :

$$T_\xi M_\alpha = e^{-i\alpha(\xi)} M_\alpha T_\xi .$$

So the T_ξ and M_α, together with scalar multiplications by elements of $\mathbb{T} = \{u \in \mathbb{C} : |u| = 1\}$, generate a $(2n+1)$-dimensional Lie group Heis, which fits into an exact sequence

$$\mathbb{T} \longrightarrow \text{Heis} \longrightarrow V \oplus V^*.$$

The group Heis depends only on the vector space $V \oplus V^* = T^*V$ with its symplectic form σ given by

$$\sigma((v_1,\alpha_1),(v_2,\alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2).$$

Theorem (2.2) $L^2(V)$ is an irreducible unitary representation of Heis, and is characterized as the unique irreducible representation in which $\mathbb{T} \subset$ Heis acts naturally.

Corollary (2.3) Up to a scalar multiplication, $L^2(V)$ depends only on the symplectic vector space $V \oplus V^*$.

The Gaussian ray $L_A \subset L^2(V)$ can also be characterized in terms of the group Heis. But first we need to know more about the space of complex Gaussian measures on V, which is traditionally called the "Siegel generalized upper $\frac{1}{2}$-plane".

Theorem (2.4) The following three manifolds are canonically isomorphic.

(i) symmetric linear maps $A : V_{\mathbb{C}} \to V_{\mathbb{C}}^*$ with $\operatorname{im}(A)$ positive-definite.

(ii) positive maximal isotropic subspaces $W \subset (V \oplus V^*)_{\mathbb{C}}$.

(iii) symplectic linear maps $J : V \oplus V^* \to V \oplus V^*$ such that $J^2 = -1$.

Here "isotropic" refers to the form σ, and W is positive if $i\sigma(\bar{w}, w) > 0$ for all non-zero $w \in W$. The theorem is trivial: the correspondence (i) $\to$ (ii) is $A \mapsto \operatorname{graph}(A)$, and (iii) $\longrightarrow$ (ii) by $J \mapsto$ ((+i)-eigenspace of J).

The Gaussian ray $L_A = L_W \subset L^2(V)$ is the unique ray fixed by the abelian subgroup defined by W in the complexified group $\mathrm{Heis}_{\mathbb{C}}$. But it is a little clear to work with the Lie algebra of $\mathrm{Heis}_{\mathbb{C}}$, which is generated by

$$\begin{cases} iD_\xi & \text{for } \xi \in V \qquad — \quad D_\xi \text{ is differentiation along } \xi \\ m_\alpha & \text{for } \alpha \in V^* \qquad — \quad m_\alpha \text{ is multiplication by } \alpha \end{cases}$$

subject to

$$[iD_\xi \, , \, m_\alpha] = i\alpha(\xi) \tag{2.5}$$

The formula

$$\left(i\frac{\partial}{\partial x_k}\right) e^{\frac{i}{2}\sum A_{pq} x_p x_q} = -\left(\sum A_{kq} x_q\right) e^{\frac{i}{2}\sum A_{pq} x_p x_q}$$

shows that $e^{\frac{i}{2}\langle x, Ax\rangle}$ is annihilated by the elements of $W = \mathrm{graph}(A) \subset (V \oplus V^*)_{\mathbb{C}}$.

The algebra (over $\mathbb{C}$) generated by symbols D_ξ and m_α satisfying (2.5) is called the *Weyl algebra* $\mathcal{A}(V \oplus V^*)$. It is a precise analogue of the *Clifford algebra* $C(V \oplus V^*)$ associated to the natural symmetric form on $V \oplus V^*$.

Once we have picked a Gaussian measure A, or — better — a splitting $(V \oplus V^*)_{\mathbb{C}} = W \oplus \overline{W}$, we get a new description of $L^2(V)$. For the symmetric algebras $S(W)$ and $S(\overline{W})$ are commutative subalgebras of the Weyl algebra $\mathcal{A}(V \oplus V^*)$, and $S(W)$ annihilates the ray L_A.

Theorem (2.6) There is a dense isometric embedding

$$S(\overline{W}) \longrightarrow L^2(V)$$

given by

$$\bar{w}_1 \bar{w}_2 \cdots \bar{w}_k \longmapsto \bar{w}_1 \bar{w}_2 \cdots \bar{w}_k \, e^{\frac{i}{2}\langle x, Ax\rangle},$$

where, on the right, the $\bar{w}_i$ are regarded as elements of the Weyl algebra.

The image of the embedding (2.6) is clearly the space of all functions

$$p(x)\, e^{\frac{i}{2}\langle x, Ax\rangle} (dx)^{1/2}, \qquad (2.7)$$

where p is a polynomial on V. Thus $L^2(V)$ is the natural Hilbert space completion of $S(\overline{W})$.

<u>Note</u>: It would be more functorial to write $S(\overline{W}) \otimes (\Lambda^n \overline{W})^{1/2} \longrightarrow L^2(V)$.

<u>The infinite dimensional case</u>

For an infinite dimensional topological vector space V there is no unique choice of a Hilbert space $\mathcal{H} = L^2(V)$. If we <u>choose</u> a Gaussian quadratic form $A : V_{\mathbb{C}} \to V_{\mathbb{C}}^*$ there is a unique Hilbert space $\mathcal{H}_A$ which contains all formal expressions of the type (2.7). It is the unique irreducible representation of $\mathrm{Heis}(V \oplus V^*)$ which contains a ray annihilated by the operators in the Weyl algebra corresponding to the subspace $W = \mathrm{graph}(A)$, and it is the natural Hilbert space completion of the symmetric algebra $S(\overline{W})$.

The main fact about this construction is

<u>Theorem</u> (2.8) (Shale) If A_1 and A_2 are two Gaussian forms then the Hilbert spaces $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$ are isomorphic as representations of the Weyl algebra $\mathcal{A}(V \oplus V^*)$ if and only if the projection

$$W_{A_1} \longrightarrow \overline{W}_{A_2},$$

coming from $(V \oplus V^*)_{\mathbb{C}} = W_{A_2} \oplus \overline{W}_{A_2}$, is a <u>Hilbert-Schmidt</u> operator.

The following definition will be basic.

A <u>polarization</u> of a topological vector space E is a <u>class</u> of splittings $E = E^+ \oplus E^-$ which are <u>close</u> in the sense of Shale's theorem.

For a polarized topological vector space E we define the
<u>restricted Grassmannian</u> $Gr(E)$ as the space of all subspaces E^+
which occur in allowable decompositions $E = E^+ \oplus E^-$.

Shale's theorem associates a Hilbert space $\mathcal{H} = L^2(V) \cong S(\bar{W})$
to each positive isotropic decomposition $(V \oplus V^*)_{\mathbb{C}} = W \oplus \bar{W}$. It also
has the important co

<u>Corollary (2.9)</u> If W_1 is a positive maximal isotropic subspace
of $(V \oplus V^*)_{\mathbb{C}}$ belonging to its restricted Grassmannian, then there
is a unique ray $L_{W_1} \subset \mathcal{H}$ which is annihilated by $W_1 \subset \mathcal{A}(V \oplus V^*)$.

§3 <u>Free bosonic field theory</u>

We can now construct an example of an n-dimensional
quantum field theory. Unfortunately it describes non-interacting
particles, although in a curved background.

Let X be a compact oriented Riemannian n-manifold, perhaps
with boundary. The space of fields is

$$\Phi(X) = \Omega^0(X) = \{\text{smooth maps } X \to \mathbb{R}\}.$$

The action $S : \Phi(X) \to \mathbb{R}$ is given by

$$S(\varphi) = \frac{1}{2} \int_X (d\varphi . * d\varphi + m^2 \varphi . * \varphi),$$

where m is the <u>mass</u>. The classical solutions are

$$\Phi_0(X) = \{\varphi : (d^* d + m^2)\varphi = 0\}.$$

The <u>germs</u> of classical solutions along ∂X, i.e. the
space $\Phi_0(\partial X)$ of "Cauchy data" along ∂X, are

$$\Phi_0(\partial X) = \Omega^0(\partial X) \oplus \Omega^{n-1}(\partial X).$$

Notice that $\Omega^0(\partial X)$ and $\Omega^{n-1}(\partial X)$ are dual spaces.

The space $\mathcal{H}_Y$ associated to a closed oriented Riemannian $(n-1)$-manifold Y will be $L^2(\Omega^0(Y))$. To define this space we must choose a subspace $W \subset (\Omega^0(Y) \oplus \Omega^{n-1}(Y))_{\mathbb{C}}$

If $\partial X = Y$ we take $W = \Phi_0(X)$, where

$$\Phi_0(X) \hookrightarrow (\Omega^0(Y) \oplus \Omega^{n-1}(Y))_{\mathbb{C}}$$

by

$$\varphi \longmapsto (\varphi|Y,\ i(*d\varphi)|Y)$$

(Notice that $(*d\varphi)|Y$ is the normal derivative of φ along Y.)

<u>Remark</u> Ultimately, the i in the preceding formula comes from the fact that X is a Riemannian rather than a Lorentzian manifold.

<u>Theorem (3.10)</u> (a) W is a positive maximal isotropic subspace of $(\Omega^0(Y) \oplus \Omega^{n-1}(Y))_{\mathbb{C}}$.

(b) The polarization defined by $W \oplus \overline{W}$ is independent of the choice of X. More precisely, it depends only on the <u>germ</u> of X along Y.

This completes the construction of the field theory, at least projectively. For we now have defined spaces $\mathcal{H}_Y$, and a ray $L_X \subset \mathcal{H}_Y$ when $Y = \partial X$. If $X : Y_0 \rightsquigarrow Y_1$ is a cobordism, then clearly

$$\mathcal{H}_{\partial X} = \mathcal{H}_{Y_0}^* \otimes \mathcal{H}_{Y_1},$$

so X defines an operator $U_X : \mathcal{H}_{Y_0} \to \mathcal{H}_{Y_1}$ up to a scalar multiple.

Of course it is very important to fix the scalar multiples which I have left completely vague. That is done by the use of ζ-function determinants, which are described in § 2.8 of the accompanying notes. But I shall not discuss this further here.

§4 Free fermionic field theory

On a real vector space V physicists regard elements of $\Lambda(V^*)$ as "anticommuting functions". By the side of $L^2(V)$ we can construct a Hilbert space $\mathcal{H} = \Lambda_{(L^2)}(V^*)$. It is, by definition, a representation of the Clifford algebra $C(V \oplus V^*)$, when $V \oplus V^*$ has now its natural symmetric quadratic form. Thus $\mathcal{H}$ has differentiation operators

$$D_\xi : \mathcal{H} \to \mathcal{H} \qquad \text{for } \xi \in V$$

and multiplication operators

$$m_\alpha : \mathcal{H} \to \mathcal{H} \qquad \text{for } \alpha \in V^*,$$

but now

$$D_\xi m_\alpha + m_\alpha D_\xi = \alpha(\xi).$$

To define $\mathcal{H}$ we choose a positive maximal isotropic subspace $W \subset (V \oplus V^*)_{\mathbb{C}}$, and take

$$\mathcal{H} = \Lambda(\overline{W}).$$

A theorem exactly analogous to Shale's theorem (2.8) holds in this situation.

The most frequent application is to the case where $W = E^- \oplus (E^-)^\circ \subset E \oplus E^*$, where $E = V_{\mathbb{C}}$. In this case we can express things much more simply, for $\mathcal{H}$ is then simply the Fock space $\mathcal{F}(E)$ associated to a polarized complex vector space E. This is described in § 2.9 of the accompanying notes.

We can now describe massless free fermionic field theory of dimension n, where n is <u>even</u>. It is defined for manifolds with a spin^c-structure.

If Y is a closed $(n-1)$-manifold, we define $\mathcal{H}_Y = \mathcal{F}(E)$, where E is the space of spinor fields $\Gamma(Y)$, which is polarized by the Dirac operator D_Y.

If $\partial X = Y$ we define $\mathcal{H}_Y^-$ to be the boundary values of the solutions of the Dirac equation $D_X \varphi = 0$, where $\varphi \in \Gamma^{even}(X)$. Everything now works as in the bosonic case.

<u>Remarks</u> (i) This theory describes particles which do not interact with each other, but which are created and annihilated by the curved background, i.e. the <u>gravitational field</u>. The number of particles created by a cobordism $X : Y_0 \rightsquigarrow Y_1$ is the <u>index</u> of the Dirac operator D_X, as is explained in Chap. 2 of the notes.

(ii) I have implicitly used two different polarizations of $E = \Gamma(Y)$, one defined by D_Y, and one by D_X, where $\partial X = Y$. These are equivalent providing the germ of X along Y agrees with $Y \times \mathbb{R}$ to order $\frac{n}{2} - 1$. Physically, this means that the propagation of fermions in a gravitational field depends only on the usual gravitational Cauchy data (i.e. metric and 1st fundamental form of Y) <u>only</u> if $n \leq 4$.

§5 Conformal field theory

There are at least three reasons for being interested in 2-dimensional conformal field theory:

(i) as non-trivial but fairly well-understood examples of quantum field theories;

(ii) as the continuum limits of statistical mechanical models describing changes of phase;

(iii) because a space with a reasonable claim to be called $\Omega^{\infty/2}(\Sigma M)$ can be the space $\mathcal{H}_{S^1}$ of a CFT only if (roughly speaking) the Riemannian manifold M satisfies Einstein's equations — in other words, a CFT is a "generalized solution of Einstein's equations.

The theory concerns compact oriented smooth surfaces with smooth boundary surfaces, equipped with a conformal structure. Call these 'Riemann surfaces'. We are interested in cobordisms $\Sigma : S_0 \rightsquigarrow S_1$, where Σ is a Riemann surface. Such cobordisms form a category, for we can sew Riemann surfaces together by diffeomorphisms of their boundary circles — but this is a non-trivial theorem.

We shall associate a vector space $\mathcal{H}_S$ to each closed oriented 1-manifold S, so that $\mathcal{H}_{S_1 \amalg S_2} = \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}$. This is equivalent to giving a single vector space $\mathcal{H}_{S^1}$ with an action of $\mathrm{Diff}^+(S^1)$, the orientation-preserving diffeomorphisms.

Then we associate an operator $U_\Sigma : \mathcal{H}_{S_0} \to \mathcal{H}_{S_1}$ to each cobordism $\Sigma : S_0 \rightsquigarrow S_1$. We want U_Σ to depend smoothly on the conformal structure of Σ. In particular, let

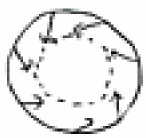
$$\mathcal{A} = \{\text{Riemann surfaces } \Sigma \text{ diffeomorphic to } S^1 \times [0,1]\}/\sim ,$$

where $\sim$ means equivalence by diffeomorphisms f such that $f|\partial\Sigma = \text{identity}$.

$\mathcal{A}$ is a semigroup which acts on $\mathcal{H}_{S^1}$. In fact we must allow it to act <u>projectively</u>. The existence of this action implies that the obvious action of $\mathrm{Diff}^+(S^1)$ on $\mathcal{H}_{S^1}$ extends to a projective action of $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$ on $\mathcal{H}_{S^1}$, with the obvious $\mathrm{Diff}^+(S^1)$ contained as the diagonal subgroup.

<u>First explanation</u> We should like to say that the action of $\mathcal{A}$ "continues analytically" to an action of the semigroup $\mathcal{A}^{\mathrm{phys}}$ of <u>Lorentzian</u> conformal cylinders. But to give a Lorentzian cylinder is roughly equivalent to giving the pair of diffeomorphisms $f_L, f_R : S^1 \to S^1$ obtained by following the left- and right-moving light lines from one end of the cylinder to the other.

<u>Better explanation</u> The Lie group $\mathrm{Diff}^+(S^1)$ does not possess a complexification. But, morally, $\mathcal{A}$ is the semigroup obtained by exponentiating the cone of "inward-pointing" elements in its complexified Lie algebra:



The relation of $\mathrm{Diff}^+(S^1)$ to $\mathcal{A}$ is precisely that of the group $PSL_2(\mathbb{R})$ to the semigroup

$$PSL_2^{<}(\mathbb{C}) = \{ g \in PSL_2(\mathbb{C}) : g(D) \subset \mathring{D} \},$$

where $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$. Thus $\mathcal{A}$ and $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$ are two "real forms" of the same complex group, and we have

<u>Theorem (5.1)</u> A projective representation of $\mathcal{A}$ by contraction operators on a Hilbert space $\mathcal{H}$ which is self-adjoint in the

sense that $U_{\bar{\Sigma}} = U_{\Sigma}^{*}$ can be continued analytically to a unitary projective representation of $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$.

To give a conformal field theory is therefore to give a representation $\mathcal{H}$ of $\mathcal{A}$ with some additional structure. Essentially, this structure is a <u>multiplication</u> law

$$U_{\Sigma} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H} \qquad (5.2)$$

induced by a "pair of pants" Σ. (To give U_{Σ} for one Σ is enough, if we already have the action of $\mathcal{A}$.)

The product (5.2) is known as the "operator product expansion". Its properties can be formalized in various ways, and reduced to pure algebra, e.g. in the theory of <u>vertex operator algebras</u>.

Two-dimensional CFT is well understood because the representations of $\mathrm{Diff}^+(S^1)$ are well understood. Two ways of constructing them are known.

(i) Let $\Omega^{a,b}$ be the space of a-forms on S^1 with coefficients in the flat bundle with holonomy $b \in \mathbb{C}^{\times}$, i.e. $\Omega^{a,b}$ consists of expressions $\varphi(\theta)(d\theta)^a$, where $\varphi(\theta + 2\pi) = b\varphi(\theta)$. Then $\Omega^{a,b}$ is a polarized vector space, with the polarization determined by the operator $i\frac{d}{d\theta}$. The polarization is preserved by diffeomorphisms of S^1: it does not depend on the choice of the parameter θ. Accordingly, $\mathrm{Diff}^+(S^1)$ acts on the Fock space $\mathcal{F}(\Omega^{a,b})$. Feigin and Fuchs proved that all (positive-energy) representations of $\mathrm{Diff}^+(S^1)$ are subquotients of the representations $\mathcal{F}(\Omega^{a,b})$.

(ii) If G is a compact Lie group then the representation theory of the loop group $\mathcal{L}G$ is well understood. In particular, every positive energy representation extends to a representation of $\mathcal{L}G_{\mathbb{C}}$. It is an important theorem that any such representation $\mathcal{H}$ admits an intertwining action of $\mathrm{Diff}^+(S^1)$, i.e. for each $f \in \mathrm{Diff}^+(S^1)$ there is an operator $U_f : \mathcal{H} \to \mathcal{H}$ such that

$$U_f \, M_\gamma \, U_f^{-1} = M_{\gamma \circ f^{-1}},$$

where M_γ is the action of $\gamma \in \mathcal{L}G$. In fact there is an operator $U_\Sigma : \mathcal{H} \to \mathcal{H}$ for each $\Sigma \in \mathcal{S}$ such that

$$U_\Sigma \, M_{\gamma_0} = M_{\gamma_1} \, U_\Sigma$$

whenever $\gamma_0, \gamma_1 \in \mathcal{L}G_{\mathbb{C}}$ are the boundary values of a holomorphic map $\gamma : \Sigma \to G_{\mathbb{C}}$.

Once again, all representations of $\mathrm{Diff}^+(S^1)$ arise in this way.

§6 Braided tensor categories

In §1.1 of the accompanying notes I have explained how a 2-dimensional topological field theory is exactly equivalent to a commutative Frobenius algebra. The best-known examples of Frobenius algebras are

(a) The representation ring of a group, and

(b) the cohomology ring of a manifold.

In this section and the next I shall explain how for a loop group one can ~~[illegible]~~ "lift" the TFT to one which takes values which are additive categories, and how for certain manifolds one can lift the TFT to one whose values are cochain complexes.

Let G be a compact Lie group, and $\mathcal{C}$ the category of positive-energy unitary representations of the loop group $\mathcal{L}G$ at a fixed level k. (The representations are projective, and, for a simply connected G, the level is the topological type of the circle bundle $\widetilde{\mathcal{L}G} \to \mathcal{L}G$ which really acts on the representation space: thus $k \in H^2(\mathcal{L}G;\mathbb{Z}) \cong H^3(G;\mathbb{Z})$.)

$\mathcal{C}$ is a $\mathbb{C}$-linear additive category, which is semisimple, with only finitely many irreducible objects. It exactly resembles the category of complex representations of a finite group.

I shall write $\mathcal{C}^{\otimes p}$ for the category of representation of $\underbrace{\mathcal{L}G \times \dots \times \mathcal{L}G}_{p}$. These are sums of representations of the form $E_1 \otimes \dots \otimes E_p$.

More functorially, I write $\mathcal{C}_S$, when S is a closed 1-manifold, for the category of representations of $\mathrm{Map}(S;G)$.

Theorem (6.1) A smooth cobordism $\Sigma : S_0 \rightsquigarrow S_1$ induces an additive functor

$$U_\Sigma : \mathcal{C}_{S_0} \longrightarrow \mathcal{C}_{S_1}.$$

The functor U_Σ is characterized by the following universal property:

for each object E of $\mathcal{C}_{S_0}$, and each complex structure σ on Σ, there is a linear map

$$T_\sigma : E \longrightarrow U_\Sigma(E)$$

such that $T_\sigma \circ \gamma_0 = \gamma_1 \circ T_\sigma$ whenever γ_0, γ_1 are the restrictions to S_0, S_1 of a σ-holomorphic map $\gamma : \Sigma \to G_{\mathbb{C}}$. (This makes sense because any representation of $\mathcal{L}G$ is automatically a representation of $\mathcal{L}G_{\mathbb{C}}$.)

When Σ is a cylinder $S^1 \times [0,1]$ we have $U_\Sigma = (\text{identity})$, and the maps $T_\sigma : E \to E$ are the action on E of the semigroup $\mathcal{A}$ described in § 5, which extends the action of $\mathrm{Diff}^+(S^1)$ on E.

The functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induced by a chosen "pair of pants" is called <u>fusion</u>. I shall write it $(E_1, E_2) \longmapsto E_1 * E_2$.

A diffeomorphism $f : \Sigma \to \Sigma'$ induces a natural transformation

$$u_f : U_\Sigma(E) \longrightarrow U_{\Sigma'}(E) \qquad (6.2)$$

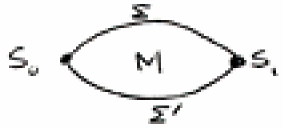
for any E. Thus $\mathrm{Diff}(\Sigma \text{ rel } \partial\Sigma)$ acts on $U_\Sigma(E)$. It must act through the <u>mapping class group</u> $\Gamma_{\Sigma, \partial\Sigma} = \pi_0 \mathrm{Diff}(\Sigma, \partial\Sigma)$, for the identity component of $\mathrm{Diff}(\Sigma, \partial\Sigma)$ has no non-trivial finite dimensional representations. In particular, the <u>coloured braid group</u> on n strings always acts on

$$E_1 * E_2 * \ldots * E_n .$$

With the example of the representations of a loop group in mind, it seems reasonable to propose that a "braided tensor category" should be <u>defined</u> as a 2-dimensional TFT with values in $\mathbb{C}$-linear categories.

For any such structure many more remarkable things are true. If $\Sigma, \Sigma' : S_0 \leadsto S_1$ then not only does a diffeomorphism $f : \Sigma \to \Sigma'$

induces a map as in (6.2), but a cobordism $M : \Sigma \rightsquigarrow \Sigma'$ (rel S_0, S_1)



induces

$$u_M : U_\Sigma(E) \to U_{\Sigma'}(E).$$

We now have a three-dimensional TFT. For $\mathcal{C}_\emptyset$ is the category of finite-dimensional vector spaces, so we can associate to a closed surface Σ the vector space $E_\Sigma = U_\Sigma(\mathbb{C})$, on which the mapping-class group Γ_Σ acts. A cobordism $M : \Sigma \rightsquigarrow \Sigma'$ induces $u_M : E_\Sigma \to E_{\Sigma'}$.

In the loop-group example the 3-dimensional theory is Chern-Simons theory at level k. That is, the number $u_M \in \mathbb{C}$ associated to a closed 3-manifold M is

$$u_M = \int_{\mathcal{A}_M/\mathcal{G}_M} e^{2\pi i k\, CS(A)}\, \mathcal{D}A,$$

where $\mathcal{A}_M$ is the space of G-connections on M, $\mathcal{G}_M$ is the corresponding gauge-group, and

$$CS(A) = \int_M \{\langle A, dA\rangle + \tfrac{1}{3}\langle A, [A,A]\rangle\}.$$

Final remarks

(i) The three-tier structure

1-manifold $\longmapsto$ category
2-manifold $\longmapsto$ vector space
3-manifold $\longmapsto$ number

is exactly of the type arising in the index theory of the Dirac operator, as described in Lecture 2 of the accompanying notes.

(ii) If we replace the categories $\mathcal{C}_S$ by their Grothendieck groups $K(\mathcal{C}_S)$ — e.g. $K(\mathcal{C}_{S^1})$ = representation-ring $R_k(\mathcal{L}G)$ of $\mathcal{L}G$ at level k — then we have a 2-dimensional TFT in the usual sense, but over $\mathbb{Z}$ rather than $\mathbb{C}$. The Frobenius algebra $K(\mathcal{C}_{S^1})$ is called the Verlinde algebra

§ 7 String algebras

The cohomology ring $H^*(M;\mathbb{C})$ of a compact oriented manifold is a Frobenius algebra : the duality is Poincaré duality. In some situations this algebra can be "lifted" to a TFT with values in cochain complexes. A theory of this kind I shall call a string algebra. But first, a general remark ...

Classical homotopy theory tells us that the category of topological spaces and homotopy classes of maps can be approximated by an additive category $\mathcal{S}p$ with a tensor product, namely the category of "spectra". Essentially, $\mathcal{S}p$ is the category of spaces and stable homotopy classes of maps. Ideally, I would describe 2-dimensional TFTs with values in $\mathcal{S}p$, but for simplicity I shall here replace $\mathcal{S}p$ by the cruder category of cochain complexes of real vector spaces. All the same, it is probably clearest to think of the objects as spaces ...

Accordingly, we consider a cochain complex $C^\cdot = C^\cdot_{S^1}$ of topological vector spaces on which $\mathrm{Diff}^+(S^1)$ acts, perhaps trivially. (See the note below for the meaning of a $\mathrm{Diff}^+(S^1)$ action.) There is an associated functor

$$(\text{closed oriented 1-manifolds}) \longrightarrow (\text{cochain complexes})$$
$$S \longmapsto C^\cdot_S$$

such that $C^\cdot_{S_1 \sqcup S_2} = C^\cdot_{S_1} \otimes C^\cdot_{S_2}$.

To each smooth cobordism $\Sigma : S_0 \rightsquigarrow S_1$ we associate a cochain map $U_\Sigma : C^\cdot_{S_0} \to C^\cdot_{S_1}$. We want this to depend smoothly on Σ. One possible way to formulate this is :

if $\mathcal{M}et_\Sigma$ is the contractible space of Riemannian metrics on Σ, there is given an element

$$\hat{U}_\Sigma \in \Omega^\cdot(\mathcal{M}et_\Sigma ; \mathrm{Hom}(C^\cdot_{S_0}, C^\cdot_{S_1}))$$

which is closed for the total differential of this double (or triple...) complex, and is basic (see below) for the action of $\mathrm{Diff}(\Sigma)$, which acts on $C^\cdot_{S_0}$ and $C^\cdot_{S_1}$ by restriction. If we restrict $\hat{U}_\Sigma$ to a point of $\mathcal{M}et_\Sigma$ we get back the previous $U_\Sigma : C^\cdot_{S_0} \to C^\cdot_{S_1}$.

Compatibility with the sewing-together of cobordisms $S_0 \overset{\Sigma}{\rightsquigarrow} S_1 \overset{\Sigma'}{\rightsquigarrow} S_2$ means that

$$\Omega^\cdot(\mathcal{M}et_\Sigma \times \mathcal{M}et_{\Sigma'} ; \mathrm{Hom}(C^\cdot_{S_0}; C^\cdot_{S_1}) \otimes \mathrm{Hom}(C^\cdot_{S_1}; C^\cdot_{S_2})) \to \Omega(\mathcal{M}et_{\Sigma\cup\Sigma'} ; \mathrm{Hom}(C^\cdot_{S_0}, C^\cdot_{S_2}))$$

$$\hat{U}_{\Sigma'} \otimes \hat{U}_\Sigma \longmapsto U_{\Sigma\cup\Sigma'}$$

where the map is obtained by combining the restriction

$$\mathcal{M}et_{\Sigma\cup\Sigma'} \longrightarrow \mathcal{M}et_\Sigma \times \mathcal{M}et_{\Sigma'}$$

with composition of maps of complexes.

A string algebra, as thus defined, comprises a great deal of information. For example

(i) The cohomology $H^*(C^\cdot_{S^1})$ is a Frobenius algebra.

(ii) For each smooth closed surface Σ we get a cohomology class $\hat{U}_\Sigma \in H^*(\mathcal{M}_\Sigma)$ of the moduli space $\mathrm{Met}_\Sigma / \mathrm{Diff}_\Sigma$.

(iii) The ring $\bigoplus_{g \geq 0} H_*(\mathcal{M}_{g,2})$ acts on $H^*(C^\cdot_{S^1})$, where $\mathcal{M}_{g,2}$ is the moduli space $\mathcal{M}et_\Sigma / \mathrm{Diff}_{\Sigma \text{ rel } \partial\Sigma}$ of a surface of genus g with two boundary circles.

<u>Note</u> To say that a Lie group G acts on a cochain complex $C^\cdot$ means that G acts on each vector space C^k in the obvious sense, and that, in addition, there is given, for each ξ in the Lie algebra $\mathfrak{g}$ of G, a linear map

$$i_\xi : C^k \to C^{k-1}$$

such that $i_\xi \circ d + d \circ i_\xi = L_\xi$, where $L_\xi : C^k \to C^k$ is the action of $\mathfrak{g}$ got by differentiating the G-action.

An element $\alpha \in C^\cdot$ is <u>basic</u> if it is invariant under G and also annihilated by i_ξ for all $\xi \in \mathfrak{g}$. If $C^\cdot$ is the de Rham complex of a manifold X on which G acts freely, then the subcomplex $C^\cdot_{\text{basic}}$ is the de Rham complex of X/G.

§8 Floer cohomology

Examples of string algebras arise in two places: conformal field theory, and the Floer cohomology of loop spaces.

Floer cohomology was invented to prove the Arnold conjecture. Arnold had observed that if (M, ω) is a compact symplectic manifold, then a function $H : M \to \mathbb{R}$ determines a Hamiltonian flow on M whose fixed points are the critical points of H. Conventional Morse theory tells us that the number of critical points of H must be at least as great as $\sum_i \dim H^i(M)$. Arnold conjectured that for a flow $f_t : M \to M$ generated by a <u>time-dependent</u> Hamiltonian $H_t : M \to \mathbb{R}$ $(0 \leq t \leq 1)$ the number of fixed-points of f_1 obeys the same estimate.

The fixed-points of $f_1 : M \to M$ are the critical points of

$$S : \mathcal{L}M \longrightarrow \mathbb{R}/(\text{periods of } \omega),$$

where $S(\gamma) = \int_\gamma (p\,dq - H_t\,dt)$. Note that $\int_\gamma p\,dq$ is defined only modulo periods of ω, for it really means $\int_\Sigma \omega$, where $\partial\Sigma = \gamma$. I shall assume the periods are integral, so that $S : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$.

Floer observed that the critical points of S generically form a finite set, but that the Hessian at each critical point has infinitely many positive and infinitely many negative eigenvalues. Nevertheless, he showed that counting gradient lines of S from one critical point to another defines a cochain complex $CF^*(\mathcal{L}M)$ whose cohomology is $H^*(M)$, and, in particular, is independent of the Hamiltonian $\{H_t\}$. (To define a gradient flow of S we must choose a Riemannian metric on M in addition to its symplectic form ω, so that M becomes "almost Kähler".)

It would be very interesting to understand properly the additional structure an infinite dimensional manifold X, such as $\mathcal{L}M$, must have if one is to define Floer cohomology $HF^*(X)$. Obviously we only need to know the function $S : X \to \mathbb{R}/\mathbb{Z}$ to within fairly generous perturbations. There seem to be two essential ingredients in the construction.

(i) At a critical point γ of S the Hessian $D^2S(\gamma)$ splits the tangent space $T_\gamma X$ into "stable" and "unstable" parts: $T_\gamma X = T_\gamma^+ \oplus T_\gamma^-$. In fact the Hessian defines such a splitting, not quite canonical, at <u>every</u> point of X, i.e. it defines a <u>polarization</u> of each tangent space $T_\gamma X$ in the sense defined on page 9. Such a structure is called a <u>polarization of X</u>. It reduces the structural group of TX from a contractible general linear group to a group GL_{res}, whose classifying space BGL_{res} is homotopy equivalent to $U_\infty = \bigcup_{n \geq 1} U_n$. Thus the polarization gives us a homotopy class of maps $X \to U_\infty$.

(ii) Besides the polarization, the function S provides an increasing family of compact subspaces $\{X_t\}_{t>0}$ of X: we define X_t as the points which lie on flow-lines of total energy $\leq t$. In the case of $\mathcal{L}M$ this means all "equators" of pseudoholomorphic maps $\varphi : S^2 \to M$ such that

$$E(\varphi) = \frac{1}{2}\int_{S^2} \langle d\varphi, *d\varphi\rangle \le t.$$

Notice that in thinking conceptually about the Floer cohomology of $\mathcal{L}M$ it is best to have in mind either the "unperturbed" case (with $H_t = 0$), when the critical points are just the point loops $M \subset \mathcal{L}M$, and the flow lines from $m_0 \in M$ to $m_\infty \in M$ are the pseudo-holomorphic maps $\varphi : S^2 \to M$ such that $\varphi(0) = m_0$ and $\varphi(\infty) = m_\infty$, or else the case of a small generic time-independent perturbation H, when the critical set $K \subset M \subset \mathcal{L}M$ is the finite set of point loops at critical points of H, and the meaning of "pseudo-holomorphic" is slightly deformed.

For the present, the main point is to understand why a cobordism $\Sigma : S_0 \rightsquigarrow S_1$, where S_0 and S_1 are circles, gives us a cochain map $U_\Sigma : CF^\cdot(\mathcal{L}M) \to CF^\cdot(\mathcal{L}M)$. For this, choose a complex structure on Σ, and let us work with a time-independent perturbation as just described. Then the matrix element of U_Σ between critical points m_0 and m_1 is the number of pseudo-holomorphic maps $\varphi : \Sigma \to M$ such that $\varphi(S_i) = m_i$. To be more precise, the moduli space of such φ has many components, each with a <u>degree</u> $\in H_2(M;\mathbb{Z})$ which determines its expected dimension. We count the points in the components of dimension 0. Evidently we could define the matrix element as an element of the group-ring R of $H_2(M;\mathbb{Z})$, so we have, if we want, a cochain-complex of R-modules.

It is fairly easy to see that ~~[illegible]~~ U_Σ is a cochain map whose cochain-homotopy class is independent of the chosen complex structure on Σ.

§9 Conformal field theory and string algebras

String algebras of two basic types arise in conformal field theory. The first comes from any CFT $S \mapsto \mathcal{H}_S$ involving the "correct" central extension of the semigroup $\mathcal{A}$. (In the usual terminology, we need $c =$ "central charge" $= 13$.) We think of $\mathcal{H}_{S^1}$ as a space of square-summable functions on $\mathcal{L}M$ for some Riemannian manifold M. We should like to study the $\mathcal{A}$-invariant part of $\mathcal{H}_{S^1}$, but that gives 0 because of the central extension. As the next best thing, we look at the Lie algebra cohomology $H^*(\mathfrak{a}_{S^1}; \mathcal{H}_{S^1})$, where $\mathfrak{a}_{S^1}$ is the Lie algebra of $\mathcal{A}$, i.e. $\mathfrak{a}_{S^1} = \mathrm{Vect}(S^1) \otimes \mathbb{C}$, but regarded as a <u>real</u> Lie algebra. This comes from the cochain complex

$$\Lambda^{\cdot}(\mathfrak{a}^*_{S^1}) \otimes \mathcal{H}_{S^1}. \qquad (9.1)$$

This still gives us nothing unless we consider the <u>middle-dimensional</u> or <u>semi-infinite</u> cohomology, i.e. we replace (9.1) by

$$\mathcal{F}(\mathfrak{a}_{S^1}) \otimes \mathcal{H}_{S^1}, \qquad (9.2)$$

where $\mathcal{F}(\mathfrak{a}_{S^1})$ is the Fock space defined using the polarization described in §5. To ensure that (9.2) really has a differential d satisfying $d^2 = 0$ we need the condition $c = 13$. It is then easy to see that $S \mapsto \mathcal{F}(\mathfrak{a}_S) \otimes \mathcal{H}_S$ is a string algebra. These are the classical algebras of bosonic string theory. They are usually called "string backgrounds". We should think of them as semiinfinite differential forms along the orbits of $\mathrm{Diff}(S^1)$ on $\mathcal{L}M$. The elements of $\mathcal{F}(\mathfrak{a}_S)$ are usually called "ghost fields", and d is the "BRS operator".

A more interesting kind of string algebra for us arises from $N=2$ supersymmetry. First let me define $N=1$ supersymmetry.

Let $S \mapsto \mathcal{H}_S$ be a CFT, and suppose that for each $\alpha \in \Omega^{\frac{1}{2}}(S^1)$ we have an operator $D_\alpha : \mathcal{H}_S \to \mathcal{H}_S$ which is natural with respect to $\mathrm{Diff}(S^1)$, and satisfies

$$D_\alpha^2 = L_{\alpha^2},$$

where L_{α^2} means the action on $\mathcal{H}_S$ of the vector field $\alpha^2 \in \Omega^{-1}(S)$ regarded as an element of the Lie algebra of $\mathrm{Diff}(S^1)$.

We require the cobordism operators $U_\Sigma : \mathcal{H}_{S_0} \to \mathcal{H}_{S_1}$ associated to Riemann surfaces $\Sigma : S_0 \rightsquigarrow S_1$ to satisfy

$$\alpha_1 \circ U_\Sigma = U_\Sigma \circ \alpha_0,$$

when α_0, α_1 are the restrictions to S_0, S_1 of a <u>holomorphic</u> $\alpha \in \Omega^{-\frac{1}{2}}_{hol}(\Sigma)$. (Cf. page 17)

A theory of this type should be thought of as modelling the square-summable <u>spinor fields</u> $\Gamma(\mathcal{L}M) = \mathcal{H}_{S^1}$ on a loop space $\mathcal{L}M$, as I tried to explain in my Bourbaki talk on Elliptic cohomology. The Dirac operator is $D_\alpha : \Gamma(\mathcal{L}M) \to \Gamma(\mathcal{L}M)$ is when $\alpha = (d\theta)^{-\frac{1}{2}}$.

An $N=2$ supersymmetric CFT arises, morally, when M is Kähler. Then $\mathcal{H}_{S^1} = \Gamma(\mathcal{L}M)$ possesses in addition an action of the loop group $\mathcal{L}\mathbb{T}$, for $\lambda \in \mathcal{L}\mathbb{T}$ acts on each complex tangent space $T_\gamma \mathcal{L}M$ by

$$(\lambda . \xi)(\theta) = \lambda(\theta)\, \xi(\theta),$$

with $\xi(\theta) \in T_{\gamma(\theta)} M$.

When we combine this action with those of $\mathrm{Vect}_{\mathbb{C}}(S^1)$ and $\mathcal{L}\mathfrak{t} = \mathrm{Lie}(\mathcal{L}\mathbb{T}) \otimes \mathbb{C}$ we obtain the action of a certain Lie superalgebra $\Omega^{(2)}$ whose even part is the complexified Lie algebra of the semidirect product $\mathrm{Diff}(S^1) \tilde{\times} \mathcal{L}\mathbb{T}$. This is the $N=2$ superalgebra.

Meanwhile, there is another Lie superalgebra $\tilde{\mathfrak{a}}^{(2)}$, with the same even part $\mathrm{Vect}_{\mathbb{C}}(S^1) \tilde{\times} \mathcal{L}\mathbb{C}$, which we expect to act on the differential forms on $\mathcal{L}M$ when M is Kähler. The action of the even part is clear. The odd part is

$$\mathrm{Vect}_{\mathbb{C}}(S^1) \oplus \mathcal{L}\mathbb{C}.$$

Here $\xi \in \mathrm{Vect}_{\mathbb{C}}(S^1)$ acts as i_ξ, the inner product with the vector field on $\mathcal{L}M$ defined by ξ (which reparametrizes the loops); and $f \in \mathcal{L}\mathbb{C}$ acts by an operator d_f characterized by

$$d_1 = d = \text{the de Rham differential}$$

$$\lambda \circ d_f \circ \lambda^{-1} = d_{\lambda f},$$

where $\lambda \in \mathcal{L}\mathbb{T}$.

When the algebra $\tilde{\mathfrak{a}}^{(2)}$ acts on $\mathcal{H}_{S^1}$ we have $d_1^2 = 0$, so we have a cochain complex, and in fact a string algebra. We also have a (p,q)-grading on $\mathcal{H}_{S^1}$ coming from the action of the constants $\mathbb{T} \subset \mathcal{L}\mathbb{T}$.

The remarkable fact is that there are two natural isomorphisms

$$\tilde{\mathfrak{a}}^{(2)} \longrightarrow \mathfrak{a}^{(2)}$$

compatible with the propagation operators of CFT. This means that an $N=2$ supersymmetric CFT gives us <u>two</u> string algebras, which are said to be related by <u>mirror symmetry</u>.