Symplectic manifolds and quantization

I shall assume you are familiar with

smooth manifolds, vector fields, and differential forms;

the relation between vector fields and 1-parameter groups of diffeomorphisms;

the Lie derivative, its relation to 1-parameter groups of diffeomorphisms,

and, above all, the Cartan relation

$$\mathcal{L}_\xi = i_\xi d + d i_\xi ;$$

de Rham's theorem;

rudimentary facts about Lie groups and Lie algebras, and about line bundles and connections in them.

§1)

A symplectic manifold is a smooth manifold X with a closed 2-form $\omega \in \Omega^2(X)$ which is nondegenerate at each point — i.e. such that ω_x induces an isomorphism $T_x X \to T_x^* X$ for each $x \in X$.
(Equivalently: $\omega^n \in \Omega^{2n}(X)$ never vanishes, where $2n = \dim(X)$.)

The basic fact is that a function f on X gives a vector field ξ_f by applying $\omega_x : T_x \cong T_x^*$ to df. More precisely,

$$i_{\xi_f} \omega = -df .$$

The Poisson bracket $\{f,g\}$ of two functions on X is

$$\{f,g\} = \xi_f\, g = \omega(\xi_f, \xi_g) = -\{g,f\}.$$

Remark On a Riemannian manifold a function f also defines a vector field: its gradient. But in the symplectic case the flow ξ_f is along the contours of f, as $\xi_f f = \{f,f\} = 0$.

What is the significance of having ω closed?

Proposition ω closed $\iff [\xi_f, \xi_g] = \xi_{\{f,g\}}$

$\iff$ the Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$

The second $\Leftrightarrow$ here is trivial, while the first comes from

$$d\omega(\xi_f, \xi_g, \xi_h) = \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\},$$

which holds without assuming ω is closed. //

Thus the functions on a symplectic manifold form a <u>Lie algebra</u> under the Poisson bracket.

<u>Definition</u> A Poisson manifold X is one with a skew tensor field $\sigma \in \Gamma(\Lambda^2 TX)$ which — via $\{f,g\} = \langle \sigma, df \wedge dg \rangle$ — makes the functions on X into a Lie algebra.

Thus a symplectic manifold is a Poisson manifold for which σ is non-degenerate at each point.

<u>Examples</u> (i) If $\mathfrak{g}$ is a Lie algebra, its dual $\mathfrak{g}^*$ is a Poisson manifold with $\{f,g\}(\alpha) = \alpha([df(\alpha), dg(\alpha)])$ for $\alpha \in \mathfrak{g}^*$. (Here for $f: \mathfrak{g}^* \to \mathbb{R}$ we regard df as a map $\mathfrak{g}^* \to (\mathfrak{g}^*)^* = \mathfrak{g}$.)

(ii) The most basic example of a symplectic manifold is the cotangent bundle T^*M of a manifold M. This has $\omega = d\alpha$, where α is the canonical 1-form which in a local coordinate chart is $\Sigma p_i dx_i$, where $\{x_i\}$ are the coordinates on X and $\{p_i\}$ are the fibrewise-linear functions on T^*M dual to the linear functions $\{dx_i\}$ on TM.

The vector field ξ_f associated to a function f on a symplectic manifold preserves the form ω, for $\mathcal{L}_\xi \omega = d i_\xi \omega + i_\xi d\omega = -ddf = 0$. Conversely if X is simply connected — or even if $H^1(X;\mathbb{R}) = 0$ — a vector field ξ which preserves ω always comes from a function f, for

$$\mathcal{L}_\xi \omega = 0 \Rightarrow d i_\xi \omega = 0 \Rightarrow i_\xi \omega = -df \text{ for some } f.$$

In general there is an exact sequence of Lie algebras

$$0 \to H^0(X;\mathbb{R}) \longrightarrow \Omega^0(X) \longrightarrow \mathrm{Vect}_\omega(X) \longrightarrow H^1(X;\mathbb{R}) \to 0, \quad (1.1)$$

where Vect_ω denotes the vector fields which preserve ω. The map $\mathrm{Vect}_\omega \to H^1$ associates to a vector field ξ and a 1-cycle γ in X the "flux" $\int_\gamma i_\xi \omega$ through γ. If X is a surface with area form ω this is the obvious flux through γ of an area-preserving flow.

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§ 2 Variational problems

Symplectic manifolds arise from variational problems.

Suppose we want to find a path $\varphi : \mathbb{R} \to M$, where M is some manifold, which makes the action

$$S_{ab}(\varphi) = \int_a^b L(\varphi(t), \dot\varphi(t), \ddot\varphi(t), \ldots)\, dt$$

stationary for all intervals $[a,b] \subset \mathbb{R}$, where $L : JM \to \mathbb{R}$ is a given smooth function on the space JM of jets of maps $\mathbb{R} \to M$. (A jet at $t \in \mathbb{R}$ is an equivalence class of maps $\varphi : U \to M$ defined in a nbd U of t, where $\varphi \sim \varphi'$ if φ and φ' touch to infinite order at t. If we coarsen this to touching to order r we get the r-jets $J_r M$, which form a manifold of dimension $(r+1)\dim(M)$ fibred over M. Thus $J_1 M = TM$.)

'Stationary' means that if $\varphi = \varphi_0$, where $\{\varphi_s\}$ is a family of paths, then

$$\frac{d}{ds} S_{ab}(\varphi)\Big|_{s=0} = 0$$

providing that for all s we have $\varphi_s = \varphi$ in the nbd of a and b.

Of course, φ is a stationary point iff the Euler-Lagrange equation $\frac{\delta L}{\delta\varphi} = 0$ holds, where

$$\frac{\delta L}{\delta \varphi} = \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) + \left(\frac{d}{dt}\right)^2\left(\frac{\partial L}{\partial \ddot\varphi}\right) - \ldots .$$

If the Lagrangian L depends only on the r-jet the Euler-Lagrange equation is an ODE of order $2r$ for φ, and generically has a solution manifold X of dimension $2rn$, where $n = \dim(M)$.

Considering variations $\delta\varphi$ with no boundary conditions we have

$$dS_{ab}(\varphi;\delta\varphi) = \int_a^b \frac{\delta L}{\delta\varphi}\,\delta\varphi\,dt + \alpha(\varphi_b;\delta\varphi_b) - \alpha(\varphi_a;\delta\varphi_a),$$

where

$$\alpha(\varphi;\delta\varphi) = \left\{\frac{\partial L}{\partial\dot\varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial\ddot\varphi}\right) + \ldots\right\}\delta\varphi + \left\{\frac{\partial L}{\partial\ddot\varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dddot\varphi}\right) + \ldots\right\}\delta\dot\varphi + \ldots .$$

Here φ_a denotes the jet of φ at $a \in \mathbb{R}$.

Thus on the solution manifold X of the Euler-Lagrange equation we have functions $S_{ab} : X \to \mathbb{R}$, and 1-forms $\alpha_x \in \Omega^1(X)$ defined by $\alpha_x(\varphi;\delta\varphi) = \alpha(\varphi_x;\delta\varphi_x)$, and

$$\alpha_b - \alpha_a = dS_{ab}. \tag{2.1}$$

In particular, $\omega = d\alpha_b = d\alpha_a$ is a naturally defined closed 2-form on X which in good cases makes it a symplectic manifold.

There is a natural flow on X defined by translating solutions in time. If ξ is the vector field which generates it, then (2.1) gives us

$$i_\xi\alpha_b - i_\xi\alpha_a = i_\xi dS_{ab} = L_b - L_a,$$

where $L_a : X \to \mathbb{R}$ is $\varphi \mapsto L(\varphi_a)$. In other words, the function $H : X \to \mathbb{R}$ defined by

$$H(\varphi) = i_\xi \alpha_a - L_a$$

is independent of $a \in \mathbb{R}$. H is called the Hamiltonian function.
Dividing (2.1) by $b-a$ and letting $b \to a$ gives

$$\mathcal{L}_\xi \alpha_a = dL_a ,$$

or equivalently

$$dH = - i_\xi \omega .$$

Thus ξ is the vector field generated by H.

The preceding formalism is not restricted to 1-dimensional problems. If for any manifold $\hat{\Sigma}$ we look for $\varphi : \hat{\Sigma} \to M$ which makes the functions

$$S_\Sigma(\varphi) = \int_\Sigma L(\varphi_x)\, dx$$

stationary for every compact submanifold Σ with boundary inside $\hat{\Sigma}$, then dS_Σ will be of the form

$$dS_\Sigma = \int_\Sigma \frac{\delta L}{\delta \varphi} \delta\varphi \, dx + \int_{\partial\Sigma} \alpha(\varphi_x ; \delta\varphi_x) .$$

On the space X of φ which satisfy $\frac{\delta L}{\delta \varphi} = 0$ we get

$$dS_\Sigma = \alpha_{\partial\Sigma} .$$

In fact for each compact oriented codimension 1 submanifold S of $\hat{\Sigma}$ we have a 1-form α_S on X, and if Σ is a cobordism from S_0 to S_1 then

$$\alpha_{S_1} - \alpha_{S_0} = dS_\Sigma ,$$

so that $\omega = d\alpha_S$ is a closed 2-form on X which depends only on the homology class of S. (The compactness of S can of course be replaced by conditions on the behaviour of φ at infinity.)

Returning to the 1-dimensional problem, the standard case is when M is a Riemannian manifold, and $L : TM \to \mathbb{R}$ is of the form

$$L(\varphi,\dot\varphi) = \tfrac{1}{2}\|\dot\varphi\|^2 - V(\varphi)$$

for some $V : M \to \mathbb{R}$. (When $V=0$ this is the problem of geodesics on M.) Then the symplectic manifold X of trajectories can be identified symplectically with T^*M by the Legendre transformation $\varphi \mapsto (\varphi(0), p(0))$, where $p(t)$ is the image of $\dot\varphi(t)$ under the map $T_{\varphi(t)}M \to T^*_{\varphi(t)}M$ given by the Riemannian metric.

§3 Darboux's theorem

Riemannian manifolds are locally quite unlike each other: the intrinsic curvature cannot be changed by a diffeomorphism. The same is true for manifolds with a skew form ω. But if ω is closed the position is quite different, for we have

Darboux's theorem Any symplectic manifold is locally isomorphic to $\mathbb{R}^{2n}$ with its canonical symplectic structure $dx_1dx_2 + dx_3dx_4 + \ldots + dx_{2n-1}dx_n$.

The idea of the proof is very simple. A vector field ξ changes ω according to $\dot\omega = \mathcal{L}_\xi\omega = d(i_\xi\omega)$. So a deformation of ω which does not change its cohomology class — say $\dot\omega = d\alpha$ — is produced by the field ξ such that $i_\xi\omega = \alpha$. More precisely, if $\{\omega_t\}$ is a family of symplectic forms which agree outside a compact region, and $\omega_t = \omega_0 + d\alpha_t$ with α_t of compact support, then integrating the time-dependent flow $\{\xi_t\}$ such that $i_{\xi_t}\omega_t = \dot\alpha_t$ gives a diffeomorphism taking ω_0 to ω_t.

To obtain Darboux's theorem it is enough to start with an arbitrary symplectic form ω on $\mathbb{R}^{2n}$ and to find a diffeomorphism which makes ω into a form with constant coefficients in a neighbourhood of the origin. For any two non-degenerate skew bilinear forms are related by a linear isomorphism. Let $\tilde{\omega}$ be the form with constant coefficients which agrees with ω at 0, and let $\omega - \tilde{\omega} = d\alpha$, where α vanishes at 0. If f is a function with compact support which is 1 near 0 then $\omega_t = \omega - t\,d(f\alpha)$ is a path joining ω to a form which is constant near 0, and for suitable f the forms will all be non-degenerate.

<u>Remarks</u> (i) The method used here was introduced by Moser to prove that two volume forms on a compact manifold are related by a diffeomorphism providing they have the same total volume.

(ii) A more "old-fashioned" proof constructs a local coordinate system $\{p_i, q_i\}$ for which $\omega = \sum dp_i dq_i$ by induction on $\frac{1}{2}\dim(X)$. We choose p_1 at random, and then choose q_1 by solving the ODE $\{p_1, q_1\} = 1$. Then consider $X_0 = \{x \in X : p_1(x) = q_1(x) = 0\}$. This is (locally) symplectic, so by induction has good coordinates $\{p_i, q_i\}_{i \geq 2}$. Extend p_i and q_i first to $X_1 = \{x \in X : p_1(x) = 0\}$ by the projection $X_1 \to X_0$ along trajectories of ξ_{q_1}, then to X by the projection $X \to X_1$ along trajectories of ξ_{p_1}.

(iii) Darboux's theorem shows that symplectic manifolds have no local invariants. A symplectic structure on a manifold X defines a cohomology class $[\omega] \in H^2(X; \mathbb{R})$ which can vary continuously. But symplectic manifolds have more subtle continuous invariants than cohomological ones. A striking indication of this is <u>Gromov's squeezing theorem</u>, which says that the unit ball in $\mathbb{R}^{2n}$ is not symplectically equivalent to any subset of $D^2_r \times \mathbb{R}^{2n-2}$ if $r < 1$, where D^2_r is the disc of radius r in $\mathbb{R}^2$.

§ 4 Quantization

We can abstract from a classical system the symplectic manifold X of its trajectories. This does not carry much information unless we know the "significance" of various functions on X. In particular, there is the Hamiltonian $H : X \to \mathbb{R}$ which generates time-translation of the trajectories.

A symplectic (or Poisson) manifold is completely encoded in the Poisson algebra of functions on X.

Definition A Poisson algebra is a commutative algebra A with a bilinear map $\{\ ,\ \} : A \times A \to A$ which makes it a Lie algebra, and satisfies

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

The states of a quantum system are the rays in a Hilbert space $\mathcal{H}$. The self-adjoint operators in $\mathcal{H}$ are the "observables", corresponding to functions $X \to \mathbb{R}$ in the classical situation. But a state $\psi \in \mathcal{H}$ and an observable A together give not a single number but a probability distribution on $\mathbb{R}$ defined by

$$\mathcal{E}(f) = \frac{\langle \psi, f(A)\psi \rangle}{\langle \psi, \psi \rangle}.$$

On this level, all quantum systems are the same; but again we might like to single out a particular operator H, the Hamiltonian, which generates the unitary group $\{e^{itH}\}$ of automorphisms of $\mathcal{H}$ corresponding to time-translation.

A classical system is meant to be an approximate description of a quantum system, but it is hard to make this precise. One general principle relates the volume of parts of X, defined by the "Liouville" volume form $\frac{1}{n!}\omega^n$, to the number of quantum states. Thus $\mathrm{vol}(X)$ should be finite precisely when $\dim(\mathcal{H})$ is, and they should be roughly equal. More generally, if $f : X \to \mathbb{R}_+$ is a classical observable such that

$f^{-1}([0,\lambda])$ is of finite volume V_λ then the corresponding quantum observable A should be a positive operator with discrete spectrum, and

$$V_\lambda \sim (\text{number of eigenvalues of } A \text{ which are } \le \lambda).$$

<u>Example</u> The geodesic flow on a Riemannian manifold M corresponds classically to $X = T^*M$, with $H : T^*M \to \mathbb{R}$ given by $H(x,\xi) = \frac{1}{2}\|\xi\|^2$. The corresponding quantum system, for which $\mathcal{H} = L^2(M)$ and $H : \mathcal{H} \to \mathcal{H}$ is the Laplacian, encodes only the spectrum of the Laplacian. This does not enable us to reconstruct M, though it does tell us a lot about it — its dimension, volume, Euler number, . . .

To say something more precise, we can consider a family of quantum systems depending on a small parameter h (Planck's constant). We observe that a Poisson algebra arises naturally when one has a family $\{\mathcal{A}_h\}$ of associative algebras over $\mathbb{C}$, defined for small h, and such that $\mathcal{A}_0$ is commutative. (I assume that $\{\mathcal{A}_h\}$ is locally trivial as a family of vector spaces.) For, given $a, b \in \mathcal{A}_0$, we can define

$$\{a,b\} = -i\frac{d}{dh}[a_h, b_h]\Big|_{h=0},$$

where $a_h, b_h \in \mathcal{A}_h$ are paths such that $a_0 = a$ and $b_0 = b$. Then we can say that the Poisson manifold defined by $\mathcal{A}_0$ is the <u>classical limit</u> of the systems $(\mathcal{H}_h, \mathcal{A}_h)$.

<u>Example</u> Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{g}_h$ be the Lie algebra obtained by multiplying the bracket of $\mathfrak{g}$ by h. Let $\mathcal{A}_h = \mathcal{U}(\mathfrak{g}_h)$, the universal enveloping algebra. Then $\mathfrak{g}_h \cong \mathfrak{g}$ if $h \neq 0$, while $\mathfrak{g}_0$ is abelian. The Poisson structure on $\mathcal{A}_0 = S(\mathfrak{g}) = \{\text{polynomials on } \mathfrak{g}^*\}$ is easily seen to be that of the Poisson manifold $\mathfrak{g}^*$ described in §1.

The natural question is: does every Poisson manifold X come from a family of non-commutative algebras $\{\mathcal{A}_h\}$, and, if so, how uniquely? Following much earlier work (Moyal, Fedosov, ...), in the last few months Kontsevich has given a remarkable proof of existence and uniqueness in complete generality, at least on the <u>formal</u> level, i.e. working with formal power series in h. A crucial point is that Kontsevich requires the algebras $\mathcal{A}_h$ to be <u>local</u> with respect to X, i.e. he defines a multiplication m_h on functions on X such that $m_h(f,g)(x)$ depends only on the jets of f and g at x.

But there is a more ambitious question. Can one associate to a symplectic manifold X a single Hilbert space $\mathcal{H}$ with a ring of operators $\mathcal{A}$ which is a canonical deformation of the Poisson algebra of X? This is the preceding question with $h = 1$ instead of h infinitesimal, but we are asking for a representation of $\mathcal{A}$ in addition. The art of constructing $\mathcal{H}$ — whose existence does not seem very well motivated from physics — is called <u>geometric quantization</u>, and has led to some very interesting mathematics.

The idea arose in particle physics. For a classical particle the space X should be a symplectic manifold with an action of the Poincaré group P of isometries of space-time. Being a particle should mean that P acts transitively on X. Symplectic manifolds homogeneous under P are easily classified: they correspond to pairs of real numbers (m, s), "mass" and "spin". A quantum particle should presumably be represented by a Hilbert space on which P acts by an <u>irreducible</u> unitary representation. These representations again correspond to pairs (m, s), though now s must be integral. This observation was the origin of Kirillov's doctrine that for any Lie group G the irreducible unitary representations correspond very closely to the homogeneous symplectic manifolds for G, which are, as we shall see, very easy to classify.

A good example to have in mind in thinking about quantization is the sphere S^2_r of radius r in $\mathbb{R}^3$, with its natural area form ω. The algebra of polynomial functions on S^2_r is $\mathbb{C}[X,Y,Z]/(X^2+Y^2+Z^2-r^2)$, and the Poisson structure is

$$\{X,Y\} = \text{[illegible]}\, Z \quad , \text{ and cyclic permutations.}$$

It should be thought of as a quotient of the Poisson algebra of $\mathfrak{g}^*$, where $\mathfrak{g} \cong \mathbb{R}^3$ is the Lie algebra of SO_3. Then $\mathcal{U}(\mathfrak{g})$ is generated by X, Y, Z with the relations $[X,Y] = Z$, etc., and has a centre generated by the Casimir element $X^2+Y^2+Z^2$. The natural choice for the algebra of operators associated to S^2_r is therefore $\mathcal{A} = \mathcal{U}_r(\mathfrak{g})$, the quotient of $\mathcal{U}(\mathfrak{g})$ by the relation $X^2+Y^2+Z^2 = r^2$. But this means that a representation of $\mathcal{A}$ is a representation of $\mathfrak{g}$ with Casimir operator r^2. There is no such representation unless $r^2 =$ [illegible]$n(n+1)$ for some integer n, and when that condition is satisfied there is a unique representation, of dimension $2n+1$, on which most of $\mathcal{A}$ acts trivially.

It quickly emerged that the proper starting points of geometric quantization, accounting for the integrality properties, are not simply symplectic manifolds, but rather pairs (X,L), where X is symplectic and L is a hermitian line bundle with a unitary connection A whose curvature form is the symplectic form ω.

If one is familiar with the theory of line bundles the following result is more or less obvious.

<u>Proposition</u> (i) For given symplectic (X,ω) one can find (L,A) if and only if X is <u>integral</u>, i.e. ω represents an element of $H^2(X;\mathbb{Z})$.

(ii) If (L,A) is one choice then any other differs from it by tensoring L with a flat hermitian line bundle. These correspond to elements of $H^1(X;\mathbb{T})$.

(iii) For integral X there is an exact sequence of Lie groups corresponding to the sequence (1.1) of Lie algebras

$$1 \longrightarrow H^0(X;\mathbb{T}) \longrightarrow \mathrm{Aut}(X;L) \longrightarrow \mathrm{Diff}_\omega(X) \longrightarrow H^1(X;\mathbb{T}) \longrightarrow 1.$$

Here $\mathrm{Aut}(X,L)$ means the pairs $(f,\tilde{f})$, with $f \in \mathrm{Diff}_\omega(X)$ and $\tilde{f}: L \to L$ a bundle map which covers f and preserves the connection.

Lecture 2

Examples of deformation quantization of a Poisson manifold X

(i) $X = \mathfrak{g}^*$ gives $A_h = \mathcal{U}(\mathfrak{g}_h)$ as above.

(ii) If X is a symplectic vector space V of dim $2n$ the Weyl algebra $A_h(V)$ is defined as the algebra over $\mathbb{C}$ generated by the real vector space V^* subject to the relations $[a,b] = i\hbar\omega^{-1}(a,b)$, where ω^{-1} is the skew form on V^* corresponding to ω on V.

Remark $A_h(V) \cong \mathcal{U}(\mathfrak{g})$ where $\mathfrak{g}$ is the Heisenberg Lie algebra fitting into an exact sequence $0 \to \mathbb{R} \to \mathfrak{g} \to V^* \to 0$.

If we choose symplectic coordinates (x_i, p_i) for V then $A_h(V)$ is generated by x_i and p_i subject to $[p_k, x_m] = i h \delta_{km}$. Now $[ih\frac{\partial}{\partial x_k}, x_m] = ih\delta_{km}$, so $A_h(V) \cong \{$differential operators on $\mathbb{R}^n$ with polyn. coeffs$\}$

More invariantly,

Proposition For every symplectic isomorphism $V \cong M \oplus M^* = T^*M$ we have a canonical isomorphism

$$A_h(V) \cong \{\text{polyn. diff. ops on } M\}.$$

In particular, the RHS depends only on the symplectic vector space T^*M.

Remarks (a) Notice that making h small amounts to making ω big, i.e. the classical limit is the limit of large symplectic manifolds.

(b) In ex.(i) and (ii) we have considered only polynomial functions on X. But if we work with formal power series in h this still gives us a deformation quantization of $C^\infty(X)$, as $m_h(f,g)(x)$ depends only on the jets of f and g at x. The same applies in the next example.

(iii) If $X = T^*M$ for any manifold M we have $A_h(X) = \{$differential operators on $M\}$. For if we associate to a function f on

T^*M which is polynomial along the fibres the differential operator $P_f = f(x, ih\frac{\partial}{\partial x})$ then $[P_f, P_g] = ih\, P_{\{f,g\}} \pmod{h^2}$. (Notice that different ordering conventions for defining $f(x, ih\frac{\partial}{\partial x})$ do not change P_f mod h.)

Geometric quantization

The particle physics background, and Kirillov's ~~[illegible]~~ doctrine have been mentioned. Recall that given a symplectic manifold (X,ω) we want to define a Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ acts, where $\mathcal{A}$ = " $\mathcal{A}_h$ when $h = 1$ ". Observe that $\mathcal{A}$ will be canonically defined if the formal power series defining $\mathcal{A}_h$ are analytic functions of h.

If $X = S^2$ with the area form of area m then ~~$\mathcal{A}$~~

$$\mathcal{A} = \mathbb{C}[X,Y,Z]/(\rho) = \mathcal{U}(\mathfrak{g})/\rho,$$

where $[X,Y] = iZ$, etc, $\rho = X^2+Y^2+Z^2 - m(m+2)$, and $\mathfrak{g} = so_3$.
We want $\mathcal{A}$ to act on $\mathcal{H}$ by a $*$-representation, i.e. X, Y, Z should be self-adjoint. This means that $\mathcal{H} = 0$ unless m is an integer, in which case $\mathcal{H}$ is the $(m+1)$-dim^l repn of $\widetilde{SO}_3$ = (double covering of SO_3).

In particular, the group $\mathrm{Diff}_\omega(S^2)$ = (area-preserving diffeomorphisms of S^2) cannot possibly act on $\mathcal{H}$, but only $\widetilde{SO}_3$. Thus there can be no functor (symplectic manifolds) $\longrightarrow$ (Hilbert spaces); at best we can hope to define

(symplectic manifolds with extra structure) $\longrightarrow$ (Hilbert spaces),

~~[illegible]~~

The first piece of extra structure we undoubtedly need is a line bundle L with a connection A, as described above.

Group actions An action $G \to \mathrm{Diff}(X)$ of a group G on a Poisson manifold X is called Hamiltonian if the Lie algebra action

$\mathfrak{g} \to \mathrm{Vect}_\omega(X)$ is lifted to $\mathfrak{g} \to C^\infty(X)$. This is certainly the case if (X,ω) comes from (X,L) and G acts on (X,L). (For then the function generating ξ is $i_\xi A$, where ξ and A are thought of on the principal bundle of L.)

In this situation the map $\mathfrak{g} \to C^\infty(X)$ is usually regarded as a map $p : X \to \mathfrak{g}^*$, called the <u>moment map</u> of the action. By its construction it is G-equivariant, and its derivative $T_x X \to \mathfrak{g}^*$ at x is the transpose of the map $\mathfrak{g} \to T_x X$ defining the G-action. (Here $T_x X$ is identified with $T_x^* X$ by ω.)

The following gives a complete description of homogeneous symplectic mfds.

<u>Proposition</u> If G acts transitively on X then $p : X \to \mathfrak{g}^*$ makes X a covering space of a G-orbit in $\mathfrak{g}^*$. (And every such orbit is symplectic)

<u>Proof</u> The derivative of p is injective at each point, for it is the transpose of $\mathfrak{g} \to T_x X$, which is surjective for a transitive action. But $X \to \mathfrak{g}^*$ is a G-map, so takes an orbit to an orbit.

To see that each orbit <u>is</u> symplectic:

$\omega(\xi\alpha, \eta\alpha) = \alpha([\xi,\eta])$ (for $\alpha \in \mathfrak{g}^*$, and $\xi,\eta \in \mathfrak{g}$) by definition, and so

$$\begin{aligned}\omega(\xi\alpha, \eta\alpha) = 0 \text{ for all } \eta \in \mathfrak{g} &\iff \alpha([\xi,\eta]) = 0 \text{ for all } \eta \\ &\iff (\xi\alpha)(\eta) = 0 \quad \text{- - - -} \\ &\iff \xi\alpha = 0 .\end{aligned}$$ //

Kirillov's doctrine can now be reformulated as a relation between irred. unitary repns of G and "coadjoint orbits" in $\mathfrak{g}^*$. It can be understood to some extent from the viewpoint of deformation quantization.

Let $\mathfrak{Z}(\mathfrak{g})$ be the centre of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. In any irreducible repn of G the centre $\mathfrak{Z}(\mathfrak{g})$ acts by a ring-homomorphism $\chi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ called its <u>infinitesimal character</u>. But we have

<u>Theorem</u> $\mathfrak{Z}(\mathfrak{g})$ is canonically isomorphic to $S(\mathfrak{g})^G$, the algebra of G-invariant polynomial functions on $\mathfrak{g}^*$.

This theorem is highly non-trivial. It was proved by Harish-Chandra for semisimple groups. Then Kirillov defined a map in general, and conjectured it to be an isomorphism. This was proved by Duflo case by case. Kontsevich's recent work gives a non-case-by-case proof. Kirillov's map is easy to define: the exponential map $\exp : \mathfrak{g} \to G$ induces a map $\exp^* : \Omega^{(1/2)}(G) \to \Omega^{(1/2)}(\mathfrak{g})$ on the spaces of $\frac{1}{2}$-densities. Now $\mathfrak{Z}(\mathfrak{g})$ acts on $\Omega^{(1/2)}(G)$, and $S(\mathfrak{g})$ acts by constant coefficient differential operators on $\Omega^{1/2}(\mathfrak{g})$. Kirillov conjectured that the respective actions of $\mathfrak{Z}(\mathfrak{g})$ and $S(\mathfrak{g})^G$ correspond under $\exp^*$. This assigns an orbit in $\mathfrak{g}^*$ to each irreducible repn of G precisely to the extent that orbits are separated by invariant polynomials on $\mathfrak{g}^*$, i.e. to the extent that ring homomorphisms $S(\mathfrak{g})^G \to \mathbb{C}$ correspond to orbits.

Linear geometric quantization

For each way of writing a symplectic vector space V as $M \oplus M^*$ we can identify the Weyl algebra $\mathcal{A}(V)$ with the polynomial differential operators on M, and so we can define an action of $\mathcal{A}(V)$ on the Hilbert space $L^2(M)$. But of course $L^2(M)$ is not naturally associated to V.

The Stone-von Neumann Theorem on the uniqueness of the representation of the "canonical commutation relations" tells us that if $\tilde{M} \oplus \tilde{M}^*$ is another decomposition then the spaces $L^2(M)$ and $L^2(\tilde{M})$ are uniquely isomorphic up to a scalar multiplication — i.e. the <u>projective</u> space $\mathbb{P}(L^2(M))$ is functorially associated to V, though not obviously so.

We should like to do better still. If we had a functor $V \mapsto \mathcal{H}_V$ then the group $Sp(V)$ would act unitarily on $\mathcal{H}_V$. This is impossible. The optimal result is that a double covering $Mp(V)$ of $Sp(V)$, called the <u>metaplectic group</u> can be made to act on $\mathcal{H}_V$. Thus there is a functor

(symplectic vector spaces with extra structure) $\to$ (Hilbert spaces)

which we want to understand fully. (The functor is called the <u>metaplectic representation</u>.)

Lecture 3

Linear quantization

We associated the Hilbert space $\mathcal{H} = L^2(M)$ to a symplectic vector space (V,ω) by choosing a decomposition $V = M \oplus M^*$. The Stone–von Neumann theorem shows that $\mathbb{P}(\mathcal{H})$ is naturally associated to (V,ω).

<u>Sketch of proof</u> Because $\mathcal{A}(V) \cong \mathcal{U}(\mathfrak{g})$ the Hilbert space $\mathcal{H}$ is a unitary representation of the Heisenberg group G associated to V whose Lie algebra is $\mathfrak{g}$. This group is a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow V \longrightarrow 1,$$

and the central $\mathbb{T}$ acts by scalar multiplication on $\mathcal{H}$. There is only one irreducible representation of this type. For, as ω vanishes on M and M^*, we can regard M and M^* as abelian subgroups of G. Decomposing $\mathcal{H}$ under the action of M^* expresses $\mathcal{H}$ as a direct integral $\int_M \mathcal{H}_m \, dm$, where M^* acts on $\mathcal{H}_m$ by e^{im} for $m \in (M^*)^* = M$. But conjugation by $\mu \in M \subset G$ changes e^{im} to $e^{i(m+\mu)}$, so μ defines an isomorphism $\mathcal{H}_m \to \mathcal{H}_{m+\mu}$, and $\mathcal{H}$ can be identified as a representation of G with $L^2(M; \mathcal{H}_0)$. But we must have $\dim(\mathcal{H}_0) = 1$ for $\mathcal{H}$ to be irreducible. ///

To go further, we introduce the space $\mathcal{J}(V)$ of complex isotropic subspaces W of $V_{\mathbb{C}}$ which are <u>positive</u> in the sense that $i\omega(\bar{w}, w) > 0$ if $w \neq 0$. For any such W we have $V_{\mathbb{C}} = W \oplus \overline{W}$, and

$$\langle w_1, w_2 \rangle = i\omega(\bar{w}_1, w_2)$$

is a Hermitian inner product.

<u>Proposition (3.1)</u> Points of $\mathcal{J}(V)$ correspond precisely to positive complex structures on V compatible with ω, i.e. to linear maps $J : V \to V$

such that (i) $J^2 = -1$,
(ii) $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$,
(iii) $\omega(v, Jv) > 0$ if $v \neq 0$.

The correspondence is : $J \mapsto W = \{w \in V_{\mathbb{C}} : Jw = iw\}$. //

$\mathcal{J}(V)$ is a complex manifold with two important descriptions.

(a) If we choose $V = M \oplus M^*$ then $\mathcal{J}(V) = \{$symmetric $\mathbb{C}$-linear $A : M \to M^*$ such that $\mathrm{Im}(A)$ is positive definite$\}$. This is the "Siegel generalized upper half-plane". The correspondence is $A \mapsto \mathrm{graph}(A) \subset M_{\mathbb{C}} \oplus M^*_{\mathbb{C}}$.

(b) If we choose $V_{\mathbb{C}} = W \oplus \bar{W}$ then

$$\mathcal{J}(V) = \{\text{symmetric } A : W \to W^* \text{ such that } \|A\| < 1\}.$$

This is the "Siegel generalized unit disc", which describes $\mathcal{J}(V)$ as a bounded open subset of $\mathbb{C}^{\frac{1}{2}n(n+1)}$, where $\dim(V) = 2n$.

The real Lagrangian (i.e. max. isotropic) subspaces $M \subset V$ form a compact manifold $\mathrm{Lag}(V)$ of real dimension $\frac{1}{2}n(n+1)$ contained in the boundary of $\mathcal{J}(V)$ — in fact $\mathrm{Lag}(V)$ is the Shilov boundary of $\mathcal{J}(V)$. The group $Sp(V)$ acts transitively on both $\mathcal{J}(V)$ and $\mathrm{Lag}(V)$, and

$$\mathcal{J}(V) \cong Sp(V)/U(W), \qquad \mathrm{Lag}(V) \cong Sp(V)/\mathrm{stab}(M)$$
$$\cong U(W)/U(W) \cap \mathrm{stab}(M) \cong U_n/O_n.$$

The functoriality of $\mathcal{H}$ is now described by

Proposition (3.2) For each $W \in \mathcal{J}(V)$ the subspace

$$\mathcal{H}^W = \{\xi \in \mathcal{H} : w\xi = 0 \text{ for all } w \in W\}$$

is 1-dimensional. (Here W is regarded as contained in $\mathcal{A}(V)$.) The lines $\mathcal{H}^W$ form a hermitian holomorphic (= h.h.) line bundle $\Lambda_{\mathcal{H}}$ on $\mathcal{J}(V)$ such that $\Lambda_{\mathcal{H}} \otimes \Lambda_{\mathcal{H}} \cong \mathrm{Det}^*$ as h.h. line bundles, where Det^* is the tautological bundle with fibre $\Lambda^n W^*$.

(The proof is given below.)

Remarks (i) If two h.h. line bundles are isomorphic, the isomorphism is unique up to multiplication by a constant element of $\mathbb{T}$, by the maximum modulus principle.

(ii) Any two h.h. line bundles Λ_1, Λ_2 on $\mathcal{J}(V)$ such that $\Lambda_i^{\otimes 2} \cong \mathrm{Det}^*$ are isomorphic, for $\Lambda_1^* \otimes \Lambda_2$ is a <u>flat</u> unitary line bundle on the contractible space $\mathcal{J}(V)$.

<u>Corollary (3.5)</u>

$\{$ choices of representation $\mathcal{H}$ $\} \longrightarrow \{$h.h. line bundles Λ such that $\Lambda^{\otimes 2} \cong \mathrm{Det}^*\}$ is an equivalence of categories.

Equivalently, $\mathcal{H}$ can be regarded as a functor of (V, Λ). In fact

<u>Proposition (3.4)</u> For any $W \in \mathcal{J}(V)$, $\mathcal{H}$ is <u>canonically</u> isomorphic to the Hilbert space completion of $S(W^*) \otimes \Lambda_{\mathcal{H},W}$, where $\Lambda_{\mathcal{H},W}$ is the fibre of $\Lambda_{\mathcal{H}}$ at W — i.e. $\Lambda_{\mathcal{H},W}$ is a "choice" of $\Lambda^n(W^*)^{\otimes \frac{1}{2}}$.

For a given choice of Λ, the group $\mathrm{Aut}(V, L)$ is a double covering of $Sp(V)$ called the <u>metaplectic group</u>. It is a non-trivial double covering, for (3.4) implies that the subgroup which preserves W is

$$\tilde{U}(W) = \{(u, \lambda) : u \in U(W),\ \lambda^2 = \det(u)\}.$$

<u>Proof of (3.2)</u> It is enough to consider the particular choice $\mathcal{H} = L^2(M)$, for any other is $\mathcal{H} \otimes (\text{line})$. If $W \in \mathcal{J}(V)$ is given by $A : M_{\mathbb{C}} \to M_{\mathbb{C}}^*$ with $\mathrm{Im}(A) > 0$ we find at once that $\mathcal{H}^W$ consists of all multiples of $\varphi_A = e^{\frac{1}{2}i\langle x, Ax\rangle}$. Thus $\Lambda_{\mathcal{H}}$ comes equipped with a holomorphic section $\{\varphi_A\}$ whose $(\text{norm})^2$ is easily calculated to be $\det(\frac{1}{2\pi}\,\mathrm{Im}\,A)^{-1/2}$. On the other hand the choice $V = M \oplus M^*$ defines a holomorphic section $A \mapsto S_A = \Lambda^n\binom{1}{A}$ of Det on $\mathcal{J}(V)$, and $\|S_A\|^2 = \det(\mathrm{Im}(A))$. //

Finally, the choice of a line bundle Λ on $\mathcal{J}(V)$ induces a <u>real</u> line bundle Λ on $\mathrm{Lag}(V)$ such that $\Lambda^{\otimes 2} = \mathrm{Det}^*$, and we have

<u>Proposition (3.5)</u> $\mathcal{H}_{V,\Lambda}$ is <u>canonically</u> isomorphic to $L^2(M)\otimes\Lambda_M$, where $L^2(M)$ denotes the Hilbert space of $\frac{1}{2}$-densities on M.

<u>Remark</u> On $\mathrm{Lag}(V)$ the relation $\Lambda^{\otimes 2} \cong \mathrm{Det}^*$ does not fix the isomorphism class of Λ, for $\pi_1 \mathrm{Lag}(V) = \mathbb{Z}$. But the extendability to $\mathcal{J}(V)$ picks out a class.

<u>Affine-linear quantization</u>

Suppose now that V is given only as an <u>affine</u> symplectic space, with underlying vector space V_0. Let $\mathcal{J}_{aff}(V)$ denote the affine positive Lagrangian subspaces $W \subset V_{\mathbb{C}}$. (Each such W meets V in a unique point.) The complex manifold $\mathcal{J}_{aff}(V)$ is a bundle of complex affine spaces over $\mathcal{J}(V_0)$ whose fibre V_J at J is V with the complex structure J. We readily check

<u>Proposition (3.6)</u> If $\mathcal{H}$ is a quantization of V then

$$\{\mathcal{H}^W\} \cong (\mathrm{Det}^*)^{\otimes \frac{1}{2}} \otimes L$$

as h.h. line bundles on $\mathcal{J}_{aff}(V)$, where Det^* is the line bundle pulled back from $\mathcal{J}(V_0)$, and $L|V_J$ is obtained from a line bundle L on V with a unitary connection with curvature ω, by giving it the holomorphic structure J.

Thus we can associate a Hilbert space $\mathcal{H}$ functorially to a symplectic affine space V equipped with L and $\Lambda = (\mathrm{Det}^*)^{\otimes \frac{1}{2}}$, and for any $J \in \mathcal{J}(V_0)$ we have a <u>canonical</u> isomorphism

$$\mathcal{H} \cong \Gamma_{L^2\,\text{holomorphic}}(V_J;\ L_J \otimes \Lambda_J).$$

<u>The enlargement of $\mathcal{H}$</u>

As $W \in \mathcal{J}(V)$ tends to a boundary point $M \in \mathrm{Lag}(V)$ the ray $\mathcal{H}^W$ does not have a limit in $\mathbb{P}(\mathcal{H})$. (For example, if $V = \mathbb{R}^2$ then $a^{\frac{1}{4}}e^{\frac{1}{2}iax^2}$ is not in L^2 for real a, and tends to the delta-function at the origin if $a \to i\infty$.) But, without spoiling the functoriality, we can enlarge $\mathcal{H}$ to a complete topological vector space $\hat{\mathcal{H}}$ which does contain a unique ray for each real Lagrangian subspace M.

To do this, we observe that the action of $\widetilde{Sp}(V)$ on $\mathcal{H}$ does not extend to an action of the complexification $\widetilde{Sp}(V)_{\mathbb{C}} = \widetilde{Sp(V_{\mathbb{C}})}$, but it does extend holomorphically to a semigroup $\widetilde{Sp}^{+}(V)_{\mathbb{C}} \subset \widetilde{Sp}(V)_{\mathbb{C}}$. This semigroup can be defined as the elements of $Sp(V_{\mathbb{C}})$ which contract the open subset $\mathfrak{J}(V)$ of $\mathrm{Lag}(V_{\mathbb{C}})$ inside itself, or else as the elements whose graphs are positive Lagrangian subspaces of $(\check{V} \oplus V)_{\mathbb{C}}$, where $\check{V}$ is V with the reversed symplectic form. The second description shows that each element defines a ray in $\overline{\mathcal{H}} \otimes \mathcal{H} = \{\text{Hilbert-Schmidt operators } \mathcal{H} \to \mathcal{H}\}$.

If g belongs to this semigroup, the image of $g : L^2(M) \to L^2(M)$ consists of real-analytic $\frac{1}{2}$-densities, and g is very strongly contracting. We can now define dual topological vector spaces $\check{\mathcal{H}}$ and $\hat{\mathcal{H}}$ such that

$$\check{\mathcal{H}} \subset \mathcal{H} \subset \hat{\mathcal{H}},$$

and such that $\hat{\mathcal{H}}$ contains a ray for each $M \in \mathrm{Lag}(V)$: we take $\check{\mathcal{H}} = \bigcup g\mathcal{H}$, and $\hat{\mathcal{H}}$ = closure of the image of $\prod: \mathcal{H} \to \prod_g \mathcal{H}$, where g runs through the semigroup.

Lecture 4

Quantization in non-linear situations

To get some feeling for the problem of extending geometric quantization beyond the linear situation, let us ask whether one can assign a ray in the quantization $\hat{\mathcal{H}}$ of a symplectic vector space V to a general Lagrangian submanifold Σ of V. If $V = M \oplus M^*$ is a linear decomposition then for any smooth $F : M \to \mathbb{R}$ the graph Σ_F of $dF : M \to M^*$ is Lagrangian, and any Lagrangian submanifold which projects isomorphically on to M is of this form. Introducing canonical coordinates $(x_1, \ldots, x_n ; p_1, \ldots, p_n)$, the manifold Σ_F is the set of zeros of the functions $p_k - \frac{\partial F}{\partial x_k}$, and the corresponding operators

$$p_k - \frac{\partial F}{\partial x_k} = i\hbar \frac{\partial}{\partial x_k} - \frac{\partial F}{\partial x_k}$$

annihilate the unique ray in $\hat{\mathcal{H}}$ consisting of the multiples of $e^{-iF/\hbar}$. On the other hand, $\mathcal{H}$ can also be described as $L^2(M^*)$, where the isomorphism $L^2(M) \to L^2(M^*)$ is the Fourier transform $\varphi \mapsto \hat{\varphi}$, where

$$\hat{\varphi}(p) = (2\pi)^{-n/2} \int \varphi(x)\, e^{i\langle p, x\rangle/\hbar}\, dx .$$

If Σ_F projects isomorphically to M^* it is the graph of $dG : M^* \to M$, where $G : M^* \to \mathbb{R}$ is the <u>Legendre transform</u> of F, i.e.

$$G(p) = \langle p, x\rangle - F(x), \quad \text{where } x \text{ satisfies } p = dF(x).$$

Now x_k acts on $L^2(M^*)$ as $-i\hbar \frac{\partial}{\partial p_k}$, so we expect Σ_F to correspond to the ray containing $e^{iG/\hbar}$ in $L^2(M^*)$. But $e^{iG/\hbar}$ is not the Fourier transform of $e^{-iF/\hbar}$. In fact the Legendre transform is the <u>classical limit</u> of the Fourier transform, in the sense that the Fourier transform of $e^{-iF/\hbar}$ is

$$(2\pi)^{-n/2} \int e^{i(\langle p, x\rangle - F(x))/\hbar}\, dx \quad = \quad e^{iG(p)/\hbar}\{1 + O(\hbar)\}.$$

(To see this we calculate the integral by stationary phase.) Thus the two candidates for the ray associated to Σ_p tend to coincidence as $h \to 0$.

<u>Quantization of cotangent bundles</u>

There is a well-developed theory of quantization for cotangent bundles. It is — essentially — what is called "micro-local analysis".

I shall consider here only the cotangent bundles T^*M of <u>compact</u> manifolds M.

Let $X = T^*M - \{\text{zero section}\}$. This should have the same quantization as T^*M, as it has the same Liouville volume. It has a free action of the multiplicative group $\mathbb{R}_+^\times$, which scales the fibres, and satisfies $t^*\omega = t\omega$.

I have already said that the algebra $\mathcal{A}_0(X)$ of smooth functions on X which are polynomial on the fibres of T^*M has the deformation quantization $\mathcal{A}_h(X)$, where $\mathcal{A}_h(X)$ is the algebra of differential operators acting on the space $\mathcal{H} = L^2(M)$ of $\frac{1}{2}$-densities on M. (Recall that in a standard chart (x_i, p_i) the function $p_k \in \mathcal{A}_h(X)$ acts by $ih\frac{\partial}{\partial x_k}$.) This picture makes strong use of the linear structure of T^*M. But we can consider the larger algebra $\hat{\mathcal{A}}_0(X)$ of smooth functions f on X which have asymptotic expansions $f \sim \sum_{k \leq m} f_k$, where f_k is homogeneous of degree k under the $\mathbb{R}_+^\times$-action — i.e. $t^*f_k = t^k f_k$. This algebra does not depend on the linear structure, but only the $\mathbb{R}_+^\times$-action, and it can be deformed to the algebra $\hat{\mathcal{A}}_h(X)$ of <u>pseudo-differential operators</u> on $L^2(M)$.

To define a pseudodifferential operator we begin by observing that a differential operator $f \mapsto Pf$ is characterized by the fact that for each $x \in M$ the map $f \mapsto Pf(x)$ is a distribution with support $\{x\}$.

Lemma (4.1) If V is a real vector space, and $\mathcal{D}ist_0(V)$ denotes the distributions with support $\{0\}$, then

$$\mathcal{D}ist_0(V) \cong S^{\cdot}(V) \cong \text{Polynomials on } V^*. \qquad //$$

(The isomorphism is given by the Fourier transform.)

Let us define a larger class of distributions $\mathcal{D}ist_0^{\psi}(V)$ with singular support $\{0\}$ — i.e. which are smooth in $V - \{0\}$. This class consists of all α whose Fourier transforms $\hat{\alpha}$ have asymptotic expansions

$$\hat{\alpha} \sim \sum_{k \leq m} \hat{\alpha}_k$$

with α_k homogeneous of degree k. The remarkable theorem of Hörmander is that $\mathcal{D}ist_0^{\psi}(V)$ depends only on V as a smooth manifold with the base-point 0. In other words, we can define $\mathcal{D}ist_x^{\psi}(M)$ for any smooth manifold M and any $x \in M$.

Definition (4.2) A pseudodifferential operator $f \mapsto Pf$ is one such that the distribution $P_x : f \mapsto (Pf)(x)$ belongs to $\mathcal{D}ist_x^{\psi}(M)$ for each $x \in M$, and depends smoothly on x.

If $\hat{P}_x \sim \sum_{k \leq m} \hat{P}_{x,k}$ then P has degree m, and the homogeneous function $\hat{P}_{x,m}$ on $X = T^*M - \{0\}$ is called its principal symbol.

These concepts can be generalized. If Σ is a submanifold of M, then a conormal distribution along Σ is one whose singular support is Σ and which, in a tubular neighbourhood of Σ looks like a smooth family $\alpha_x \in \mathrm{Dist}_x^{\psi}(N_x\Sigma)$, where $N_x\Sigma$ is the normal slice to Σ at x. Again, it can be shown that this class of distributions depends only on Σ. Each member has a degree, and a principal symbol which is a homogeneous function on $N^*\Sigma$. The obvious conormal distribution of degree 0 on Σ with principal symbol 1

is the δ-function δ_Σ along Σ, but there are many others.

For any submanifold Σ of M the conormal bundle N^*_Σ is a conical (i.e. invariant under the $\mathbb{R}^*_+$-action) Lagrangian submanifold of T^*M. Conversely, if Γ is a conical Lagrangian submanifold of T^*M then in any open set where the projection $\Gamma \to M$ has constant rank Γ is of the form N^*_Σ. With a little more care, we can define a class of <u>Lagrangian distributions</u> on M for each conical Lagrangian submanifold Γ of X, so that when $\Gamma = N^*_\Sigma$ we get the conormal distributions above. Any Lagrangian distribution for Λ with degree 0 and principal symbol 1 defines a candidate for the ray in $\hat{\mathcal{H}} = L^2(M)^\wedge$ associated to Γ.

The theory of Lagrangian distributions reveals the functorial properties of the quantization of cotangent bundles. Let $\mathrm{Diff}_{hs}(X)$ denote the homogeneous symplectic isomorphisms $f : X \to X$, i.e. those which commute with the $\mathbb{R}^\times_+$-action. The graph Γ_f of such an f is a conical Lagrangian submanifold of $\tilde{X} \times X$. Let $\mathcal{F}_f$ denote the distributions on $M \times M$ which are Lagrangian with respect to Γ_f, and of degree 0 with principal symbol 1. These distributions define operators $L^2(M) \to L^2(M)$ which are called <u>Fourier integral operators</u>. As f varies they form a group $\mathcal{F}(X)$ which fits into an exact sequence

$$1 \to \mathcal{G}(X) \to \mathcal{F}(X) \to \mathrm{Diff}_{hs}(X) \to 1, \qquad (14.3)$$

where $\mathcal{G}(X)$ denotes the invertible pseudodifferential operators of order 0 with principal symbol 1. (For a pseudodifferential operator is one whose kernel is conormal with respect to the diagonal in $M \times M$.) The sequence (14.3) expresses the precise functoriality of the quantization $X \mapsto \mathcal{H} = L^2(M)$: $\mathcal{H}$ is associated to X only up to the action of an element of the group $\mathcal{G}(X)$.

Nevertheless, this gives us quite a lot of information, as the action of $\mathcal{G}(X)$ preserves the singular support (and even the wave-front set) of a distribution in M. A beautiful example of this is the following, discussed by Guillemin in Duke Math. J. $\underline{44}$ (1977).

For a Riemannian manifold M, the <u>geodesic flow</u> on X is generated by the Hamiltonian function $\frac{1}{2}\|p\|^2$, and corresponds to the 1-parameter unitary group $e^{it\Delta}$ on $\mathcal{H} = L^2(M)$, where Δ is the Laplacian. Although this flow does not belong to $\mathrm{Diff}_{hs}(X)$ the rescaled flow $\{f_t\}$ generated by $\|p\|$ <u>does</u> commute with $\mathbb{R}_+^\times$. It corresponds to $e^{it\Delta^{1/2}}$ on $\mathcal{H}$. We conclude that the singular support of the kernel of $e^{it\Delta^{1/2}}$ is (the projection of) the graph of f_t to $M\times M$, and hence that the singular support of $\mathrm{Trace}\,(e^{it\Delta^{1/2}})$, as a distribution in t, is the set of points t such that the graph of f_t meets the diagonal in $X\times X$, i.e. is the set of periods of <u>periodic geodesics</u> in M.

<u>Quantization of compact symplectic manifolds</u>

Suppose that (X, L) is a compact symplectic manifold with a hermitian line bundle with a connection whose curvature is the symplectic form ω of X. For each $x \in X$ we have the complex manifold $\mathcal{J}(T_xX)$ described in Lecture 3. These manifolds fit together to form a bundle $\mathcal{J}(TX)$ on X with contractible fibres. A section of this bundle is an almost-complex structure on X compatible with ω. (It defines in particular a Riemannian metric on X.) On the total space $\mathcal{J}(TX)$ there is a tautological complex line bundle Det^* whose first Chern class $c_1 \in H^2(\mathcal{J}(TX); \mathbb{Z}) = H^2(X; \mathbb{Z})$ is the first Chern class of X, and is a lift of the Stiefel-Whitney class $w_2 \in H^2(X; \mathbb{Z}/2)$.

If c_1 is divisible by two (i.e. if $w_2 = 0$), we can find a line bundle Λ on $\mathcal{J}(TX)$ with an isomorphism $\Lambda^{\otimes 2} \cong \mathrm{Det}^*$. A choice of Λ is called a <u>metaplectic structure</u> on X.

A section σ of $\mathcal{J}(TX)$ which is defined by an actual complex structure on X will be called a <u>compatible Kähler structure</u> on X. It automatically makes the line bundles L and $\sigma^*\Lambda$ on X into holomorphic bundles. Comparison with the linear case makes it natural to define a "quantization"

$$\mathcal{H}_\sigma = \Gamma_{hol}(X; L \otimes \Lambda),$$

where Λ denotes $\sigma^*\Lambda$, so that $\Lambda^{\otimes 2}$ is the canonical bundle of X. Probably this is only a good definition when $L \otimes \Lambda^*$ is a <u>positive</u> line bundle on X (i.e. its curvature is a Kähler form), for then the Kodaira vanishing theorem tells us that the cohomology groups $H^i(X; L \otimes \Lambda)$ vanish for $i > 0$. As L is a positive bundle by definition, we certainly know that $L^{\otimes k} \otimes \Lambda^*$ is positive for all sufficiently large k.

When the vanishing theorem applies the dimension of $\mathcal{H}_\sigma$ can be calculated by the Riemann-Roch theorem, and is

$$\dim(\mathcal{H}_\sigma) = \int_X e^{\omega} \hat{A}_X .$$

Here the $\hat{A}$-genus $\hat{A}_X$ of X is the rational cohomology class of mixed dimension defined in terms of the Pontrjagin classes $p_i(X)$ by

$$\hat{A}_X = \prod \frac{x_i/2}{\sinh(x_i/2)},$$

where $\sum p_i(X) = \prod (1 - x_j)^2$. Because $\hat{A}_X = 1 + (\text{higher terms})$, if we apply this formula to (X, kL) we find that the dimension of $\mathcal{H}_\sigma$ is a polynomial in k of degree $n = \frac{1}{2}\dim X$, whose leading term is

$$k^n \int_X \frac{\omega^n}{n!},$$

i.e. the Liouville volume of (X, kL). As usual, the "classical" volume is the leading term.

As the complex structure σ varies, the spaces $\mathcal{H}_\sigma$ form a vector bundle on the parameter space, providing the higher cohomology always vanishes. One might ask whether there is a natural connection in this bundle which would allow one to identify the different $\mathcal{H}_\sigma$, perhaps only as projective spaces. But that is impossible.

<u>Example</u>

Let $\mathcal{C}(S^2)$ denote the space of complex structures on the sphere S^2. In dimension 2 all complex structures are compatible with the symplectic structure, providing they define the same orientation. Up to diffeomorphism, there is only one complex structure on S^2, so

$$\mathcal{C}(S^2) \cong \mathrm{Diff}(S^2; \mathbb{P}^1_{\mathbb{C}})/\mathrm{PSL}_2\mathbb{C}.$$

If $\int_{S^2}\omega = k$ then $\mathcal{H}_\sigma$ is the k-dimensional representation of the group SO_3 of diffeomorphisms of S^2 which preserve the Kähler structure, and the bundle $\{\mathcal{H}_\sigma\}$ on the contractible space $\mathcal{C}(S^2)$ is

$$\mathrm{Diff}(S^2; \mathbb{P}^1_{\mathbb{C}}) \times_{\mathrm{PSL}_2\mathbb{C}} \mathbb{C}^k.$$

This is a trivial bundle: a trivialization corresponds to a map $\mathcal{C}(S^2) \longrightarrow \mathrm{Diff}(S^2; \mathbb{P}^1_{\mathbb{C}})$, and can be defined, for example, by requiring three chosen points of S^2 to map to $\{0, 1, \infty\}$ in $\mathbb{P}^1_{\mathbb{C}}$. The relevant question, however, is whether the bundle $\{\mathcal{H}_\sigma\}$ — or $\{\mathbb{P}\mathcal{H}_\sigma\}$ — is <u>naturally</u> trivial, i.e. whether the trivialization is compatible with the action of the group $\mathcal{G}$ of symplectic diffeomorphisms of S^2. If so, then $\mathcal{G}$ would act projectively on the k-dimensional space of flat sections, extending the obvious action of SO_3. That is impossible. For the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ is $C^\infty(S^2)/\mathbb{R}$, which is the sum of all irreducible representations of SO_3, each with multiplicity 1, and so there is only one possible SO_3-invariant map $\mathfrak{g} \longrightarrow \mathrm{Aut}(\mathcal{H}_\sigma)$, and it is not a homomorphism of Lie algebras.

There is one important observation to make about the spaces $\mathcal{H}_\sigma$. When the vanishing theorem applies, $\mathcal{H}_\sigma$ is the cohomology of the Dolbeault complex

$$\Omega^0(X; L\otimes\Lambda) \xrightarrow{\bar\partial} \Omega^{0,1}(X; L\otimes\Lambda) \xrightarrow{\bar\partial} \Omega^{0,2}(X; L\otimes\Lambda) \longrightarrow \cdots .$$

Equivalently, $\mathcal{H}_\sigma$ is the kernel of the Fredholm operator

$$\bar\partial + \bar\partial^* : \bigoplus_{i \text{ even}} \Omega^{0,i}(X; L\otimes\Lambda) \longrightarrow \bigoplus_{i \text{ odd}} \Omega^{0,i}(X; L\otimes\Lambda). \quad (14\cdot4)$$

Now, for a hermitian vector space W of dimension n, the space

$$(\Lambda^n W^*)^{\otimes \frac{1}{2}} \otimes \bigoplus \Lambda^i W^*,$$

as a representation of the double covering of the unitary group $U(W)$, is precisely the spin representation of the orthogonal group of the real vector space underlying W. So the source and target of (14·4) are the sections of $L\otimes\Delta^+$ and $L\otimes\Delta^-$, where $\Delta^\pm$ are the bundles of $\frac{1}{2}$-spinors on the Kähler manifold X_σ. Furthermore, it is well known that $\bar\partial+\bar\partial^*$ is the Dirac operator of X_σ, coupled to the line bundle L. This means that $\mathcal{H}_\sigma$ is defined for any almost complex structure σ, i.e. any section σ of $\mathcal{J}(TX)$, and not only for sections coming from complex structures.

We cannot assume that the Dirac operator is surjective for all σ, so its index is only a virtual vector space, but we do have a canonical family of Fredholm operators whose index is the "quantization" of X, parametrized by the contractible space $\Gamma(\mathcal{J}(TX))$.

The Fredholm operators represent the class $[L].[X]$ in K-theory, where $[X] \in K_{2n}(X)$ is the fundamental class of X defined by the K-orientation. Thus choosing a K-orientation defines a "virtual quantization" which is the image of $[L]$ under the associated Gysin map $K^0(X) \longrightarrow K^{-2n}(pt) \cong K^0(pt)$. This is essentially the same kind of functoriality that arose in the quantization of cotangent bundles, for the ambiguity in the choice of the K-theory fundamental class (i.e. of the Dirac operator, whose principal symbol is canonical up to canonical homotopy) is closely analogous to the choice of a pseudo-differential operator on X of degree 0 and principal symbol 1, which was the ambiguity we encountered in quantizing a cotangent bundle.