Lecture 3

Braided tensor categories

3.1 Representations of loop groups

To make this lecture as concrete as possible I shall begin by describing some remarkable properties of the representations of loop groups. A few preliminary explanations are needed, though they are irrelevant to the main idea.

We consider unitary representations of the group $\mathcal{L}G$ of smooth loops in a compact connected Lie group G. We are only interested in so-called *positive-energy* representations: I shall explain the meaning of that presently. Positive energy representations are necessarily projective, i.e. they are really representations of a central extension of $\mathcal{L}G$ by the circle group $\mathbb{T}$. The isomorphism class of the extension which acts is called the *level* of the representation. If G is simply connected it is completely determined by the topological type of the central extension as a circle-bundle on $\mathcal{L}G$, i.e. by its Chern class in $H^2(\mathcal{L}G;\mathbb{Z}) \cong H^3(G;\mathbb{Z})$. There are only finitely many different irreducible representations of $\mathcal{L}G$ at each level.

Let us fix a level k, and let $\mathcal{R}$ be the category of all level k representations of $\mathcal{L}G$ which are finite sums of irreducibles. Then the surprising fact is that a smooth cobordism $\Sigma : S^1 \rightsquigarrow S^1$ induces an additive functor $U_\Sigma : \mathcal{R} \to \mathcal{R}$, and, more generally, a cobordism

$$\Sigma \quad : \quad \underset{\longleftarrow\, p \,\longrightarrow}{S^1 \amalg \cdots \amalg S^1} \quad \rightsquigarrow \quad \underset{\longleftarrow\, q \,\longrightarrow}{S^1 \amalg \cdots \amalg S^1}$$

induces a functor $U_\Sigma : \mathcal{R}_p \to \mathcal{R}_q$, where $\mathcal{R}_p$ is the category of representations of $(\mathcal{L}G)^p = \mathcal{L}G \times \cdots \times \mathcal{L}G$ which are of level k on each factor. Composition of cobordisms Σ corresponds to composition of the operators U_Σ, and a "trivial" cobordism $\Sigma = S \times I$ induces the identity functor. Furthermore, a diffeomorphism $f : \Sigma \to \Sigma'$ between two cobordisms $\Sigma, \Sigma' : S_0 \rightsquigarrow S_1$ which is the identity on S_0 and S_1 induces a transformation of functors $T_f : U_\Sigma \to U_{\Sigma'}$.

In the last sentence I have oversimplified slightly. I should have required the cobordisms Σ to be *rigged* surfaces. A rigging of a surface Σ is something analogous to a choice of a simply connected covering space of Σ — I shall give a precise definition below. The choice of rigging does not affect the functor U_Σ up to isomorphism, but the diffeomorphisms f can be lifted to the rigging in more than one way (in fact, a sequence of ways forming a $\mathbb{Z}$-torsor, if Σ is connected), and changing the lift changes T_f by multiplication by a root of unity.

Postponing discussion of technicalities, let us consider the general implications of the existence of the functors U_Σ.

The first observation is that there is a bi-additive functor $\mathcal{R}\times\mathcal{R} \to \mathcal{R}$ induced by a pair-of-pants cobordism $\Sigma : S^1 \amalg S^1 \to S^1$. It is called *fusion*, and is unique up to an isomorphism of functors, as Σ is unique up to diffeomorphism. It is a kind of tensor product on the category $\mathcal{R}$, and I shall write it $(H_1, H_2) \mapsto H_1 * H_2$. (The usual tensor product of Hilbert spaces does not give us a tensor product on $\mathcal{R}$, for the level of $H_1 \otimes H_2$ is the sum of the levels of H_1 and of H_2). The same arguments that we used in Lecture 1 tell us that fusion is associative and commutative up to isomorphism. The commutativity isomorphism

$$T_\theta \;:\; H_1 * H_2 \to H_2 * H_1,$$

however, is induced by a diffeomorphism θ of the pair of pants which interchanges the two incoming circles while being the identity on the outgoing one. Thus $T_\theta^2 = T_{\theta^2}$ need not be the identity. We can, however, say something about it. For this we need a general observation.

Proposition 3.1.1 *A finite dimensional representation of the group* $\mathrm{Diff}\,(\Sigma \text{ rel } \partial\Sigma)$, *for any manifold* Σ, *is necessarily trivial on the identity component, i.e. it factorizes through the mapping-class group* $\Gamma_{\Sigma \text{ rel } \partial\Sigma} = \pi_0\, \mathrm{Diff}\,(\Sigma \text{ rel } \partial\Sigma)$.

I shall give the proof at the end of this section. We apply the result as follows. As any object H of our category $\mathcal{R}$ is a finite sum of irreducibles, its group of automorphisms is a product of finite dimensional general linear groups, and so 3.1.1 tells us that for any action of $\mathrm{Diff}(\Sigma \text{ rel } \partial\Sigma)$ on H the operator T_f depends only on the isotopy class of f. In particular every H has a canonical automorphism ρ_H which is obtained by choosing an isomorphism between H and $U_A(H)$, where $A = S^1 \times I$, and transferring to H the action of the standard "Dehn twist" $f : A \to A$ which generates the mapping-class group $\Gamma_{A \text{ rel } \partial A} \;\cong\; \mathbb{Z}$. The diffeomorphism θ^2 of Σ is isotopic to the Dehn twist in an annulus around the outgoing boundary circle of Σ, and so

$$T_\theta^2 \;=\; \rho_{H_1 * H_2}. \tag{3.1.2}$$

Of course ρ_h must simply be multiplication by some scalar λ on each irreducible summand P of H.

More generally, for any H the object

$$H^{*p} = H \underbrace{* \cdots *}_{\longleftarrow p \longrightarrow} H$$

depends on a choice of a surface Σ_p which is a disc with p holes. The relevant mapping-class group is the group diffeomorphisms of Σ_p which are allowed to permute the p incoming circles. This is the *braid group* Br_p on p strings, and it acts on H^{*p}. The fact that it is the braid group Br_p rather than the symmetric group S_p which permutes the factors of H^{*p} expresses the less-than-perfect commutativity of the operation of fusion, and explains the name "braided tensor category".

If Σ is a surface with p boundary circles, all outgoing, then U_Σ is a functor $\mathcal{R}^{\otimes 0} \to \mathcal{R}^{\otimes p}$. As $\mathcal{R}^{\otimes 0}$ is the category of finite dimensional vector spaces it is natural to identify U_Σ with the object $U_\Sigma(\mathbb{C})$ of $\mathcal{R}^{\otimes p}$: it is a unitary representation of $(\mathcal{L}G)^p$ with an interwining action of $\Gamma_{\Sigma \text{ rel } \partial\Sigma}$. If Σ is closed then U_Σ is simply a finite dimensional unitary representation of Γ_Σ (actually a projective representation because of the question of rigging).

The representation theory of loop groups thus provides us unexpectedly with a panoply of representations of braid groups and mapping-class groups. But in fact much more is true. If Σ and Σ' are cobordisms from S_0 to S_1 then not only does a diffeomorphism $f : \Sigma \to \Sigma'$ give us a transformation T_f from U_Σ to $U_{\Sigma'}$, but a cobordism $M : \Sigma \rightsquigarrow \Sigma'$ does so too. (A cobordism between $\Sigma, \Sigma' : S_0 \rightsquigarrow S_1$ means a 3-manifold M with an isomorphism

$$\partial M \;\cong\; \bar{\Sigma} \cup ((\bar{S}_0 \amalg S_1) \times I) \cup \Sigma'$$

where $\bar{\Sigma}$ and Σ' are joined by a cylinder along their common boundary

$$\partial\Sigma = \partial\Sigma' = ((\bar{S}_0 \amalg S_1) \times I.)$$

In particular, if Σ is a closed surface, so that U_Σ is a vector space, then a 3-manifold M with $\partial M = \Sigma$ gives us a vector $T_M \in U_\Sigma$, and a closed 3$-$manifold M gives us a *number* $T_M \in \mathbb{C}$. This is the next surprise: we now have a 3-dimensional topological field theory in the sense of Lecture 1, and it turns out to be Chern-Simons theory, i.e.

$$T_M = \int e^{2\pi i CS(A)} \mathcal{D}A, \tag{3.1.3}$$

where, for a connection A on a G-bundle on the 3-manifold M,

$$CS(A) = \int_M (\langle A, dA\rangle \;+\; \langle A, [A, A]\rangle). \tag{3.1.4}$$

The integral in (3.1.4) is over all G-bundles-with-connection A on M, while in (3.1.4) the connection A is written as a Lie-algebro-valued 1-form, which is only legitimate if the bundle is trivial. Like the category $\mathcal{R}$ of representations, the Chern-Simons action (3.1.4) depends on a choice of level, which enters into the formula implicitly as the choice of the inner product on the Lie algebra.

3.2 Category-valued field theories

Let us now try to give a definition of a category-valued topological field theory modelled on the properties of the representations of loop groups. The values will be assumed to be $\mathbb{C}$-linear categories. I shall take this to mean not only that the morphisms between any two objects form a complex vector space, and composition is bilinear, but also that

(i) any two objects have a direct sum, and

(ii) idempotents split, i.e. if $f : A \to A$ is a morphism such that $f^2 = f$ then $A = \ker(f) \oplus \mathrm{im}(f)$

Two such categories $\mathcal{C}_1$ and $\mathcal{C}_2$ have a tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$, which is unique up to canonical equivalence. Any object of $\mathcal{C}_1 \otimes \mathcal{C}_2$ is a summand in a finite sum

$$\bigoplus P_i \otimes Q_i \ ,$$

where the P_i are objects of $\mathcal{C}_1$ and the Q_i are objects of $\mathcal{C}_2$. As an example, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are the categories of finite dimensional representations of compact groups G_1 and G_2 then $\mathcal{C}_1 \otimes \mathcal{C}_2$ is the corresponding category for $G_1 \times G_2$.

Let $\mathcal{V}ect$ denote the category of finite dimensional complex vector spaces. Then there is an equivalence

$$\mathcal{V}ect \otimes \mathcal{C} \longrightarrow \mathcal{C}$$

for any linear category $\mathcal{C}$ (for it makes sense to tensor an object of $\mathcal{C}$ by an object of $\mathcal{V}ect$). I shall denote the category of additive functors $\mathcal{C} \to \mathcal{V}ect$ by $\mathcal{C}^*$.

I shall define a *category-valued n-dimensional topological field theory* as the following data.

(i) A functor $\mathcal{E}$ from closed oriented $(n-1)$-manifolds (and diffeomorphisms) to linear categories. It is required to take disjoint unions to tensor products, which implies that $\Psi(\varnothing) = \mathcal{V}ect$.

(ii) An additive functor $\Psi_Y : \mathcal{E}(X_0) \to \mathcal{E}(X_1)$ for each cobordism $Y : X_0 \rightsquigarrow X_1$, such that

$$\Psi_{Y_2} \circ \Psi_{Y_1} = \Psi_{Y_2 \circ Y_1} . \tag{3.2.1}$$

If S_0 is empty then Ψ_Y will be identified with the object $\Psi_Y(\mathbb{C})$ of $\mathcal{E}(X_1)$: thus Ψ_Y is a finite dimensional vector space if Y is closed.

I shall assume the same kind of compatibility between the data as in Lecture 1. In any discussion of categories it is quite difficult to decide how much to make explicit, and how much is better left to the reader's imagination. In the present situation it does seem important to say that the cobordisms $Y : X_0 \rightsquigarrow X_1$ between given manifolds X_0, X_1 themselves form a category $\mathcal{C}ob(X_0; X_1)$, in which the morphisms are diffeomorphisms $Y \to Y'$, and that $Y \mapsto \Psi_Y$ is a functor from $\mathcal{C}ob(X_0; X_1)$ to the category of functors $\mathcal{E}(X_0) \to \mathcal{E}(X_1)$. Composition of cobordisms is really only defined up to isomorphism: to say that Y_3 is $Y_2 \circ Y_1$ really means that there is a map $f : Y_1 \amalg Y_2 \to Y_3$ with certain properties, and (3.2.1) really means that f induces an isomorphism of functors

$$f_* \ : \ \Psi_{Y_2} \ \circ \ \Psi_{Y_1} \ \to \ \Psi_{Y_3}.$$

I do not think there are any interesting subtleties hidden here. The point to keep in mind is that the group of diffeomorphisms of a closed n-manifold Y acts on the vector space Ψ_Y, and if $Y : X_0 \rightsquigarrow X_1$ then $\mathrm{Diff}(Y \text{ rel } \partial Y)$ — the group of diffeomorphisms which fix the boundary — acts on $\Psi_Y(E)$ for each object E of $\mathcal{E}(X_0)$.

The analogue of the non-degeneracy assumption (1.1.7) is

Assumption 3.2.2 *The cobordism $X \times I : X \rightsquigarrow X$ induces a functor $\mathcal{E}(X) \to \mathcal{E}(X)$ which is equivalent to the identity.*

Notice that we do not want to assume that the equivalence is given: in the loop group example we get an equivalence for each choice of a complex structure on $X \times I$.

As in each of the situations we considered in Lecture 1 there is an analogue of the finiteness - and - duality result . I shall say that two linear categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are in duality if there is a bi-additive functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{V}ect$ and on object Δ of $\mathcal{C}_1 \otimes \mathcal{C}_2$ which induce inverse equivalences

$$\mathcal{C}_1 \rightleftarrows \mathcal{C}_2^* \quad \text{and} \quad \mathcal{C}_2 \rightleftarrows \mathcal{C}_1^*.$$

The existence of a duality if very restrictive (cf. (1.1.8)):

Proposition 3.2.3 *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are in duality then each is semisimple — i.e. each object is a finite sum of irreducible objects — and has, up to isomorphism, only finitely many irreducible objects.*

Corollary 3.2.4 *If $\mathcal{E}$ is a category-valued topological field theory then each category $\mathcal{E}(X)$ is semisimple with finitely many irreducible objects, and the cobordism $X \times I : \bar{X} \amalg X \to \varnothing$ induces a duality*

$$\mathcal{E}(\bar{X}) \times \mathcal{E}(X) \to \mathcal{V}ect,$$

which will be denoted $(A, B) \mapsto \langle A, B \rangle$.

Proof of (3.2.3) The object Δ of $\mathcal{C}_2 \otimes \mathcal{C}_1$ is a summand in a finite sum $\bigoplus P_i \otimes Q_i$, and it follows at once that each object A of $\mathcal{C}_1$ is a summand in

$$\bigoplus \langle A, P_i \rangle \otimes Q_i.$$

Thus each A is a summand in a finite sum of copies of Q, where $Q = \oplus Q_i$. Replacing Q by $P = \oplus P_i$, the same holds for the objects of $\mathcal{C}_2$.

Because $\mathcal{C}_1 \to \mathcal{C}_2^*$ is an equivalence to functor $A \mapsto \langle P, A \rangle$ identifies $\mathcal{C}_1$ with a subcategory of the category of finite dimensional vector spaces. Let R denote the ring $\mathrm{End}_{\mathcal{C}_1}(Q)$, which is a finite dimensional algebra over $\mathbb{C}$. The functor

$$A \mapsto \mathrm{Hom}_{\mathcal{C}_1}(\mathrm{Q}; \mathrm{A})$$

from $\mathcal{C}_1$ to finitely generated projective right R-modules is faithful, and it is easy to see that it is an equivalence of categories. On the other hand $\mathcal{C}_1$ is equivalent to $\mathcal{C}_2^*$, which is an abelian category, and an abelian category in which every object is projective is clearly semisimple. ■

For semisimple categories $\mathcal{C}$ with finitely many irreducibles the contravariant functor $\mathcal{C}^{op} \to \mathcal{C}^*$ given by

$$A \mapsto \{B \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathrm{A}; \mathrm{B})\}$$

is an equivalence. Putting $\mathcal{C}(X)^{op} \simeq \mathcal{C}(X)^*$ together with $\mathcal{C}(\bar{X}) \simeq \mathcal{C}(X)^*$ we see that one can define a functor $A \mapsto A^*$ which is an equivalence $\mathcal{C}(X)^{op} \to \mathcal{C}(\bar{X})$. It is characterized by

$$\langle A^*, B \rangle \cong \mathrm{Hom}_{\mathcal{C}(X)}(A; B).$$

Because the functor $A \mapsto A^*$ must induce an isomorphism

$$\mathrm{Hom}_{\mathcal{C}(X)}(A; B) \to \mathrm{Hom}_{\mathcal{C}(\bar{X})}(B^*; A^*),$$

while the symmetry of $\langle A^*, C^* \rangle$ gives us

$$\mathrm{Hom}_{\mathcal{C}(X)}(A; C^*) \cong \mathrm{Hom}_{\mathcal{C}(\bar{X})}(C; A^*),$$

we have a natural isomorphism $A \to A^{**}$.