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On the Moduli of Todorov Surfaces

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Let Z be a canonical surface, that is, a compact complex surface with only rational double points as singularities whose canonical divisor K_Z is ample. A theorem of Bombieri [5] says that if $c_1^2(Z) \geq 10$ and $p_g(Z) \geq 6$ then the bicanonical map $\phi_{|2K_Z|}$ is either birational, or maps Z two to one onto a (birationally) ruled surface. On the other hand, a recent theorem of Francia [13] says that if $\chi(\mathcal{O}_Z) \geq 2$ then the linear system $|2K_Z|$ is free; in particular, the birational image has dimension 2. It is then natural to study the canonical surfaces Z with $\chi(\mathcal{O}_Z) \geq 2$ whose bicanonical map is not birational, but whose bicanonical image is a non-ruled surface.

The first observation about such surfaces rests on the classical consequence of Clifford's theorem that a non-ruled surface X of degree d in $\mathbf{P}^{n-1}$ must satisfy $d \geq 2n - 4$, with equality if and only if X is a K3 surface with rational double points (cf. [2], for example). If X is the bicanonical image of a canonical surface Z and the bicanonical map has degree m, then the degree of X is $4c_1^2(Z)/m$ while $n = h^0(2K_Z) = \chi(\mathcal{O}_Z) + c_1^2(Z)$. If we assume that $m \geq 2$, X is non-ruled, and $\chi(\mathcal{O}_Z) \geq 2$ we get a chain of inequalities

$$2c_1^2(Z) \geq 4c_1^2(Z)/m \geq 2\chi(\mathcal{O}_Z) + 2c_1^2(Z) - 4 \geq 2c_1^2(Z)$$

which imply that $m = \chi(\mathcal{O}_Z) = 2$ and X is a K3 surface with rational double points. More generally, we may consider canonical surfaces Z with $\chi(\mathcal{O}_Z) = 2$ for which the bicanonical map factors through a finite map of degree 2 $Z \longrightarrow X$ with X a K3 surface with rational double points; it is these which we call *Todorov surfaces.*

The first example of canonical surfaces whose bicanonical map has degree 2 and whose bicanonical image is a K3 surface were given by Todorov [41], as counterexamples for the Torelli problem. Earlier examples of Todorov surfaces in the more general sense were given by Kunev [20] for the same purpose: Kunev's surfaces have $p_g = c_1^2 = 1$ and $q = 0$ so that the bicanonical map has degree 4 onto $\mathbf{P}^2$; when that map is Galois, there is an intermediate K3 surface.

Our goal here is to give an explicit description of the moduli spaces of Todorov surfaces. We show that Todorov surfaces from 11 irreducible families, which are characterized by the order of the 2-torsion subgroup of $\mathrm{Pic}(Z)$

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and by $c_1^2(Z)$. (Examples of surfaces belonging to each of the 11 families were essentially constructed by Todorov [41].) For each family, we give a description of the (coarse) moduli space in the form $\mathcal{V}/\Gamma$, where $\mathcal{V}$ is an open subset of a projective bundle over a Hermitian symmetric space D, and Γ is an arithmetic group acting on D and on $\mathcal{V}$. The main tools (which were also employed by Todorov) are the global Torelli theorem [36] and the surjectivity of the period map [19] for algebraic $K3$ surfaces. Our description of the moduli spaces is more precise than the one given by Todorov: the inaccuracies in appendix 1 of [41] arise in part from Todorov's failure to allow the surfaces to acquire rational double points. One of the themes of this paper is therefore the systematic extension of certain known results about smooth surfaces to include the case of surfaces with rational double points.

It is possible for a Todorov surface to have deformations which are not Todorov surfaces. We have not considered those deformations here, but the larger families which include the Todorov surfaces have been studied in the case $c_1^2(Z) = 1$ or 2 by several people, including Catanese [9], [10], [11], Catanese-Debarre [12], Oliverio [34], Todorov [40] and Usui [42], [43], [44].

We begin with some combinatorial preliminaries in section 1; we have borrowed an idea from Beauville [3] and phrased this section in the language of binary linear codes. The coding theory is applied to the study of ordinary double points on $K3$ surfaces in section 2; we obtain a characterization of these surfaces (theorem (2.5)) which generalizes a result of Nikulin [29] about Kummer surfaces. In sections 3 and 4 we study double covers of surfaces in the presence of rational double points; at the end of section 3 we give a modification of the classical "canonical resolution" process for desingularizing the double cover of a smooth surface branched along a singular curve, which has the advantage of producing the *minimal* desingularization of the double cover. Todorov surfaces are introduced in section 5, and their moduli spaces are studied in section 7. (Further information about the moduli spaces can be found in our related joint paper with M.-H. Saito [27].) Section 6 and the appendix constitute a technical digression concerning embeddings of certain even integral symmetric bilinear forms (called *lattices* in the text); we assume throughout the paper a general familiarity on the part of the reader with the theory of such embeddings (as presented in [31] or [24], for example). The paper closes with a few miscellaneous remarks about Todorov surfaces.

A few words about notation. When S is a smooth surface, we regard the first Chern class as defining maps $c_1 : \mathrm{Pic}(S) \longrightarrow H^2(S, \mathbf{Z})$ and $c_1 : \mathrm{Div}(S) \otimes \mathbf{Q} \longrightarrow H^2(S, \mathbf{Z})$; we also regard the Néron-Severi group $\mathrm{NS}(S)$ as the image in $H^2(S, \mathbf{Z})$ of $\mathrm{Pic}(S)$ under c_1. If $\Gamma \in \mathrm{Div}(S) \otimes \mathbf{Q}$ has the property that $c_1(\Gamma) \in H^2(S, \mathbf{Z})$ and the denominators of the coefficients of Γ are relatively prime to the orders of all torsion elements in $\mathrm{Pic}(S)$, we write $\mathcal{O}_S(\Gamma)$ for the unique line bundle such that $c_1(\Gamma) = c_1(\mathcal{O}_S(\Gamma))$. In particular, when S is simply connected, $\mathcal{O}_S(\Gamma)$ is defined for any $\Gamma \in (\mathrm{Div}(S) \otimes \mathbf{Q}) \cap c_1^{-1}(H^2(S, \mathbf{Z}))$.

We also use a certain amount of "modern" terminology: a linear system is *free* if it has neither fixed components nor base points; a line bundle $\mathcal{L}$ on a normal surface X is *nef* if $\mathcal{L} \cdot C \geq 0$ for all curves C on X, and is *big* if $\mathcal{L} \cdot \mathcal{L} > 0$; and a birational morphism $\pi : X \longrightarrow Y$ is *non-discrepant* [1] if $\pi^*(K_Y) = K_X$. A non-discrepant birational morphism $\pi : X \longrightarrow Y$ between surfaces with rational double points is called a *partial desingularization.*

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§1. Equidistant binary linear codes.

A *binary linear code* $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ is a finite set I together with a vector subspace V of the $\mathbf{F}_2$-vector space $\mathbf{F}_2^I$ of maps from I to $\mathbf{F}_2$. (We allow $I = \phi$ and use the convention that $\mathbf{F}_2^\phi$ is a zero-dimensional vector space.) An isomorphism between two binary linear codes $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ and $\mathcal{D} = (W \subset \mathbf{F}_2^J)$ is an isomorphism of sets $\sigma : I \longrightarrow J$ such that the induced map $\sigma^* : \mathbf{F}_2^J \longrightarrow \mathbf{F}_2^I$ restricts to an isomorphism of $\mathbf{F}_2$-vector spaces $\sigma^*|_W : W \xrightarrow{\sim} V$. The group of automorphisms of C is denoted by $\mathrm{Aut}(C)$.

If $\phi \in \mathbf{F}_2^I$, the *weight* of ϕ is the natural number $\#\{i \in I | \phi(i) = 1\}$. A binary linear code $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ is *equidistant* if all nonzero elements of V have the same weight; this common weight is called the *distance* of the code. [2]

The easiest examples of binary linear codes are the *trivial codes* $\mathcal{T}_k = (\{0\} \subset \mathbf{F}_2^{\{1,\dots,k\}})$ for $k \geq 0$. More interesting examples are given by the following construction (cf. [3]): let W be an $\mathbf{F}_2$-vector space of dimension $\alpha < \infty$, and let $W^* = \mathrm{Hom}_{\mathbf{F}_2}(W, \mathbf{F}_2)$ be the dual space. Regarding elements of W^* as maps of sets produces a binary linear code $\mathcal{U}_\alpha = (W^* \subset \mathbf{F}_2^W)$. Moreover, since $\phi(0) = 0$ for all $\phi \in W^*$, we may regard W^* as a vector subspace of $\mathbf{F}_2^{W \setminus \{0\}}$, producing a second code $\mathcal{C}_\alpha = (W^* \subset \mathbf{F}_2^{W \setminus \{0\}})$. It is easily seen that $\mathcal{C}_\alpha$ and $\mathcal{U}_\alpha$ are equidistant with distance $2^{\alpha-1}$.

Closely related to these is another binary linear code $\mathcal{D}_{\alpha+1} = (V \subset \mathbf{F}_2^W)$ defined as follows (cf. [3]): V is spanned by the image of W^* in $\mathbf{F}_2^W$ and by the element $\phi_0 \in \mathbf{F}_2^W$, where $\phi_0(w) = 1$ for all $w \in W$. $\mathcal{D}_{\alpha+1}$ is not equidistant, for ϕ_0 has weight 2^α while all other nonzero elements of V have weight $2^{\alpha-1}$.

There are several constructions which can be used to produce new codes from given ones.

(i) If $(V \subset \mathbf{F}_2^I)$ is a binary linear code and J is a subset of I, let $V_J = \{\phi \in V \mid \phi|_{I \setminus J} \equiv 0\}$ and call the resulting binary linear code $(V_J \subset \mathbf{F}_2^J)$ the *linear subcode* associated to J. For example, the linear subcode of $\mathcal{D}_{\alpha+1}$ associated to $W \setminus \{0\}$ is the binary linear code $\mathcal{C}_\alpha$.

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[1] Those who have read Reid's footnote [38; p. 133] will recognize the present author as a linguistic conservative.

[2] This terminology derives from the *Hamming distance* (cf. [4; p.16]) between two elements $\phi, \psi \in \mathbf{F}_2^I$, which is defined to be $\#\{i \in I | \phi(i) \neq \psi(i)\}$; $\mathcal{C}$ is equidistant with distance ω if and only if the Hamming distance between every pair of distinct elements is ω.

(ii) If $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ and $\mathcal{D} = (W \subset \mathbf{F}_2^J)$ are binary linear codes, let $\mathcal{C} \oplus \mathcal{D}$ denote the *direct sum code* $(V \oplus W \subset \mathbf{F}_2^{I \cup J})$. For example, $\mathcal{U}_\alpha$ is isomorphic to the direct sum $\mathcal{C}_\alpha \oplus \mathcal{T}_1$, with the isomorphism given by $W \cong (W \setminus \{0\}) \cup \{0\}$.

(iii) If $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ is a binary linear code and $\sigma : J \longrightarrow I$ is an arbitrary map, we call $\sigma^*(\mathcal{C}) = (\sigma^*(V) \subset \mathbf{F}_2^J)$ the *pullback code.*

(iv) As a special case of (iii), if n is a natural number and $\sigma : J \longrightarrow I$ satisfies $\#\{j \in J | \sigma(j) = i\} = n$ for each $i \in I$, we call $\sigma^*(\mathcal{C})$ the *n-fold repetition of* $\mathcal{C}$ and denote it by $\mathcal{C}^n$.

The pullback construction (iii) enables us to regard $\mathcal{U}_\alpha$ as a "universal binary linear code of dimension α", in the following way. Given a binary linear code $\mathcal{C} = (V \subset \mathbf{F}_2^I)$, we let $W = \mathrm{Hom}(V, \mathbf{F}_2)$ and define a tautological map $\tau : I \longrightarrow W$ by $\tau(i)(\phi) = \phi(i)$ for all $\phi \in V$. Then τ^* establishes an isomorphism between W^* and V, so that the original binary linear code $\mathcal{C}$ is the pullback by τ of the code $(W^* \subset \mathbf{F}_2^W)$; this latter code is isomorphic to U_α.

Theorem (1.1). *Let $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ be an equidistant binary linear code with distance ω, let $W = \mathrm{Hom}(V, \mathbf{F}_2)$ with $\tau : I \longrightarrow W$ the tautological map, and let $\alpha = \dim V$ and $k = \#(I)$. Then $2^\alpha | 2\omega$, $k \geq (\omega/2^{\alpha-1})(2^\alpha - 1)$, and for all nonzero $w \in W$, $\#\{i \in I | \tau(i) = w\} = \omega/2^{\alpha-1}$. In particular, if we let $l = k - (\omega/2^{\alpha-1})(2^\alpha - 1)$, then*

$$\mathcal{C} \cong (\mathcal{C}_\alpha)^{\omega/2^{\alpha-1}} \oplus \mathcal{T}_l.$$

Proof. For each $w \in W$, let

$$a_w = \#\{i \in I | \tau(i) = w\}.$$

It suffices to show that $a_0 = l$ and that for $w \neq 0$, $a_w = \omega/2^{\alpha-1}$, for in that case, writing $\mathcal{U}_\alpha = \mathcal{C}_\alpha \oplus \mathcal{T}_\alpha$ we will have $\tau^*(\mathcal{C}_\alpha) = (\mathcal{C}_\alpha)^{\omega/2^{\alpha-1}}$ and $\tau^*(\mathcal{T}_1) = \mathcal{T}_l$.

For each $\phi \in V$ we have

$$\{i \in I | \phi(i) = 1\} = \bigcup_{\substack{w \in W \\ w(\phi)=1}} \{i \in I | \tau(i) = w\}.$$

Thus if $\phi \neq 0$ we get $\sum_{w(\phi)=1} a_w = \omega$ and $\sum_{w(\phi)=0} a_w = k - \omega$ by the assumption of equidistance, while if $\phi = 0$ these sums are 0 and k respectively. We combine these as

$$(*) \qquad \sum_w (-1)^{w(\phi)} a_w = \begin{cases} k - 2\omega & \text{if } \phi \neq 0 \\ k & \text{if } \phi = 0 \end{cases}.$$

Let $A_{w,\phi} = (-1)^{w(\phi)}$. The matrix $A = (A_{w,\phi})_{w \in W, \phi \in V}$ is a *Hadamard matrix* (cf. [4; p. 144]): this means that $AA^T = diag(2^\alpha, \ldots, 2^\alpha)$, as is easily verified.

If we compose the matrix of equations (*) with A^T, we find for a fixed $w_0 \in W$,

$$2^\alpha a_{w_0} = \sum_\phi (-1)^{w_0}(\phi) \sum_w (-1)^{w(\phi)} a_w = \begin{cases} k - (k - 2\omega) & \text{if } w_0 \neq 0 \\ k + (2^\alpha - 1)(k - 2\omega) & \text{if } w_0 = 0 \end{cases}$$

which immediately implies $a_0 = k - (\omega/2^{\alpha-1})(2^\alpha - 1)$ and $a_{w_0} = \omega/2^{\alpha-1}$ for $w_0 \neq 0$, as required. *Q.E.D.*

Corollary (1.2). *Let β be a natural number, let $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ be a binary code, let $\alpha = \dim V$, and let $k = \#(I)$. Then $\mathcal{C}$ is isomorphic to a linear subcode of $\mathcal{D}_{\beta+1}$ if and only if*

(i) *either $\mathcal{C}$ is equidistant with distance $2^{\beta-1}$ or $\mathcal{C} \cong \mathcal{D}_{\beta+1}$, and*

(ii) $2^{\beta-\alpha}(2^\alpha - 1) \leq k \leq 2^\beta - \beta + \alpha - 1$.

Moreover, given α and k satisfying (ii) *with $\alpha \leq \beta + 1$, there is a linear subcode $\mathcal{C} = (V \subset \mathbf{F}_2^I)$ of $\mathcal{D}_{\beta+1}$ with $\alpha = \dim V$ and $k = \#(I)$.*

Proof. First suppose that $\mathcal{C}$ is isomorphic to the linear subcode of $\mathcal{D}_{\beta+1}$ associated to $J \subset W$. If $J = W$ then (i) and (ii) are immediate. If $J \subset W \setminus \{w_0\}$, consider the affine translation T of W defined by $T(w) = w + w_0$; it is easy to check that $T \in \operatorname{Aut} \mathcal{D}_{\beta+1}$. Since $T(w_0) = 0$, by replacing J by $T(J)$ we may assume that $J \subset W \setminus \{0\}$. But in that case, $\mathcal{C} \cong (V_J \subset \mathbf{F}_2^J)$ is a linear subcode of $\mathcal{C}_\beta$ and hence equidistant with distance $2^{\beta-1}$, proving (i).

By theorem (1.1), $k \geq 2^{\beta-\alpha}(2^\alpha - 1)$; on the other hand, since $V_J = \{\phi \in W^* \mid \phi|_{W \setminus J} \equiv 0\}$ we see that the linear span of the subset $W \setminus J$ in W has dimension $\beta - \alpha = \operatorname{codim} V_J$. In particular, $W \setminus J$ must contain at least $\beta - \alpha + 1$ points, so that $2^\beta - k \geq \beta - \alpha + 1$, finishing the proof of (ii).

Conversely, suppose that $\mathcal{C}$ satisfies (i) and (ii). If $\mathcal{C} \cong \mathcal{D}_{\beta+1}$ the statement is immediate, so we may assume without loss of generality that $\mathcal{C}$ is equidistant with distance $2^{\beta-1}$. By theorem (1.1), $\mathcal{C} \cong (\mathcal{C}_\alpha)^{2^{\beta-\alpha}} \oplus \mathcal{T}_l$, where $l = k - 2^{\beta-\alpha}(2^\alpha - 1)$; in particular, $\alpha \leq \beta$. We will exhibit a linear subcode of $\mathcal{C}_\beta$ which is isomorphic to this latter code.

Let $0, w_1, \ldots, w_{\beta-\alpha}$ be $\beta - \alpha + 1$ points of W in general position, and let $K = W \setminus \{0, w_1, \ldots, w_{\beta-\alpha}\}$. By theorem (1.1), there is an isomorphism $\sigma : (V_K \subset \mathbf{F}_2^K) \xrightarrow{\sim} (\mathcal{C}_\alpha)^{2^{\beta-\alpha}} \oplus \mathcal{T}_m$, where $m = 2^{\beta-\alpha} - \beta + \alpha - 1$. Let L be the subset of K mapping to $(\mathcal{C}_\alpha)^{2^{\beta-\alpha}}$ under σ. Since $\#(L) = 2^{\beta-\alpha}(2^\alpha - 1) \leq k \leq 2^\beta - \beta + \alpha - 1 = \#(K)$, there is a set J with $\#(J) = k$ such that $L \subset J \subset K$. But now σ induces an isomorphism $(V_J \subset \mathbf{F}_2^J) \cong (\mathcal{C}_\alpha)^{2^{\beta-\alpha}} \oplus \mathcal{T}_l$, which proves the converse.

Note that we have proved the last statement as well, in our explicit construction of $(V_J \subset \mathbf{F}_2^J)$, once we observe that when $\alpha = \beta + 1$ the only solution of (ii) is $k = 2^\beta$. *Q.E.D.*

We can use theorem (1.1) to compute the automorphism group of an equidistant code.

Lemma (1.3). *Let* $\mathcal{C} = (V \subset \mathbf{F}_2^I) \cong \mathcal{C}_\alpha^{\omega/2^{\alpha-1}} \oplus \mathcal{T}_l$ *be an equidistant binary linear code with distance* ω, *where* $\alpha = \dim V$, $k = \#(I)$ *and* $l = k - (\omega/2^{\alpha-1})(2^\alpha - 1)$. *Then there is an exact sequence:*

$$1 \longrightarrow (\mathfrak{S}_{\omega/2^{\alpha-1}})^{2^\alpha-1} \times \mathfrak{S}_l \longrightarrow \operatorname{Aut}\mathcal{C} \longrightarrow \operatorname{GL}(\alpha, \mathbf{F}_2) \longrightarrow 1$$

where $\mathfrak{S}_n$ *denotes the symmetric group on* n *letters.*

Proof. Let $\tau : I \longrightarrow W = \operatorname{Hom}(V, \mathbf{F}_2)$ be the tautological map. If $\sigma \in \operatorname{Aut}\mathcal{C}$ then σ induces a linear automorphism $\sigma^* \in \operatorname{GL}(V)$, and this in turn induces $\sigma^{**} \in \operatorname{GL}(W)$; we thus get a homomorphism $\operatorname{Aut}\mathcal{C} \longrightarrow \operatorname{GL}(W)$. The kernel of this homomorphism is easily identified: $\sigma^{**} = 1_W$ if and only if $\sigma^* = 1_V$, and in that case, for every $i \in I$ and $\phi \in V$ we have

$$\tau(\sigma(i))(\phi) = \phi(\sigma(i)) = \sigma^*(\phi)(i) = \phi(i) = \tau(i)(\phi)$$

so that $\tau(\sigma(i)) = \tau(i)$. (In other words, σ acts as a permutation of each of the fibers of τ.) The same computation shows that any permutation σ of I which acts as a permutation of each fibers of τ acts as the identity on V and so lies in the kernel of $\operatorname{Aut}\mathcal{C} \longrightarrow \operatorname{GL}(W)$. Since τ has $2^\alpha - 1$ fibers of cardinality $\omega/2^{\alpha-1}$ and 1 fiber of cardinality l, we see that the kernel of $\operatorname{Aut}\mathcal{C} \longrightarrow \operatorname{GL}(W)$ is exactly $(\mathfrak{S}_{\omega/2^{\alpha-1}})^{2^\alpha-1} \times \mathfrak{S}_l$.

It remains to show that the homomorphism $\operatorname{Aut}\mathcal{C} \longrightarrow \operatorname{GL}(W)$ is surjective. Fix a nonzero $w_0 \in W$ and for each nonzero $w \in W$ choose an isomorphism

$$f_w : \{i \in I | \tau(i) = w\} \xrightarrow{\sim} \{i \in I | \tau(i) = w_0\}.$$

(This is possible because each of these sets has cardinality $\omega/2^{\alpha-1}$.) Given $\gamma \in \operatorname{GL}(W)$, define σ by

$$\sigma(i) = \begin{cases} i & \text{if } \tau(i) = 0 \\ f_{\gamma(w)}{}^{-1} \circ f_w(i) & \text{if } \tau(i) = w \neq 0. \end{cases}$$

Then $\sigma \in \operatorname{Aut}\mathcal{C}$ and $\sigma^{**} = \gamma$. *Q.E.D.*

We will also need the automorphism groups of the codes $\mathcal{D}_{\alpha+1}$.

Lemma (1.4). *If* $\mathcal{D}_{\alpha+1} = (V \subset \mathbf{F}_2^W)$ *then* $\operatorname{Aut}(\mathcal{D}_{\alpha+1})$ *is the affine general linear group* $AGL(W)$.

Proof. As we noted in the proof of corollary (1.2), if $w_0 \in W$ then the affine translation $T : w \longmapsto w + w_0$ lies in $\operatorname{Aut}(\mathcal{D}_{\alpha+1})$. Moreover it is easy to see that $\operatorname{Aut}(\mathcal{U}_\alpha) \subset \operatorname{Aut}(\mathcal{D}_{\alpha+1})$. Since $\operatorname{Aut}(\mathcal{U}_\alpha) = \operatorname{GL}(W)$ by the previous lemma, we have $AGL(W) \subset \operatorname{Aut}(\mathcal{D}_{\alpha+1})$.

On the other hand, for any $\gamma \in \operatorname{Aut}(\mathcal{D}_{\alpha+1})$ after composing with an affine translation we may assume $\gamma(0) = 0$. But then $\gamma \in \operatorname{Aut}(\mathcal{U}_\alpha) = \operatorname{GL}(W)$. Thus, $\operatorname{Aut}(\mathcal{D}_{\alpha+1}) \subset \operatorname{AGL}(W)$. *Q.E.D.*

§2. K3 surfaces with ordinary double points.

A K3 *surface with rational double points* is a compact complex surface X, all of whose singular points are rational double points, such that $q(X) = 0$ and the canonical divisor K_X is trivial. (This is a slight generalization of the usual definition.) We say that X has *ordinary double points* if all singular points of X have type A_1.

Let Σ be a K3 surface with ordinary double points, let $\pi : S \longrightarrow \Sigma$ be the minimal desingularization, and let E_Σ be the subgroup of $\mathrm{Div}(S)$ generated by $\{\pi^{-1}(P) \mid P \in \mathrm{Sing}\,\Sigma\}$. The *double point lattice of* Σ is the saturation $L_\Sigma = (E_\Sigma \otimes \mathbf{Q}) \cap c_1^{-1}(H^2(S,\mathbf{Z}))$ of E_Σ in the free abelian group $(\mathrm{Div}(S) \otimes \mathbf{Q}) \cap c_1^{-1}(H^2(S,\mathbf{Z}))$. The intersection form gives L_Σ the structure of an even lattice, and determines an adjoint map ad : $L_\Sigma \longrightarrow L_\Sigma^* = \mathrm{Hom}(L_\Sigma, \mathbf{Z})$. (We shall usually suppress the adjoint map, and regard L_Σ as a subgroup of L_Σ^*.) There is then a chain of embeddings (cf. [31]) $E_\Sigma \subset L_\Sigma \subset L_\Sigma^* \subset E_\Sigma^* \subset (E_\Sigma \otimes \mathbf{Q})$; since E_Σ^*/E_Σ is a 2-elementary group, we see that $L_\Sigma \subset E_\Sigma \otimes (1/2)\mathbf{Z}$ and hence that $[L_\Sigma : E_\Sigma]$ is a power of 2. We call the integer $\alpha = \log_2[L_\Sigma : E_\Sigma]$ the *2-index of* Σ.

The basis $\{\pi^{-1}(P) \mid P \in \mathrm{Sing}\,\Sigma\}$ of E_Σ furnishes a natural identification of $\mathrm{Hom}(E_\Sigma, \mathbf{Z}/2\mathbf{Z})$ with $\mathbf{F}_2{}^{\mathrm{Sing}\,\Sigma}$. We let $f : L_\Sigma \longrightarrow \mathbf{F}_2{}^{\mathrm{Sing}\,\Sigma}$ denote the composite homomorphism

$$L_\Sigma \hookrightarrow \mathrm{Hom}(E_\Sigma, \mathbf{Z}) \longrightarrow \mathrm{Hom}(E_\Sigma, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\sim} \mathbf{F}_2{}^{\mathrm{Sing}\,\Sigma}$$

The kernel of f is exactly E_Σ, and f induces a binary linear code $\mathcal{C}_\Sigma = (\mathrm{Im}\, f \subset \mathbf{F}_2^{\mathrm{Sing}\,\Sigma}$, called the *double point code of* Σ. Since $\mathrm{Im}\, f \cong L_\Sigma/E_\Sigma$, the 2-index of Σ is $\dim(\mathrm{Im}\, f)$. Note that the double point lattice L_Σ can be recovered from the double point code $\mathcal{C}_\Sigma$ via the natural identification of $\mathbf{F}_2{}^{\mathrm{Sing}\,\Sigma}$ with $\mathrm{Hom}(E_\Sigma, \mathbf{Z}/2\mathbf{Z})$: if for $\phi \in \mathbf{F}_2{}^{\mathrm{Sing}\,\Sigma}$ we let

$$\Gamma_\phi = \frac{1}{2} \sum_{\substack{P \in \mathrm{Sing}\,\Sigma \\ \phi(P)=1}} \pi^{-1}(P) \in E_\Sigma \otimes \mathbf{Q},$$

then L_Σ is generated by E_Σ and $\{\Gamma_\phi \mid \phi \in \mathrm{Im}\, f\}$.

Important examples of K3 surfaces with ordinary double points are furnished by the Kummer surfaces: by definition, a *Kummer surface* is a surface of the form $\Sigma = T/(\pm 1)$, where T is a complex torus of complex dimension 2, so that a Kummer surface always has 16 ordinary double points[3]: the images of the 2-torsion subgroup of T. D. B. Fuks (cf. [36; appendix to section 5]), V. V. Nikulin [29], and A. Beauville [3] have shown that the double point code of a Kummer surface Σ is isomorphic to the binary linear code $\mathcal{D}_5$ described in section 1; identifying $\mathrm{Sing}\,\Sigma$ with the 2-torsion subgroup of T gives $\mathrm{Sing}\,\Sigma$ the

[3]Some authors use the term "Kummer surface " to denote the minimal desingularization of $T/(\pm 1)$.

structure of an $\mathbf{F}_2$-vertor space of dimension 4, and applying the construction of $\mathcal{D}_5$ to that vector space produces the double point code of Σ.

We can generalize these examples in the following way: let Σ_0 be a Kummer surface and let $\Sigma \longrightarrow \Sigma_0$ be a partial desingularization. There is a natural inclusion $\operatorname{Sing}\Sigma \subset \operatorname{Sing}\Sigma_0$ which induces an inclusion $E_\Sigma \subset E_{\Sigma_0}$. Hence $L_\Sigma = L_{\Sigma_0} \cap (E_\Sigma \otimes \mathbf{Q})$, and $\mathcal{C}_\Sigma$ is the linear subcode of $\mathcal{C}_{\Sigma_0}$ associated to the subset $\operatorname{Sing}\Sigma \subset \operatorname{Sing}\Sigma_0$. If $k = \#(\operatorname{Sing}\Sigma)$ and α is the 2-index of Σ, corollary (1.2) then implies that $0 \le \alpha \le 5$ and $2^{4-\alpha}(2^\alpha - 1) \le k \le \alpha + 11$.

For each α and k such that $0 \le \alpha \le 5$ and $2^{4-\alpha}(2^\alpha - 1) \le k \le \alpha + 11$, let us fix once and for all a partial desingularization $\Sigma_{\alpha,k}$ of a Kummer surface Σ_0 such that $k = \#(\operatorname{Sing}\Sigma_{\alpha,k})$ and the 2-index of $\Sigma_{\alpha,k}$ is α, and let us define $E_{\alpha,k} = E_{\Sigma_{\alpha,k}}$, $L_{\alpha,k} = L_{\Sigma_{\alpha,k}}$, and $\mathcal{C}_{\alpha,k} = \mathcal{C}_{\Sigma_{\alpha,k}}$. Such a $\Sigma_{\alpha,k}$ is easy to construct, since corollary (1.2) guarantees the existence of a subset $J_{\alpha,k} \subset \operatorname{Sing}\Sigma_0$ whose linear subcode has the correct properties; we simply blow up the points in $\operatorname{Sing}\Sigma_0 \backslash J_{\alpha,k}$. For later convenience, we choose Σ_0 to be the Kummer surface of a principally polarized abelian surface whose endomorphism ring is $\mathbf{Z}$.

Theorem (2.1). *Let Σ be a K3 surface with ordinary double points, let $k = \#(\operatorname{Sing}\Sigma)$ and let α be the 2-index of Σ. Then*

(i) *$0 \le \alpha \le 5$ and $2^{4-\alpha}(2^\alpha - 1) \le k \le \alpha + 11$, and*

(ii) *there exists an isomorphism of binary linear codes $\mathcal{C}_\Sigma \xrightarrow{\sim} \mathcal{C}_{\alpha,k}$ which induces an isometry $L_\Sigma \xrightarrow{\sim} L_{\alpha,k}$.*

Proof. By a theorem of V. V. Nikulin [29; theorem 1 and corollary 1] if $k \ge 16$ then Σ is a Kummer surface. In particular, by the result of Fuks-Nikulin-Beauville noted before, $\mathcal{C}_\Sigma \cong \mathcal{D}_5$, while $\mathcal{C}_{5,16} \cong \mathcal{D}_5$ as well. A map $\operatorname{Sing}\Sigma \longrightarrow \operatorname{Sing}\Sigma_{5,16}$ inducing the composite isomorphism of binary linear codes must clearly give an isometry $L_\Sigma \xrightarrow{\sim} L_{5,16}$.

We may thus assume that $k \le 15$ so that Σ is not a Kummer surface. Write $\mathcal{C}_\Sigma = (V \subset \mathbf{F}_2{}^{\operatorname{Sing}\Sigma})$; if $\phi \in V$ then

$$c_1(\Gamma_\phi) = \frac{1}{2} \sum_{\substack{P \in \operatorname{Sing}\Sigma \\ \phi(P)=1}} c_1(\pi^{-1}(P)) \in H^2(S, \mathbf{Z}),$$

where $\pi : S \longrightarrow \Sigma$ is the minimal desingularization. We now recall the following fundamental fact, proved for example in [29; lemma 3] or [24; lemma 3.3].

Lemma (2.2). *If $C_1, \ldots C_n$ are smooth disjoint rational curves in a smooth K3 surface S such that $c_1((1/2)\sum C_i) \in H^2(S, \mathbf{Z})$, then $n = 0, 8$ or 16.*

Applying this lemma to our situation, we have $n \le k < 16$ so that $n = 0$ or 8; this implies that the weight of each nonzero $\phi \in V$ is 8 so that $\mathcal{C}_\Sigma$ is equidistant with distance 8. Now by theorem (1.1), $2^\alpha \mid 16$ (so that $\alpha \le 4$) and $k \ge 2^{4-\alpha}(2^\alpha - 1)$.

Let N_Σ be the orthogonal complement of $c_1(L_\Sigma)$ in $H^2(S, \mathbf{Z})$. Then by a standard argument [31], $N_\Sigma{}^*/N_\Sigma \cong L_\Sigma{}^*/L_\Sigma$, while $L_\Sigma{}^*/L_\Sigma$ is a subquotient of $E_\Sigma{}^*/E_\Sigma \cong (\mathbf{Z}/2\mathbf{Z})^k$ with $\#(E_\Sigma{}^*/E_\Sigma) = 2^{2\alpha} \cdot \#(L_\Sigma{}^*/L_\Sigma)$. We conclude that $N_\Sigma{}^*/N_\Sigma \cong (\mathbf{Z}/2\mathbf{Z})^{k-2\alpha}$, but since $\mathrm{rank}(N_\Sigma) = 22-k$ we see that $k-2\alpha \leq 22-k$, or in other words, $k \leq \alpha + 11$.

We may now apply corollary (1.2) to conclude that $\mathcal{C}_\Sigma$ is isomorphic to a linear subcode of $\mathcal{D}_5$, and thus (by theorem (1.1)), $\mathcal{C}_\Sigma \cong \mathcal{C}_{\alpha,k}$. But as in the previous case, such an isomorphism induces an isometry $L_\Sigma \xrightarrow{\sim} L_{\alpha,k}$. *Q.E.D.*

We will several additional results about K3 surfaces with rational double points, and their minimal desingularizations.

Proposition (2.3). *Let S be a smooth* K3 *surface, let $C_1, \cdots C_k$ be smooth rational curves on S with $C_i \cdot C_j = -2\delta_{ij}$, let σ be a permutation of $\{1, \ldots, k\}$, and let ϕ be an automorphism of S. Suppose that $\phi(C_i) = C_{\sigma(i)}$, and that for all $x \in H^2(S, \mathbf{Z})$ with $x \cdot C_i = 0$ for all i we have $\phi^*(x) = x$. If $k \leq 15$, then ϕ and σ are trivial.*

Proof. ϕ^* acts trivially on $H^{2,0}(S)$, so that $\phi^*(\omega) = \omega$ for any holomorphic 2-form ω on S, that is, ϕ is a *symplectic* automorphism of S; these have been classified by Nikulin [30].

It suffices to prove the proposition for ϕ of prime order $p \geq 2$. In that case, Nikulin's analysis shows that ϕ has $24(p+1)^{-1}$ isolated fixed points on S which become A_{p-1} singularities on the quotient S/ϕ; moreover, the fixed part of $H^2(S, \mathbf{Z})$ under the action of ϕ has rank $22 - 24(p+1)^{-1}(p-1)$.

Let the cycle decomposition of σ consist of l fixed elements and m p-cycles, so that $k = l + mp$ and $m \leq k/p$. The fixed part of $H^2(S, \mathbf{Z})$ under the action of ϕ then has rank $(22-k)+l+m = 22+m(1-p)$; hence $22 - 24(p+1)^{-1}(p-1) = 22 + m(1-p)$ so that $m = 24(p+1)^{-1}$. But now since $k \leq 15$,

$$\frac{24}{p+1} \leq \frac{k}{p} \leq \frac{15}{p}$$

which is not possible for $p \geq 2$. *Q.E.D.*

Notice that in contrast to proposition (2.3), the translation by a point of order 2 on a complex torus T induces an automorphism of order 2 on the minimal desingularization S of the Kummer surface $T/(\pm 1)$ which permutes the 16 exceptional curves non-trivially, and acts as the identity on their orthogonal complement in $H^2(S, \mathbf{Z})$.

We turn now to the problem of characterizing the set of smooth K3 surfaces which are minimal desingularizatons of K3 surfaces with ordinary double points. The result (theorem (2.5) below), generalizes a theorem of Nikulin [29; theorem 3] in the case of Kummer surfaces.

Let S be a smooth K3 surface, and let $C_1, \ldots, C_n$ be a collection of smooth rational curves on S whose intersection form is negative-definite. We define the *root system spanned by* $C_1, \ldots, C_n$ to be the set

$$R = R(C_1, \ldots, C_n) = \{\sum a_i C_i \mid a_i \in \mathbf{Z},\ (\sum a_i C_i)^2 = -2\} \subset \mathrm{Div}(S).$$

R satisfies the usual axioms of a root system, with a change in sign on the quadratic form (cf. [6]). In particular, for $E_1, E_2 \in R$ we have $|E_1 \cdot E_2| \leq 2$, with equality if and only if $E_1 = \pm E_2$.

The *Weyl group of the root system* R is the subgroup $W(R) \subset \mathrm{Aut}(\mathrm{Div}(S))$ generated by the reflections s_E for all $E \in R$, where

$$s_E(D) = D + (E \cdot D)E$$

for $D \in \mathrm{Div}(S)$.

Proposition (2.4). *Let R be a root system on S, let $E_1, \ldots, E_k$ be a collection of effective divisors on S such that $E_i \in R$ for each i, and $E_i \cdot E_j = -2\delta_{ij}$. Suppose that for each $I \subset \{1, \ldots, k\}$ with $|I| = 4$, $(1/2)c_1(\sum_{i \in I} E_i) \notin H^2(S, \mathbf{Z})$. Then there exists some $w \in W(R)$ such that $w(E_i)$ is an effective irreducible divisor (and hence a smooth rational curve) for each i.*

Proof. Let $C_1, \ldots, C_n$ span the root system R. For any effective $E \in R$ we write $E = \sum a_i C_i$ and define the *length of* E to be the positive integer $l(E) = \sum a_i$. Thus $l(E) = 1$ if and only if E is irreducible.

Suppose that $E_1, \ldots, E_{r-1}$ are irreducible, and let $l = l(E_r)$. By a double induction on r and l, it suffices to show; if $l > 1$, there is some $w \in W(R)$ such that $w(E_1), \ldots, w(E_{r-1})$ are effective and irreducible, $w(E_r)$ is effective, and $l(w(E_r)) < l$.

We thus assume $l \geq 2$, so that E_r is reducible. There must be some component C of E_r with $C \cdot E_r < 0$; since $E_r \neq \pm C$, we have $E_r \cdot C = -1$. Since $E_i \cdot E_r = 0$ for $i < r$, we see that C does not coincide with any such E_i. Thus, since C and E_i are both effective, $E_i \cdot C = 0$ or 1 for each $i < r$.

Re-order the set $\{E_1, \ldots, E_{r-1}\}$ so that $E_i \cdot C = 1$ for $1 \leq i \leq m$, $E_i \cdot C = 0$ for $m + 1 \leq i \leq r - 1$. Let $w_0 = s_C$; one easily computes

$$w_0(E_i) = \begin{cases} E_i + C & \text{if } 1 \leq i \leq m \\ E_i & \text{if } m+1 \leq i \leq r-1 \\ E_r - C & \text{if } i = r. \end{cases}$$

$E_r - C$ is effective since C is a component of E_r; hence, if $m = 0$ then $w = w_0$ is the desired element of $W(R)$.

If $m \geq 1$, since $E_1 \cdot (E_r - C) = -1$, E_1 must be a component of $E_r - C$ so that $E_r - C - E_1$ is effective. Let $w_1 = s_{E_1} w_0$ and compute again:

$$w_1(E_i) = \begin{cases} C & \text{if } i = 1 \\ E_i + C + E_1 & \text{if } 2 \le i \le m \\ E_i & \text{if } m+1 \le i \le r-1 \\ E_r - C - E_1 & \text{if } i = r. \end{cases}$$

If $m = 1$, $w = w_1$ is the desired element.

If $m \ge 2$ then $E_2 \cdot (E_r - C - E_1) = -1$ so that $E_r - C - E_1 - E_2$ is effective. Moreover, $C \cdot (E_r - C - E_1 - E_2) = -1$ so that $E_r - 2C - E_1 - E_2$ is also effective. Let $w_2 = s_C s_{E_1} w_1$ and make one final computation:

$$w_2(E_i) = \begin{cases} E_2 & \text{if } i = 1 \\ E_1 & \text{if } i = 2 \\ E_i + 2C + E_1 + E_2 & \text{if } 3 \le i \le m \\ E_i & \text{if } m+1 \le i \le r-1 \\ E_r - 2C - E_1 - E_2 & \text{if } i = r. \end{cases}$$

If $m = 2$, $w = w_2$ is the desired element. The case $m \ge 3$, however, will lead to a contradiction: in the case, $E_3 \cdot (E_r - 2C - E_1 - E_2) = -2$ and both curves are effective, which implies $E_r - 2C - E_1 - E_2 = E_3$. But then $(1/2)c_1(E_1 + E_2 + E_3 + E_r) = c_1(C + E_1 + E_2 + E_3) \in H^2(S, \mathbf{Z})$, contrary to our assumptions. *Q.E.D.*

Theorem (2.5). *Let S be a smooth* K3 *surface. The following are equivalent:*

(i) *There exists a* K3 *surface Σ with k ordinary double points whose 2-index is α and a map $\pi : S \longrightarrow \Sigma$ which is the minimal desingularization of Σ.*

(ii) *There exists a primitive embedding of the lattice $L_{\alpha,k}$ into $NS(S)$.*

Proof. Since $NS(S)$ is the image of the first Chern class map, (i) $\Rightarrow$ (ii) follows immediately from the definition of the double point lattice L_Σ and theorem (2.1). To show that (ii) $\Rightarrow$ (i), let $\phi : L_{\alpha,k} \hookrightarrow NS(S)$ be a primitive embedding, and let $\{\pi^{-1}(P) \mid P \in \operatorname{Sing} \Sigma_{\alpha,k}\}$ be the natural basis of $E_{\alpha,k}$.

Let $NA(S) = \{ x \in H^{1,1}(S) \cap H^2(S, \mathbf{R}) \mid x \cdot x > 0 \text{ and } x \cdot C > 0 \text{ for every effective curve } C \text{ on } S \}$, let V be the component of $\{x \in H^{1,1}(S) \cap H^2(S, \mathbf{R}) \mid x \cdot x > 0\}$ which contains $NA(S)$ and let W be the subgroup of $\operatorname{Aut} H^2(S, \mathbf{Z})$ generated by the reflections s_δ associated to all $\delta \in NS(S)$ with $\delta \cdot \delta = -2$. Choose some $\kappa \in ((\operatorname{Im} \phi)^\perp \otimes \mathbf{R}) \cap V$; the standard facts about W (cf. [45] or [33; proposition 1.10]) imply that there is some $w_1 \in W$ such that $w_1(\kappa) \in \overline{NA(S)}$.

For each $P \in \operatorname{Sing} \Sigma_{\alpha,k}$, $w_1\phi(\pi^{-1}(P))$ is an element of $NS(S)$ with self-intersection -2, so by Riemann-Roch, $\pm w_1\phi(\pi^{-1}(P))$ is the class of an effective divisor E_P. We let w_2 be the product of all reflections $s_{w_1\phi(\pi^{-1}(P))}$ such that $-w_1\phi(\pi^{-1}(P))$ is effective; then $w_2 w_1 \phi(\pi^{-1}(P)) = c_1(E_P)$ is effective for each P. Since $w_2 w_1(\kappa) = w_1(\kappa) \in \overline{NA(S)}$, we see that $w_2 w_1(\kappa) \cdot C \ge 0$ for every effective curve C on S.

Write $E_P = \sum a_{i,P} C_{i,P}$ with $C_{i,P}$ irreducible and $a_{i,P} > 0$. Since $w_2 w_1(\kappa) \cdot E_P = \kappa \cdot \phi(\pi^{-1}(P)) = 0$ while $w_2 w_1(\kappa) \cdot C_{i,P} \geq 0$ we see that $w_2 w_1(\kappa) \cdot C_{i,P} = 0$ for each i and P. Hence, for each irreducible component $C_{i,P}$ of E_P, $c_1(C_{i,P})$ lies in $(w_2 w_1(\kappa)^\perp) \cap NS(S)$.

Since $(w_2 w_1(\kappa)^\perp) \cap NS(S)$ has a negative-definite intersection form, the elements of self-intersection -2 in that set form a root system R. We apply proposition (2.4) to this root system and the set of effective curves $\{E_P \mid P \in \operatorname{Sing} \Sigma_{\alpha,k}\}$ and obtain some $w_3 \in W(R)$ such that $w_3(E_P)$ is an effective irreducible divisor for each P. Contracting these smooth rational curves $w_3(E_P)$ gives the desired surface Σ. *Q.E.D.*

We will need one additional lemma about root systems on K3 surfaces.

Lemma (2.6). *Let R be a root system on the smooth* K3 *surface S, and let $C_1, \ldots C_k$ be distinct irreducible curves in R such that for all $D \in R$, $(\sum C_i) \cdot D \in 2\mathbf{Z}$. If w is an element of the Weyl group $W(R)$ such that $w(C_i)$ is effective and irreducible for each i, then there is a permutation σ of $\{1, \ldots, k\}$ such that $w(C_i) = C_{\sigma(i)}$.*

Proof. We first claim that for any $w_1 \in W(R)$, $(1/2)(w_1(\sum C_i) - \sum C_i)$ is contained in the $\mathbf{Z}$-linear span L of R. We write w_1 as a product of reflections in elements of R, and prove our claim by induction on the number of reflections.

If w_1 is the identity, there is nothing to prove. Otherwise, write $w_1 = s_E w_2$ for some $E \in R$; by induction hypothesis, $(1/2)(w_2(\sum C_i) - \sum C_i) \in L$. Now

$$w_2(\sum C_i) \cdot E = (\sum C_i) \cdot w_2^{-1}(E) \in 2\mathbf{Z}$$

so that

$$\frac{1}{2}(s_E w_2(\sum C_i) - w_2(\sum C_i)) = \frac{1}{2}(w_2(\sum C_i) \cdot E)E \in L.$$

Hence

$$\begin{aligned} \frac{1}{2}(w_1(\sum C_i) - \sum C_i) &= \frac{1}{2}(s_E w_2(\sum C_i) - w_2(\sum C_i)) \\ &\quad + \frac{1}{2}(w_2(\sum C_i) - \sum C_i) \in L, \end{aligned}$$

proving the claim.

We apply this when $w_1 = w$, and find that

$$\frac{1}{2}(\sum w(C_i) - \sum C_i) \in L.$$

But $\{C_1, \ldots, C_k\}$ is part of a basis of L (a complete basis is formed by all irreducible effective curves in R), and each $w(C_i)$ either coincides with some C_j, or is also part of the basis. Since there is a linear relation among $\{C_i\} \cup \{w(C_i)\}$, each $w(C_i)$ must coincide with some C_j and the lemma follows. *Q.E.D.*

§3. Double covers of surfaces with ordinary double points.

Let W be a normal compact complex surface, S be a smooth compact complex surface, and $\rho : W \to S$ be a proper finite holomorphic map of degree 2. There is a natural involution [4] $j : W \to W$ comuting with ρ whose eigenspace decomposition induces a splitting $\rho_* \mathcal{O}_W \cong \mathcal{O}_S \oplus \mathcal{M}^{-1}$ for some line bundle $\mathcal{M}$ on S. If Δ is the branch locus of ρ (a reduced Cartier divisor on S) then $\mathcal{O}_S(\Delta) = \mathcal{M}^{\otimes 2}$; note that Δ is the image of the fixed point set of j.

Conversely, given a reduced Cartier divisor Δ on S and a line bundle $\mathcal{M}$ such that $\mathcal{O}_S(\Delta) = \mathcal{M}^{\otimes 2}$, there is a standard construction for a double cover W : in each sufficiently small open set $U \subset S$, one chooses a local section $y \in \Gamma(U, \mathcal{M}^{\otimes 2})$ which vanishes on $\Delta \cap U$ and defines $\rho^{-1}(U) = \{x \in \Gamma(U, \mathcal{M}) \mid x^2 = y\}$. The double cover $\rho : W \to S$ is then uniquely specified by the data $(S, \mathcal{M}, \Delta)$, and is called the *double cover of S branched on Δ with associated line bundle $\mathcal{M}$.*

If Q is a smooth point of a surface S, ξ is a projectivized tangent vector to S at Q, and Δ is a curve on S containing Q, Δ is said to have an *infinitely near m-ple point at Q in direction ξ* if Δ has multiplicity m at Q, and on the blowup $\tilde{S}$ of S at Q, the proper transform of Δ has multiplicity m at the point of the exceptional divisor corresponding to ξ. If $\rho : W \to S$ is a double cover branched on the reduced Cartier divisor Δ, then W has only rationaly double points if and only if Δ has neither infinitely near triple points nor points of multiplicity greater than three.

The construction outlined above is quite well known (more details can be found in [16; section 2] or [35; section II]); we wish to generalize it to the case in which the quotient may have ordinary double points.

Let Y be a normal compact complex surface, let $j : Y \to Y$ be an involution, and let $\sigma : Y \to \Sigma$ be the quotient by j. We call j an *ODP-involution* if all singularities of Σ are ordinary double points, and for each $P \in \operatorname{Sing}\Sigma, \sigma^{-1}(P)_{\text{red}}$ is a smooth point of Y which is an isolated fixed point of j. (Note that when Y is smooth, any involution of Y is an ODP-involution.) Given such an involution, we let $\pi : S \to \Sigma$ be the minimal desingularization, let $W = S \times_\Sigma Y$ be the fiber product, and let $\rho : W \to S$ and $\gamma : W \to Y$ be the natural projections. The map γ is simply the blowup of the set $\cup_{P \in \operatorname{Sing}\Sigma} \sigma^{-1}(P)_{\text{red}}$, and each curve $\pi^{-1}(P)$ is a component of the branch locus of ρ. Moreover, since each point $\sigma^{-1}(P)_{\text{red}}$ is an *isolated* fixed point of j, each such component is disjoint from the rest of the branch locus. We may thus write the branch locus of ρ in the form $\Delta = \pi^*(B) + \sum_{P \in \operatorname{Sing}\Sigma} \pi^{-1}(P)$ for some reduced Cartier divisor B on Σ with $B \cap \operatorname{Sing}\Sigma = \phi$.

Conversely, suppose that Σ is a surface with ordinary double points, $\pi : S \to \Sigma$ is the minimal desingularization and $(\mathcal{M}, B)$ is a pair satisfying:

Condition (3.1). *$\mathcal{M}$ is a line bundle on S and B is a reduced Cartier*

[4] An *involution* is an automorphism of degree 2.

divisor on Σ *such that* $B \cap \operatorname{Sing}\Sigma = \phi$ *and* $\mathcal{O}_S(\pi^*(B) + \sum_{P \in \operatorname{Sing}\Sigma} \pi^{-1}(P)) = \mathcal{M}^{\otimes 2}$.

Then we define the *double cover of* Σ *branched on* $B \cup \operatorname{Sing}\Sigma$ *with associated line bundle* $\mathcal{M}$ as follows: first form the double cover $\rho : W \to S$ branched on $\Delta = \pi^*(B) + \sum P \in \operatorname{Sing}\Sigma \pi^{-1}(P)$ with associated line bundle $\mathcal{M}$. For each $P \in \operatorname{Sing}\Sigma$, $(1/2)\rho^{-1}(\pi^{-1}(P))$ is an exceptional curve of the first kind on W, and these curves are mutually disjoint; let Y be the surface obtained by contracting these curves. The induced meromorphic map $Y \dashrightarrow \Sigma$ extends to a holomorphic map $\sigma : Y \to \Sigma$, which is the desired double cover. It is easy to check that $\sigma : Y \to \Sigma$ is uniquely specified by the data $(\Sigma, \mathcal{M}, B)$ and that this gives a one to one correspondence between ODP-involutions and such triples. As in the previous case, Y has only rational double points if and only if B has neither infinitely near triple points nor points of mulitiplicity greater than three.

Lemma (3.2). *Let* $\sigma : Y \to \Sigma$ *be the quotient by an ODP-involution, and let* B *be the divisorial part of the branch locus of* σ. *Then* $2K_Y = \sigma^*(2K_\Sigma + B)$.

Proof. Let $\pi : S \to \Sigma$ be the minimal desingularization, and let $W = S \times_\Sigma Y$ be the fiber product with projections $\rho : W \to S$ and $\gamma : W \to Y$. ρ is branched on $\Delta = \pi^*(B) + \sum_{P \in \operatorname{Sing}\Sigma} \pi^{-1}(P)$ so that by the usual canonical divisor formula for a double cover, we have

$$K_W = \rho^* K_S + \frac{1}{2}\rho^*\Delta$$

and hence

$$2K_W = \rho^*(2K_S) + \rho^*\pi^*(B) + \sum P \in \operatorname{Sing}\Sigma \rho^*\pi^{-1}(P).$$

On the other hand, by the canonical divisor formula for a blowup applied to γ, we have

$$K_W = \rho^* K_Y + \sum_{P \in \operatorname{Sing}\Sigma} \gamma^{-01}\sigma^{-1}(P).$$

Since $2\gamma^{-1}\sigma^{-1}(P) = \rho^*\pi^{-1}(P)$ and $\rho^*\pi^*(B) = \gamma^*\sigma^*(B)$, comparing these shows that

$$\gamma^*(2K_Y) = \rho^*(2K_S) + \gamma^*\sigma^*(B).$$

But now since π is non-discrepant,

$$\rho^*(2K_S) = \rho^*\pi^*(2K_\Sigma) = \gamma^*\sigma^*(2k_\Sigma)$$

which implies $2K_Y = \sigma^*(2K_\Sigma) + \sigma^*(B)$ as claimed. *Q.E.D.*

We now recall the following definition from [25]: let Σ' be a complex surface, let $Q \in \Sigma'$ be a smooth point and let ξ be a projectivized tangent vector at Q. Choose local coordinates y and z at Q such that $\xi(z) = 0$. The *directed blowup* *of Q in direction ξ of weight* 1 is the blowup $\Sigma \to \Sigma'$ of the ideal generated by y and z^2. The inverse image of Q under such a blowup is a smooth rational curve on Σ; Σ has an ordinary double point at one point of this curve, but is otherwise smooth in a neighborhood of the exceptional curve.

Lemma (3.3). *Let Σ' be a compact complex surface with ordinary double points, let $Q \in \Sigma'$ be a smooth point, and let ξ be a projectivized tangent vector at Q. Let $\delta : \Sigma \to \Sigma'$ be the ordinary blowup of Q or the directed blowup of Q in direction ξ of weight 1, let C be the exceptional divisor of δ and let $Y \to \Sigma$ be the double cover branched along $B \cup \operatorname{Sing}\Sigma$ with associated line bundle $\mathcal{M}$, where $(\mathcal{M}, B)$ satisfies condition* (3.1). *Then $B' = \delta(B)$ is a reduced Cartier divisor on Σ' and there exists a unique line bundle $\mathcal{M}'$on the minimal desingularization S' of Σ' such that*

(i) *$(\mathcal{M}', B')$ satisfies condition* (3.1),

(ii) *if $Y' \to \Sigma'$ is the double cover of Σ' branched on $B' \cup \operatorname{Sing}\Sigma'$ with associated line bundle $\mathcal{M}'$, and $\overline{Y} = \Sigma \times_{\Sigma'} Y'$ is the fiber product, then the normalization of $\overline{Y}$ is Y, and*

(iii) *if δ is an ordinary blowup then for some $m \geq 0$, B' has a point of mulitiplicity m at Q and*

$$\delta^*(B') = B + 2[m/2]C$$

while if δ is a directed blowup then for some $n \geq 0$, B' has an infinitely near $(2n+1)$-ple point at Q in direction ξ and

$$\delta^*(B') = B + (4n+2)C.$$

Proof. We prove this in case δ is a directed blowup, leaving the easier (and well-known) case of an ordinary blowup to the reader. Let C be the exceptional divisor of δ, let R be the singular point of Σ contained in C, and let $E = \pi^{-1}(R)$ where $\pi : S \to \Sigma$ is the minimal desingularization. $2C$ is then a Cartier divisor on Σ, and $\pi^*(2C) = 2\tilde{C} + E$ for some $\tilde{C}$ which is an exceptional curve of the first kind on S; the induced map $\tilde{\delta} : S \to S'$ blows down $\tilde{C}$ and E. Since Σ' is smooth at Q, the Weil divisor B' (which is Cartier on $\Sigma' \setminus Q \cong \Sigma \setminus C$) must be Cartier on all of Σ', and $B' \cap \operatorname{Sing}\Sigma = \phi$.

Let $\mathcal{M}' = \tilde{\delta}_*(\mathcal{M})$; there are then nonnegative integers m, n and r such that $\tilde{\delta}^*(\mathcal{M}') = \mathcal{M} \otimes \mathcal{O}_S(m\tilde{C} + nE)$ and $\delta^*(B') = B + rC$. Intersecting the first equation with E and with $\tilde{C}$ shows that $\mathcal{M} \cdot E = -m + 2n$ and $\mathcal{M} \cdot \tilde{C} = m - n$. From the second equation we deduce $\pi^*\delta^*(B') = \pi^*(B) + r\tilde{C} + (r/2)E$; again intersecting with E and with $\tilde{C}$ we see that $\pi^*(B) \cdot E = 0$ and $\pi^*(B) \cdot \tilde{C} = r/2$. By (3.1), $\mathcal{M}^{\otimes 2} = \mathcal{O}_S(\pi^*(B) + E + \sum_{P \in \operatorname{Sing}\Sigma} \pi^{-1}(P))$, so that $2(\mathcal{M} \cdot E) = \pi^*(B) \cdot e - 2$ and $2(\mathcal{M} \cdot \tilde{C}) = \pi^*(B) \cdot \tilde{C} + 1$. Combining these computations, we see that $r = 4n + 2$ and $m = 2n + 1$.

Now

$$\begin{aligned}(\tilde{\delta}^*(\mathcal{M}'))^{\otimes 2} &= \mathcal{O}_S(\pi^*(B) + E + \sum_{\substack{P\in \mathrm{Sing}\,\Sigma\\ P\neq R}} \pi^{-1}(P) + 2(2n+1)\tilde{C} + 2nE)\\ &= \mathcal{O}_S(\pi^*\delta^*(B') + \sum_{\substack{P\in \mathrm{Sing}\,\Sigma\\ P\neq R}} \pi^{-1}(P)),\end{aligned}$$

which implies that $(\mathcal{M}', B')$ satisfies (3.1). Moreover, the pullback $\overline{Y}$ of the double cover Y' is a double cover of Σ branched on

$$\delta^*(B') + \sum_{\substack{P\in \mathrm{Sing}\,\Sigma\\ P\neq R}} \pi^{-1}(P) = B + (4n+2)C + \sum_{\substack{P\in \mathrm{Sing}\,\Sigma\\ P\neq R}} \pi^{-1}(P);$$

pulling this branch locus back to S gives

$$\pi^*\delta^*(B') = \pi^*(B) + E + \sum_{\substack{P\in \mathrm{Sing}\,\Sigma\\ P\neq R}} \pi^{-1}(P) + 2((2n+1)\tilde{C} + nE)$$

so that the normalization is the double cover branched on

$$\pi^*(B) + E + \sum_{\substack{P\in \mathrm{Sing}\,\Sigma\\ P\neq R}} \pi^{-1}(P),$$

and coincides with the original double cover.

Finally, note that if $\alpha : S \to \overline{S}$ is the contraction of $\tilde{C}$ and $\beta : \overline{S} \to S'$ the contraction of $\alpha(E)$, then $\beta^*(B') = \alpha(\pi^*(B)) + (2n+1)\alpha(E)$ while $\alpha^*(\alpha(\pi^*(B))) = \pi^*(B) + (2n+1)\tilde{C}$, so that B' has an infinitely near $(2n+1)$-ple point in direction ξ (corresponding to $\tilde{C}\cap E$), as required. *Q.E.D.*

As an application of lemma (3.3), we exhibit a method for finding the minimal desingularization of a singularity arising from a double cover of a smooth surface branched along a singular curve. One standard method of desingularizing in this case is the "canonical resolution" process (cf. [16;section 2] or [1; section III.7]): given a double cover $Y_1 \to S_1$ with branch locus B_1 such that S_1 is smooth, one blows up a singular point of B_1 via $S_2 \to S_1$ and defines $Y_2 \to S_2$ to be the normalization of the fiber product $S_2 \times_{S_1} Y_1$; repeating this process eventually desingularizes Y_1.

Our variant of this, which produces the minimal desingularization, is as follows. If we are inductively given a surface Σ_k with ordinary double point and a (normal) double cover $Y_k \to \Sigma_k$ branched on $B_k \cap \mathrm{Sing}\,\Sigma_k$, we choose some $Q \in \Sigma_k$ which is a sinular point of B_k. (If no such Q exists, then Y_k is smooth and we stop.) If Q is an infinitely near $(2n+1)$-ple point of B_k in direction ξ, let $\Sigma_{k+1} \to \Sigma_k$ be the directed blowup (of weight 1) of Σ_k at Q in direction ξ;

otherwise, let $\Sigma_{k+1} \to \Sigma_k$ be the ordinary blowup of Σ_k at Q. In either case, we get a new (noemal) double cover $Y_{k+1} \to \Sigma_{k+1}$ branched along $B_{k+1} \cup \operatorname{Sing} \Sigma_{k+1}$ if we choose B_{k+1} according to (3.3)(iii) (setting $B = B_{k+1}$ and $B' = B_k$), so that we may continue the process.

To see that this process terminates and produces the minimal desingularization of the original surface Y_1, let $Y \to Y_1$ be the minimal desingularization, let j be the induced ODP-involution on Y and let $\Sigma = Y/j$. The natural map $\Sigma \to \Sigma_1$ is a birational morphism between surfaces with ordinary double points and $\Sigma_1 = S_1$ is smooth; theorem 1.4 of [25] (cf. also [39]) then implies that $\Sigma \to \Sigma_1$ can be factored as a composite $\Sigma = \Sigma_n \to \Sigma_{n-1} \to \cdots \to \Sigma_1$ of ordinary blowups and directed blowups of weight 1. If we use descending induction and lemma (3.3), we see that there are double covers $Y_k \to \Sigma_k$ branched on $B_k \cup \operatorname{Sing} \Sigma_k$ with Y_{k+1} the normalization of the fiber product $\Sigma_{k+1} \times_{\Sigma_k} Y_k$. If some $\Sigma_{k+1} \to \Sigma_k$ were an ordinary blowup at either an infinitely near $(2n+1)$-ple point of B_k or a point not in $\operatorname{Sing} B_k$, then Y would have an exceptional curve of the first kind mapping to $\operatorname{Sing} Y_1$; hence, such blowups are not allowed, and the process described above is the one which produces the minimal desingularization.

§4. Involutions on canonical surfaces.

A *canonical surface* is a complex surface Z with rational double points whose canonical divisor K_Z is ample. We call the involution $j : Z \to Z$ an *RDP -involution* if Z is a canonical surface, j is an involution, and the quotient Z/j has rational double points. To each RDP-involution $j : Z \to Z$ we associate a triple $(X, \mathcal{M}, B)$ as follows: $X = A/j$, B (which is a Weil divisor on X) is the divisorial part of the branch locus of the quotient map $Z \to X$, and if $\mu : S \to X$ is the minimal desingularization, $W = S \times_X Z$ is the fiber product and $\eta : W \to S$ is the projection, then $\mathcal{M}$ is the line bundle on S such that $\eta_* \mathcal{O}_W = \mathcal{O}_S \oplus \mathcal{M}^{-1}$.

Theorem (4.1). *The association described above gives a one to one correspondence between isomorphism classes of RDP-involutions and isomorphism classes of triples* $(X, \mathcal{M}, B)$ *with the following properties:*

(i) *X is a surface with rational double points, $\mathcal{M}$ is a line bundle on the minimal desingularization $\mu : S \to X$, and B is a Cartier divisor on X such that $2K_X + B$ is ample*

(ii) *there exist smooth disjoint rational curves $C_1, \ldots, C_k$ on S such that*

$$\mathcal{M}^{\otimes 2} = \mathcal{O}_S(\mu^*(B) + \sum C_i)$$

(iii) *$\mu^*(B)$ is disjoint from the curves C_i, and is a reduced divisor with neither infinitely near triple points nor points of multiplicity greater than three.*

Note that property (ii) determines a natural partial desingularization $\nu : \Sigma \to X$, where $\pi : S \to \Sigma$ is the contraction of the curves $C_1, \ldots, C_k$; we call this the *distinguished partial desingularization* of X.

Proof. Let $j : Z \to Z$ be an RDP-involution, and let $\varepsilon : \tilde{Z} \to Z$ be the minimal desingularization. The involution j lifts to an involution on $\tilde{Z}$ (which we shall again denote by j); we let $\tilde{X} = \tilde{Z}/j$ and let $\phi : \tilde{X} \to X$ be the induced map.

ϕ is a birational morphism between surfaces with rational double points, so we may apply theorem (1.4) of [25] (cf. also [39]). Specialized to the present situation in which $\tilde{X}$ has only ordinary double points, that theorem guarantees that ϕ can be factored as

$$\tilde{X} = \Sigma_1 \xrightarrow{\psi_1} \Sigma_2 \longrightarrow \cdots \xrightarrow{\psi_{n-1}} \Sigma_n = \Sigma \xrightarrow{\nu} X$$

where each Σ_i has ordinary double points, each ψ_i is either an ordinary blowup or a directed blowup of weight 1, and ν is a partial desingularization. (The referee points out that each ψ_i is actually an ordinary blowup, as follows from the analysis at the end of section 3.)

Let $Y_1 = \tilde{Z}$; since Y_1 is smooth, the map $Y_1 \to \Sigma_1$ is the quotient by an ODP-involution, and is therefore the double cover of Σ_1 brached on $B_1 \cup \operatorname{Sing} \Sigma_1$ with associated line bundle $\mathcal{M}_1$ for some $(\mathcal{M}_1, B_1)$ satisfying (3.1). Let $B_{i+1} = \psi_i(B_i)$ for each 1. By induction using lemma (3.3), we see that there is a sequence of double covers Y_1 of Σ_1 brached on $B_i \cup \operatorname{Sing} \Sigma_i$ with associated line bundles $\mathcal{M}_i$ fitting into a diagram

$$\begin{array}{ccccccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n & = & Y & \xrightarrow{\beta} & Z \\ \downarrow & & \downarrow & & & & \downarrow & & \sigma\downarrow & & \tau\downarrow \\ \Sigma_1 & \longrightarrow & \Sigma_2 & \longrightarrow & \cdots & \longrightarrow & \Sigma_n & = & \Sigma & \xrightarrow{\nu} & X \end{array}$$

in which each square is the normalization of the Cartesian product; the induction also shows that B_i is a Cartier divisor disjoint from $\operatorname{Sing} \Sigma_i$.

Let C be a component of the exceptional locus of ν. Then $\sigma^*(C)$ is contracted by β; since Z is canonical, this implies that $\sigma^*(C) \cdot K_Y = 0$ and hence, by lemma (3.2), $C \cdot (2K_\Sigma + B_n) = 0$. On the other hand, since ν is non-discrepant $C \cdot K_\Sigma = 0$, so that $C \cdot B_n = 0$. Thus, $\nu(B_n)$ is Cartier divisor on X with $\nu^*(\nu(B_n)) = B_n$. Since $mB_n - \nu^*(mB)$ is supported on the exceptional locus of ν for any integer m for which mB is Cartier, we conclude that $B = \nu(B_n)$ is Cartier and $\nu^*(B) = B_n$.

Properties (ii) and (iii) now follow from the description of ODP-involutions given in section 3 (applied to $\sigma : Y \to \Sigma$), once we set $\mathcal{M} = \mathcal{M}_n$. It remains to show that $2K_X + B$ is ample. But $\nu^*(2K_X + B) = 2K_\Sigma + B_n$ while $\sigma^*(2K_\Sigma + B_n) = 2K_Y$ by lemma (3.2). Hence, since β is non-discrepant, $2K_\Sigma + B_n$ is nef; moreover, $C \cdot (2K_\Sigma + B_n) = 0$ if and only if $\sigma^*(C) \cdot K_Y = 0$. Since Z is canonical, this holds if and only if $\sigma^*(C)$ is contracted by β, that is, if and only if C is contracted by ν. It follows that $2K_X + B = \nu(2K_\Sigma + B_n)$ is ample.

Conversely, if we are given $(X, \mathcal{M}, B)$, we let $\pi : S \to \Sigma$ be the contraction of the curves $C_1, \ldots, C_k$ and let $\nu : \Sigma \to X$ be the induced non-discrepant map. Properties (ii) and (iii) guarantee the existence of a double cover $\sigma : Y \to \Sigma$ branched on $\nu^*(B) \cup \operatorname{Sing}\Sigma$ with associated line bundle $\mathcal{M}$; we let Z be the canonical model of Y, which exists since $K_Y = \sigma^*(2K_\Sigma + \nu^*(B)) = \sigma^*\nu^*(2K_X + B)$ is nef and big. The argument in the preceding paragraph now shows that (since $2K_X + B$ is ample), the rational map $Z \dashrightarrow X$ is a finite morphism (since curves are contracted by $\beta : Y \to Z$ if and only if their images are contracted by $\nu : \Sigma \to X$.) *Q.E.D.*

§5. Todorov surfaces.

Let Z be a canonical surface, let $j : Z \to Z$ be an RDP-involution and suppose that $X = Z/j$ is a K3 surface with rational double points. As in section 4, we associate a triple $(X, \mathcal{M}, B)$ to j, which determines a distinguished partial desingularization $\nu : \Sigma \to X$ in which Σ has only ordinary double points. Note that since K_X is trivial, B is an ample Cartier divisor on X. Moreover, since the minimal desingularization S of Σ is simply connected,

$$\mathcal{M} = \mathcal{O}_S\Big(\frac{1}{2}\mu^*(B) + \frac{1}{2}\sum_{P \in \operatorname{Sing}\Sigma} \pi^{-1}(P)\Big).$$

Lemma (5.1). *Let X be a K3 surface with rational double points, B be an ample Cartier divisor on X, $\mu : S \to X$ be the minimal desingularization and $\nu : \Sigma \to X$ be a partial disingularization with induced map $\pi : S \to \Sigma$ such that Σ has only ordinary double points and*

$$\frac{1}{2}c_1\Big(\mu^*(B) + \sum_{P \in \operatorname{Sing}\Sigma} \pi^{-1}(P)\Big) \in H^2(S, \mathbf{Z}).$$

Then

(i) *If $B^2 = 2$, $\#(\operatorname{Sing}\Sigma) = 1$ or 9, and there is a point $Q \in \operatorname{Sing}\Sigma$ such that*

$$\frac{1}{2}c_1\Big(\sum_{\substack{P \in \operatorname{Sing}\Sigma \\ P \neq Q}} \pi^{-1}(P)\Big) \in H^2(S, \mathbf{Z}),$$

then the linear system $|\nu^*(B)|$ has a single base point, at Q.

(ii) *In all other cases, the linear system $|B|$ is free.*

Proof. $\mu^*(B)$ is a nef and big divisor on the smooth K3 surface S, so that by a theorem of Mayer [22], either the linear system $|\mu^*(B)|$ is free (and hence $|B|$ is free), or $\mu^*(B) \equiv nE + C$ for some elliptic pencil $|E|$ on S, and some section C of $|E|$, where $n = (1/2)(1 + \mu^*(B)^2) \geq 2$. In the latter case; C is the unique fixed component of $|\mu^*(B)|$ and we compute:

$$E \cdot (\mu^*(B) + \sum_{P \in \operatorname{Sing} \Sigma} \pi^{-1}(P)) = 1 + \sum_{P \in \operatorname{Sing} \Sigma} E \cdot \pi^{-1}(P).$$

Now $(1/2)c_1(\mu^*(B) + \sum_{P \in \operatorname{Sing} \Sigma} \pi^{-1}(P)) \in H^2(S, \mathbf{Z})$ so that this intersection number must be even; hence, there is some $Q \in \operatorname{Sing} \Sigma$ with $E \cdot \pi^{-1}(Q) \geq 1$. For each Q, we have

$$0 = \pi^{-1}(Q) \cdot \mu^*(B) = \pi^{-1}(Q) \cdot (nE + C) \geq n + \pi^{-1}(Q) \cdot C$$

so that $\pi^{-1}(Q) = C$ and $n = 2$. Thus, $|\nu^*(B)|$ has a unique base point at $Q = \pi(C)$, and $B^2 = \mu^*(B)^2 = 2$.

To finish the proof of (ii), note that when $|\mu^*(B)|$ is not free,

$$\frac{1}{2}c_1(\sum_{\substack{P \in \operatorname{Sing} \Sigma \\ P \neq Q}} \pi^{-1}(P)) = \frac{1}{2}c_1(\mu^*(B) + \sum_{P \in \operatorname{Sing} \Sigma} \pi^{-1}(P)) - c_1(E + C)$$

lies in $H^2(S, \mathbf{Z})$. Moreover, by lemma (2.2),

$$\#(\operatorname{Sing} \Sigma \backslash \{Q\}) = 0, \quad 8, \text{or} \quad 16;$$

since $\#(\operatorname{Sing} \Sigma) \leq 16$, we see that $\#(\operatorname{Sing} \Sigma) = 1$ or 9.

It remains to show that when the hypotheses of (i) are satisfied, $|\mu^*(B)|$ cannot be free. Since $\mu^*(B)$ is nef and big, Mayer's vanishing theorem [22] implies $H^1(\mu^*(B)) = 0$, so that $h^0(\mu^*(B)) = 3$ by Riemann-Roch. Let $Q \in \operatorname{Sing} \Sigma$ be the point such that

$$\frac{1}{2}c_1(\sum_{\substack{P \in \operatorname{Sing} \Sigma \\ P \neq Q}} \pi^{-1}(P)) \in H^2(S, \mathbf{Z}),$$

and define

$$\mathcal{E} = \mathcal{O}_S(\frac{1}{2}(\mu^*(B) + \sum_{P \in \operatorname{Sing} \Sigma} \pi^{-1}(P)) - \frac{1}{2} \sum_{\substack{P \in \operatorname{Sing} \Sigma \\ P \neq Q}} \pi^{-1}(P) - \pi^{-1}(Q)).$$

Then $\mathcal{E}^{\otimes 2} = \mathcal{O}_S(\mu^*(B) - \pi^{-1}(Q))$ and $\mathcal{E} \cdot \mathcal{E} = 0$. If $\mathcal{E}^{-1} \otimes \Omega_S^2 = \mathcal{E}^{-1}$ were effective, we would have

$$1 = h^0(\mathcal{O}_S(\pi^{-1}(Q))) = h^0((\mathcal{E}^{-1})^{\otimes 2} \otimes \mathcal{O}_S(\mu^*(B))) \geq h^0(\mathcal{O}_S(\mu^*(B))) = 3,$$

a contradiction. Hence, $H^2(\mathcal{E}) = H^0(\mathcal{E}^{-1} \otimes \Omega_S^2)^* = 0$, so that by Riemann-Roch, $h^0(\mathcal{E}) \geq 2$. In particular, $\mathcal{E} = \mathcal{O}_S(E)$ for some effective divisor E with $2E \equiv \mu^*(B) - \pi^{-1}(Q)$. Now $h^0(E) \geq 2$ implies that $h^0(2E) \geq 3$; since

$h^0(2E + \pi^{-1}(Q)) = 3$, we see that $\pi^{-1}(Q)$ is a fixed component of $|2E + \pi^{-1}(Q)| = |\mu^*(B)|$, which is therefore not free. *Q.E.D.*

The following computation of the invariants of a double cover of a K3 surface is essentially due to Todorov [41].

Theorem (5.2). *Let Z be a canonical surface and let $j : Z \to Z$ be an RDP-involution such that Z/j is a K3 surface with rational double points. Let $(X, \mathcal{M}, B)$ be the associated triple, let $\nu : \Sigma \to X$ be the distinguished partial desingularization, let $k = \#(\mathrm{Sing}\,\Sigma)$, let α be the 2-index of Σ, and let $d = c_1^2(Z) = (1/2)B^2$. Then*

(i) *the linear system $|B|$ is free, and $h^0(B) = d + 2$,*

(ii) *$p_g(Z) = (d-k)/4 + 3$ and $q(Z) = 0$,*

(iii) *$h^0(2K_Z) = d + (d-k)/4 + 4$, and*

(iv) *if $\tilde{Z}$ is the minimal desingularization of Z, then the 2-torsion subgroup of $\mathrm{Pic}(\tilde{Z})$ has order 2^α.*

Proof. Since $\nu^*(B) \cap \mathrm{Sing}\,\Sigma = \emptyset$, $|\nu^*(B)|$ cannot have a base point at a point of $\mathrm{Sing}\,\Sigma$, so $|B|$ is free by lemma (5.1). Let $\mu : S \to X$ and $\epsilon : \tilde{Z} \to Z$ be the minimal desingularizations. Since B and K_Z are ample on X and Z respectively, $\mu^*(B)$ and $\epsilon^*(K_Z) = K_{\tilde{Z}}$ are nef and big on S and $\tilde{Z}$ respectively. By the Kodaira-Ramanujam vanishing theorem[5] [37], keeping in mind that $K_S = 0$, we have

$$h^1(\mu^*(B)) = h^2(\mu^*(B)) = h^1(2K_{\tilde{Z}}) = h^2(2K_{\tilde{Z}}) = 0$$

so that by Riemann-Roch,

$$h^0(B) = h^0(\mu^*(B)) = \chi(\mathcal{O}_S) + \frac{1}{2}(\mu^*(B))^2 = 2 + d$$

and

$$h^0(2K_Z) = h^0(2K_{\tilde{Z}}) = \chi(\mathcal{O}_{\tilde{Z}}) + c_1^2(\tilde{Z}) = \chi(\mathcal{O}_{\tilde{Z}}) + d.$$

This proves (i), and shows that (iii) is a consequence of (ii).

To prove (ii) and (iv), note that by Bertini's theorem there is some $B_1 \in |B|$ such that $\nu^*(B_1)$ is smooth and disjoint from $\mathrm{Sing}\,\Sigma$. If we deform B to B_1 the double cover Z deforms to some surface Z_1 ; since Z and Z_1 both have rational double points, $p_g(Z) = p_g(Z_1)$ and $q(Z) = q(Z_1)$. Moreover, by the theory of "simultaneous resolution" of rational double points [7], $\tilde{Z}$ is diffeomorphic to $\tilde{Z}_1$. Since the 2-torsion subgroup of $\mathrm{Pic}(\tilde{Z})$ is a topological invariant, these subgroups of $\mathrm{Pic}(\tilde{Z})$ and $\mathrm{Pic}(\tilde{Z}_1)$ coinside. It thus suffices to prove (ii) and (iv) in the case that $\nu^*(B)$ is smooth, which we now assume.

Let $\sigma : Y \to \Sigma$ be the double cover branched on $\nu^*(B) \cup \mathrm{Sing}\,\Sigma$, and let $W = S \times_\Sigma Y$ be the fiber product with projection maps $\rho : W \to S$ and $\gamma : W \to Y$.

[5] In the cases in which we use it, this theorem was proved earlier by Mayer [22] and Kodaira [18].

Since $\nu^*(B)$ is smooth, Y coincides with the minimal desingularization $\tilde{Z}$ of Z. The map γ is the blowup of k points on Y, so that $c_1^2(W) = c_1^2(Y) - k$; since $h^0(2K_W) = h^0(2K_Y)$, Riemann-Roch immediately yields $h^1(2K_W) = k$.

On the other hand, since $\mathcal{M}^{\otimes 2} = \mathcal{O}_S(\mu^*(B) + \sum_{P\in \mathrm{Sing}\,\Sigma} \pi^{-1}(P))$, we have $\mathcal{M}^{\otimes 2} \cdot \pi^{-1}(P) < 0$ for each $P \in \mathrm{Sing}\,\Sigma$; hence, $\sum_{P\in \mathrm{Sing}\,\Sigma} \pi^{-1}(P)$ is contained in the fixed locus of $|\mathcal{M}^{\otimes 2}|$ so that $h^0(\mathcal{M}^{\otimes 2}) = h^0(\mu^*(B)) = d+2$. Since $\mathcal{M}^{\otimes 2} \cdot \mathcal{M}^{\otimes 2} = 2d - 2k$ and $h^2(\mathcal{M}^{\otimes 2}) = 0$, Riemann-Roch now shows that $h^1(\mathcal{M}^{\otimes 2}) = k$.

Now $\sigma_*\mathcal{O}_W = \mathcal{O}_S \oplus \mathcal{M}^{-1}$ and $\sigma_*\mathcal{O}_W(2K_W) = \sigma_*\sigma^*\mathcal{M}^{\otimes 2} = \mathcal{M}^{\otimes 2} \oplus \mathcal{M}$ so that

$$h^1(\mathcal{O}_W) = h^1(\mathcal{O}_S) + h^1(\mathcal{M}^{-1})$$

and

$$h^1(2K_W) = h^1(\mathcal{M}^{\otimes 2}) + h^1(\mathcal{M}).$$

Since $h^1(\mathcal{O}_S) = 0$ and $K_S = 0$, Serre duality implies that $h^1(\mathcal{O}_W) = h^1(\mathcal{M} \otimes K_S) = h^1(\mathcal{M})$. But now

$$q(Z) = h^1(\mathcal{O}_W) = h^1(\mathcal{M}^{\otimes 2}) - h^1(2K_W) = k - k = 0.$$

To compute the geometric genus, we compute the topological Euler characteristic of W . The branch curve of $\rho : W \to S$ has topological Euler characteristec $2 - 2g(\mu^*(B)) + 2k = 2k - 2d$, while S has topological Euler characteristic 24, so that

$$c_2(W) = 2 \cdot 24 - (2k - 2d) = 48 + 2d - 2k.$$

Thus,

$$12\chi(\mathcal{O}_W) = c_1^2(W) + c_2(W) = 48 + 3d - 3k$$

which immediately implies that $p_g(Z) = p_g(W) = (d-k)/4 + 3$, proving (ii).

To prove (iv), note that $\rho : W \to S$ is a double cover of smooth surfaces branched along $\alpha + 1$ smooth disjoint irreducible curves (the inverse images of B and $\mathrm{Sing}\,\Sigma$). Since $\mathrm{Pic}(S)$ has no 2-torsion, by [3; lemme 2] the 2-torsion subgroup of $\mathrm{Pic}(W)$ (which coincides with the 2-torsion subgroup of $\mathrm{Pic}(\tilde{Z})$) has order 2^α. Q.E.D

A *Todorov surface* is a canonical surface Z with $\chi(\mathcal{O}_Z) = 2$ which has an involution $j : Z \to Z$ shch that $X = Z/j$ is a K3 surface with rational double points. If Z is a Todorov surface and $\nu : \Sigma \to X$ is the distinguished partial desingularization, theorem (5.2) shows that $p_g(Z) = 1$ and $q(Z) = 0$. We call $\alpha = \log_2|\text{2-torsion in } \mathrm{Pic}(\tilde{Z})|$ and $k = c_1^2(Z) + 8$ the *fundamental invariants* of Z : by theorem (5.2) , α is the 2-index of Σ and $k = \#(\mathrm{Sing}\,\Sigma)$. Since $c_1^2(Z) > 0$, we have $k \geq 9$, while lemma (5.1) and theorem (5.2) (i) show that $(\alpha, k) \neq (1, 9)$. In addition, theorem (2.1) implies that $0 \leq \alpha \leq 5$ and

$2^{4-\alpha}(2^\alpha - 1) \le k \le \alpha + 1$. Combining all of these conditions, we find that there are 11 possible values for (α, k):

$$(\alpha, k) \in \{(0,9),(0,10),(0,11),(1,10),(1,11),(1,12),(2,12),\\ (2,13),(3,14),(4,15),(5,16)\}.$$

Examples of Todorov surfaces with each of these possible values for (α, k) can in principle be given by a method due to Tokorov [41], as follows: embed the Kummer surface Σ_0 (which was chosen in section 2) as a quartic surface in $\mathbf{P}^3$, and let $J_{\alpha,k} \subset \operatorname{Sing}\Sigma_0$ be the subset such that $\Sigma_{\alpha,k} \to \Sigma_0$ is the blowup of $\operatorname{Sing}\Sigma_0 \backslash J_{\alpha,k}$. If there is a quadric surface $Q \subset \mathbf{P}^3$ containing $\operatorname{Sing}\Sigma_0\backslash J_{\alpha,k}$ such that the proper transform B of $Q \cap \Sigma_0$ on $\Sigma_{\alpha,k}$ is smooth and disjoint from $\operatorname{Sing}\Sigma_{\alpha,k}$, then double cover of $\Sigma_{\alpha,k}$ branched of $B \cup \operatorname{Sing}\Sigma_{\alpha,k}$ will be a Todorov surface with fundamental invariants (α, k) . To guarantee that such a Q exists, one must take some care in selecting the sets $J_{\alpha,k}$, to ensure that the points in $\operatorname{Sing}\Sigma_0\backslash J_{\alpha,k}$ satisfy appropriate "general position" conditions with respect to linear systems of quadric surfaces in $\mathbf{P}^3$.

We will not pursue this method here[6], but instead will appeal to the surjectivity of the period map for K3 surfaces to show in section 7 that there is a nonempty irreducible family of Todorov surfaces with fundamental invariants (α, k) for each of the 11 possible values of (α, k) .

We need a few additional properties of Todorov surfaces.

Lemma (5.3). *Let Z be a Todorov surface, and let $j : Z \to Z$ be an involution such that $X = Z/j$ is a K3 surface with rational double points. Then*

(i) *the bicanonical map $\phi_{|2K_Z|}$ is a morphism,*

(ii) *$\phi_{|2K_Z|}$ factors through the quotient $\tau : Z \to X$, and*

(iii) *the involution j is uniquely determined by Z .*

Proof. (i) By theorem (5.2)(i), $|B|$ is free; hence, $|2K_Z| = |\tau^*(B)|$ is free so that $\phi_{|2K_Z|}$ is a morphism. (Alternatively, we could use a theorem of Francia [13] which guarantees that any canonical surface with $p_g = 1$ and $q = 0$ has a free bicanonical system).

(ii) By parts (i) and (iii) of theorem (5.2), $h^0(B) = d + 2 = h^0(2K_Z)$ so that $\phi_{|2K_Z|}$ factors through $\tau : Z \to X$. (Compare Persson [35; prop. 3.1].)

(iii) By Mayer's analysis of free linear systems on K3 surfaces [22], $\phi_{|B|}$ either embeds X , or maps X two to one onto a rational surface U . In the first case, $\deg \phi_{|2K_Z|} = 2$, so that there can be only one involution of Z through which $\phi_{|2K_Z|}$ factors. In the second case, $\deg \phi_{|2K_Z|} = 4$; by Galois theory for a biquadratic extension, if there were a second involution i through which

[6]The reader who wishes to try the exercise of selecting the sets $J_{\alpha,k}$ appropriately will need to know that the subsets of $\operatorname{Sing}\Sigma_0$ of cardinality 6 which lie in a hyperplane in $\mathbf{P}3$ occurring in section 1 of [41] and chapter 6 of [15] coincide with the subsets J occurring in proposition (6.1)(ii) below.

$\phi_{|2K_Z|}$ factored, the map $\phi_{|2K_Z|} : Z \to U$ would be a Galois cover with Galois group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and all involutions through which $\phi_{|2K_Z|}$ factored would be contained in G . The natural representation $G \to \mathrm{Aut}(H^0(K_Z)) \cong \mathbf{C}^*$ would then be non-trivial (since $p_g(U) = 0$) , but any involution whose quotient is a K3 surface would lie in the kernel of that representation. Since G is generated by such involutions, we get a contradiction; hence, the involution j is unique. Q.E.D

The free linear system $|B|$ on X is called *hyperelliptic* when $\phi_{|B|}$ maps X two to one onto a rational surface U ; this happens exactly when the bicanonical map of the corresponding Todorov surface has degree 4 . The following lemma gives a sufficient condition for this to happen; a partial converse will be proved in section 7.

Lemma (5.4). *Let Z be a Todorov surface with fundamental invariants (α, k) . If $(\alpha, k) = (0,9)$ or $(1,10)$ then the bicanonical map $\phi_{|2K_Z|}$ has degree 4 and the linear system $|B|$ on $X = Z/j$ is hyperelliptic.*

Proof. If $k = 9$, then $B^2 = 2$ and $h^0(2K_Z) = h^0(B) = 3$; it follows that $\phi_{|2K_Z|}$ and $\phi_{|B|}$ are maps of degree 4 and 2 respectively onto $\mathbf{P}^2$.

If $(\alpha, k) = (1,10)$, let $\nu : \Sigma \to X$ be the distinguished partial desingularization, and $\pi : S \to \Sigma$ be the minimal desingularization. There are points $P_1, P_2 \in \mathrm{Sing}\,\Sigma$ such that

$$\frac{1}{2} c_1 \Big(\sum_{\substack{P \in \mathrm{Sing}\,\Sigma \\ P \neq P_1, P_2}} \pi^{-1}(P) \Big) \in H^2(S, \mathbf{Z}).$$

This implies that $(1/2)c_1(B - \pi^{-1}(P_1) - \pi^{-1}(P_2)) \in H^2(S, \mathbf{Z})$ as well. Let $\mathcal{E} = \mathcal{O}_S((1/2)(B - \pi^{-1}(P_1) - \pi^{-1}(P_2)))$. Then $\mathcal{E} \cdot \mathcal{E} = 0$ so by Riemann-Roch $\mathcal{E}$ or $\mathcal{E}^{-1}$ is effective. Since $B \cdot \mathcal{E} = 2$ and B is nef, $\mathcal{E}^{-1}$ cannot be effective; thus, $\mathcal{E} = \mathcal{O}_S(E)$ for some effective divisor E , and the linear system $|E|$ has (projective) dimension at least 1 .

Let $|E_0|$ be the moving part of $|E|$. Since B is nef, $B \cdot (E - E_0) \geq 0$ so that $0 \leq B \cdot E_0 \leq 2$. If $B \cdot E_0 = 0$ then the Hodge index theorem implies $E_0^2 < 0$, but a curve of negative self-intersection on a K3 surface cannot move. Thus, $B \cdot E_0 = 1$ or 2 . But then every smooth curve $B' \in |B|$ has a linear system $|E_0 \cap B'|_{B'}$ of dimension at least 1 and degree 1 or 2 . Since B' is not rational, the degree must be 2 and B' must be hyperelliptic. Hence, $\phi_{|B|}$ (which induces the canonical map on B') must have degree 2 . Q.E.D

§6. Embeddings of Todorov lattices.

Let $(\Sigma, \mathcal{L})$ be a pair consisting of K3 surface Σ with ordinary double points and a line bundle $\mathcal{L}$ on Σ with $\mathcal{L} \cdot \mathcal{L} = 2(\#\,\mathrm{Sing}\,\Sigma) - 16 > 0$ such that if $\pi : S \to \Sigma$

is the minimal desigularization, then

$$\frac{1}{2}c_1(\pi^*(\mathcal{L})\otimes \mathcal{O}_S(\sum_{P\in \mathrm{Sing}\,\Sigma}\pi^{-1}(P)))\in H^2(S,\mathbf{Z}).$$

(Such a pair can be obtained from the quotient $X = Z/j$ of a Todorov surface by taking $\nu : \Sigma \to X$ to be the distinguished partial desingularization, and $\mathcal{L} = \mathcal{O}_\Sigma(\nu^*(B))$.) For such a pair $(\Sigma, \mathcal{L})$, we let $M_{\Sigma,\mathcal{L}}$ be the saturation in $H^2(S,\mathbf{Z})$ of the lattice generated by $c_1(L_\Sigma)$ and $c_1(\mathcal{L})$.

Let $\Sigma_{\alpha,k}$ be the fixed partial desingularization of a Kummer surface Σ_0 with 2-index α and k ordinary double points which was chosen in section 2. Recall that we chose Σ_0 to be the Kummer surface of a principally polarized abelian surface A_0, so that there is a map $f : A_0 \to \Sigma_0$ and a line bundle $\mathcal{N}_0$ on Σ_0 such that $f^*\mathcal{N}_0 = \Theta_0^{\otimes 2}$, where Θ_0 is the principal polarization on A_0.

Let $\pi_0 : S_0 \to \Sigma_0$ be the minimal desingularization, and let $\pi_{\alpha,k} : S_0 \to \Sigma_{\alpha,k}$ and $\nu_{\alpha,k} : \Sigma_{\alpha,k} \to \Sigma_0$ be the natural maps. If we let

$$\mathcal{L}_{\alpha,k} = \nu_{\alpha,k}^*(\mathcal{N}_0^{\otimes 2})\otimes \mathcal{O}_{\Sigma_{\alpha,k}}(\sum_{\substack{P\in \mathrm{Sing}\,\Sigma_0\\ P\notin \mathrm{Sing}\,\Sigma_{\alpha,k}}}\nu_{\alpha,k}^{-1}(P)),$$

then $\mathcal{L}_{\alpha,k}\cdot\mathcal{L}_{\alpha,k} = 2k-16$. Moreover, since Σ_0 is a Kummer surface,

$$\frac{1}{2}c_1(\pi_{\alpha,k}^*(\mathcal{L}_{\alpha,k})\otimes \mathcal{O}_{s_0}(\sum_{P\in \mathrm{Sing}\,\Sigma_{\alpha,k}}\pi_{\alpha,k}^{-1}(P)) =$$

$$c_1(\pi_0^*\mathcal{N}_0) + \frac{1}{2}c_1(\sum_{P\in \mathrm{Sing}\,\Sigma_0}\pi_0^{-1}(P)) \in H^2(S_0,\mathbf{Z}).$$

Thus, when $k \geq 9$ the pair $(\Sigma_{\alpha,k}, \mathcal{L}_{\alpha,k})$ satisfies the hypotheses above. We define $M_{\alpha,k} = M_{\Sigma_{\alpha,k},\mathcal{L}_{\alpha,k}}$, and call this a *Todorov lattice* when $k \geq 9$ and $(\alpha, k) \neq (1,9)$.

If $(\Sigma,\mathcal{L})$ is a pair satisfing the hypotheses above, we let $\lambda = c_1(\pi^*\mathcal{L})$, let $e_P = c_1(\pi^{-1}(P))$ for each $P \in \mathrm{Sing}\,\Sigma$, and let $\mu = (1/2)(\lambda + \sum_{P\in\mathrm{Sing}\,\Sigma} e_P) \in H^2(S,\mathbf{Z})$.

Proposition (6.1). *Let $(\Sigma,\mathcal{L})$ be a pair satisfying the hypotheses above, let α be the 2-index of Σ, and let $k = \#(\mathrm{Sing}\,\Sigma)$.*

(i) *If $(\alpha,k) \neq (5,16)$ then $M_{\Sigma,\mathcal{L}}$ is generated by $c_1(L_\Sigma)$ and μ.*

(ii) *If $(\alpha,k) = (5,6)$, then $M_{\Sigma,\mathcal{L}}$ is generated by $c_1(L_\Sigma)$, μ, and an element of the form $(1/4)+(1/2)\sum_{P\in J} e_P$, where $J \subset \mathrm{Sing}\,\Sigma$ is a collection of 6 points such that for any choice of $\mathbf{F}_2$-vector space structure on $\mathrm{Sing}\,\Sigma$ compatible with the code $\mathbf{C}_\Sigma$, every hyperplane in $\mathrm{Sing}\,\Sigma$ contain either 2 or 4 points of J.*

(iii) *There is an isomorphism of binary linear codes $\mathbf{C}_\Sigma \xrightarrow{\sim} \mathbf{C}_{\alpha,k}$ such that the induced isometry $c_1(L_\Sigma) \xrightarrow{\sim} c_1(L_{\alpha,k})$ extends to an isometry $M_{\Sigma,\mathcal{L}} \xrightarrow{\sim} M_{\alpha,k}$ which sends $c_1(\pi^*\mathcal{L})$ to $c_1(\pi^*\mathcal{L}_{\alpha,k})$.*

Proof. Let $\tilde{M}_{\Sigma,\mathcal{L}}$ be the lattice generated by $c_1(L_\Sigma)$ and μ. We first claim that if $(\alpha, k) = (5, 16)$, then $M_{\Sigma,\mathcal{L}}$ is strictly larger than $\tilde{M}_{\Sigma,\mathcal{L}}$. For in the case, $c_1(L_\Sigma)$ contains $1/2 \sum_{P \in \mathrm{Sing}\,\Sigma} e_P$ so that $\tilde{M}_{\Sigma,\mathcal{L}}$ is generated by $c_1(L_\Sigma)$ and $\nu = \mu - 1/2 \sum_{P \in \mathrm{Sing}\,\Sigma} e_P$. Now $\nu \cdot \nu = 4$ and $\nu \cdot e_P = 0$ for each P, so $\tilde{M}_{\Sigma,\mathcal{L}}$ splits as an orthogonal direct sum $< \nu > \oplus c_1(L_\Sigma)$, where $< \nu >$ denotes the span of ν. This implies that $\tilde{M}^*_{\Sigma,\mathcal{L}}/\tilde{M}_{\Sigma,\mathcal{L}}$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} \oplus L^*_\Sigma/L_\Sigma$. Now $E^*_\Sigma/E_\Sigma \cong (\mathbf{Z}/2\mathbf{Z})^{16}$ while $\log_2[L_\Sigma : E_\Sigma] = \alpha = 5$ so that $L^*_\Sigma/L_\Sigma \cong (\mathbf{Z}/2\mathbf{Z})^6$; in particular, the finite abelian group $\tilde{M}^*_{\Sigma,\mathcal{L}}/\tilde{M}_{\Sigma,\mathcal{L}}$ has 7 generators. But since $\tilde{M}_{\Sigma,\mathcal{L}}$ has rank 17, this impiles that $\tilde{M}_{\Sigma,\mathcal{L}}$ admits no primitive embedding into K3 lattice Λ (which has rank 22). But $M_{\Sigma,\mathcal{L}}$ is primitively embedded in $H^2(S, \mathbf{Z}) \cong \Lambda$; hence, $M_{\Sigma,\mathcal{L}} \neq \tilde{M}_{\Sigma,\mathcal{L}}$.

Returning to the general case, let x be an element of $M_{\Sigma,\mathcal{L}}$ and write

$$x = \frac{a}{2k-16}\lambda + \frac{1}{2}\sum b_P e_P$$

with $a, b_P \in \mathbf{Q}$. Now

$$\begin{aligned} x \cdot e_P &= -b_P \\ x \cdot \mu &= \frac{1}{2}a - \frac{1}{2}\sum b_P \\ \frac{1}{2}x \cdot x &= \frac{a^2}{4(k-8)} - \frac{1}{4}\sum (b_P)^2 \end{aligned}$$

and all of these quantities must be integers. In paticular, a and each b_P are integers as is $a^2/(k-8)$. Since $1 \leq k - 8 \leq 8$, there are three cases:

(1) $a \equiv 0 \bmod k - 8$
(2) $k = 12$ and $a \equiv 2 \bmod 4$
(3) $k = 16$ and $a \equiv 4 \bmod 8$.

In the first case, write $a = (k-8)c$ with $c \in \mathbf{Z}$. Then

$$x - c\mu = \frac{1}{2}\sum (b_P - c)e_P$$

is in the saturation of $c_1(E_\Sigma)$, and hence lies in $c_1(L_\Sigma)$, so that $x \in \tilde{M}_{\Sigma,\mathcal{L}}$. In the second case, write $a = 4c + 2$ with $c \in \mathbf{Z}$. Then $a \in 2\mathbf{Z}$ so that $\sum b_P \in 2\mathbf{Z}$; on the other hand,

$$\frac{a^2}{4(k-8)} - \frac{1}{4} \in \frac{1}{2}\mathbf{Z}$$

so that $(1/4)\sum (b_P)^2 \notin (1/2)\mathbf{Z}$ a contradition. Thus, whenever $k \leq 15$ we have $M_{\Sigma,\mathcal{L}} = \tilde{M}_{\Sigma,\mathcal{L}}$, which proves (i). (Recall that $k = 16$ implies $\alpha = 5$).

We now assume $(\alpha, k) = (5, 16)$; if x falls in the case (1) above then $x \in \tilde{M}_{\Sigma,\mathcal{L}}$. If x falls in case (3), write $a = 8c + 4$ with $c \in \mathbf{Z}$, and let $J = \{P \mid$

b_P is odd}. Then

$$x - c\mu - \sum[\frac{b_P - c}{2}]e_P = \frac{1}{4}\lambda + \frac{1}{2}\sum_{P\in J} e_P$$

so that element of $M_{\Sigma,\mathcal{L}}\setminus\tilde{M}_{\Sigma,\mathcal{L}}$ is congruent mod $\tilde{M}_{\Sigma,\mathcal{L}}$ to an element of the form $y_J = (1/4)\lambda + (1/2)\sum_{P\in J} e_P$. Moreover, if $y_{J'} = (1/4)\lambda + (1/2)\sum_{P\in J'} e_P$ is another such element, then their difference is

$$\frac{1}{2}\sum_{P\in J} e_P - \frac{1}{2}\sum_{P\in J'} e_P \in c_1(L_\Sigma).$$

Thus, $[M_{\Sigma,\mathcal{L}} : \tilde{M}_{\Sigma,\mathcal{L}}] = 2$.

Fix a subset $J \subset \operatorname{Sing}\Sigma$ such that y_J generates $M_{\Sigma,\mathcal{L}} \setminus \tilde{M}_{\Sigma,\mathcal{L}}$; by adding $(1/2)\sum_{P\in\operatorname{Sing}\Sigma} e_P$ if necessary, we may assume $\#(J) \geq 8$. Now $(1/2)y_J \cdot y_J = (1/2) - (1/4)\#(J)$ so that $\#(J) \equiv \text{mod} 4$. If $J = \{Q, R\}$ contains of 2 points, let $H \subset \operatorname{Sing}\Sigma$ be a hyperplane (with respect to some fixed $\mathbf{F}_2$-vector space structre) which contains Q but does not contain R. Then $(1/2)\sum e_P \in c_1(L_\Sigma)$ so that $y_{J'} = y_J + (1/2)\sum_{P\notin H} e_p - e_R \in M_{\Sigma,\mathcal{L}}$, where $J' = \{Q\}\cup(H\setminus\{R\})$. But then $\#(J') = 8 \not\equiv \text{mod} 4$, which is a contradiction. Thus, $\#(J) = 6$; moreover, for any J' with $y_{J'} \in M_{\Sigma,\mathcal{L}}$ we have $\#(J') = 6$ or 10.

To finish the proof of (ii), let H be any hyperplane in $\operatorname{Sing}\Sigma$ (so that $(1/2)\sum_{P\notin H} e_P \in c_1(L_\Sigma)$) and let $J' = (H \cap J) \cup (\operatorname{Sing}\Sigma \setminus (H \cup J))$. Then

$$y_{J'} = y_J = \frac{1}{2}\sum_{P\notin H} e_P - \sum_{P\notin J\setminus H} e_P \in M_{\Sigma,\mathcal{L}}$$

while $\#(J') = 2 + 2\#(H \cap J)$. Thus, $\#(H \cap J) = 2$ or 4, proving (ii).

Finally, by theorem (2.1) there is an isomorphism of binary linear codes $\mathcal{C}_\Sigma \xrightarrow{\sim} \mathcal{C}_{\alpha,k}$ which induces an isometry $L_\Sigma \xrightarrow{\sim} L_{\alpha,k}$; this clearly extends to an isometry $\tilde{M}_{\Sigma,\mathcal{L}} \xrightarrow{\sim} \tilde{M}_{\Sigma_{\alpha,k},\mathcal{L}_{\alpha,k}}$. When $k \leq 15$, (iii) now follows from (i). In the remaining case $(\alpha, k) = (5, 16)$ we must show: for any two subsets J, J' of $\operatorname{Sing}\Sigma_{5,16}$ satisfying the condition in (ii), there is an automorphism σ of the binary linear code $\mathcal{C}_{5,16} \cong \mathcal{D}_5$ such that $\sigma(J) = J'$. Recall that by lemma (1.4), $\operatorname{Aut}(\mathcal{D}_5) = \operatorname{AGL}(\operatorname{Sing}\Sigma_{5,16})$.

Fix a vector space structure on $\operatorname{Sing}\Sigma_{5,16}$ with zero-vector P_0, fix a basis $P_1, \ldots, P_4$ of the vector space, and let $P_5 = P_1 + P_2 + P_3 + P_4$ (vector space addition). We will show that any J satisfying (ii) can be mapped to $\{P_0, \ldots, P_5\}$ by an affine linear transformation. (It is easy to see that $\{P_0, \ldots, P_5\}$ satisfies (ii).) First, choose an affine transformation T so that the zero-vector P_0 lies in $T(J)$. Let $R_1, \ldots, R_4$ be 4 elements of $T(J)$ distinct from P_0; since no hyperplane contains 5 or 6 elements of $T(J)$, the set $\{P_0, R_1, \ldots, R_4\}$ is not contained in a hyperplane; hence, $R_1, \ldots, R_4$ is a basis of $\operatorname{Sing}\Sigma_{5,16}$. Let γ be the linear transformation such that $\gamma(R_i) = P_i$ for $i = 1, \ldots, 4$ for some point Q, $Q \neq P_i$.

For each $i = 1, \ldots, 4$ consider the hyperplane H_i spanned by the set of 4 point $\{P_j \mid j \neq i\}$. Since $\#(H_i \cap J) = 2$ or 4 we see thst $Q \notin H_i$. But the only point of $\operatorname{Sing}\Sigma_{5,16}$ not lying in $H_1 \cup \cdots \cup H_4$ is P_5; hence $Q = P_5$ and $\gamma \circ T(J) = \{P_0, \ldots, P_5\}$. *Q.E.D.*

Recall that a *finite quadratic form* is a finite abelian group G together with a map $q : G \to \mathbf{Q}/2\mathbf{Z}$ such that (1) $q(nx) = n^2 q(x)$ for $x \in G$, $n \in \mathbf{Z}$ and (2) $b(x,y) = (1/2)(q(x+y) - q(x) - q(y))$ defines a bilinear map of $\mathbf{Z}$-modules $b : G \times G \to \mathbf{Q}/\mathbf{Z}$. We call a finite quadratic form *special* if $2x = 0$ impiles $q(x) \in \mathbf{Z} \bmod 2\mathbf{Z}$ for all $x \in G$. Note that if G is a 2-elementary group and $x_1, \ldots, x_n$ is a generating set for G, then q is special if and only if $q(x_i) \in \mathbf{Z} \bmod 2\mathbf{Z}$ for all $i = 1, \ldots, n$.

If a and l are natural numbers with $2|al$ and $(a,l) = 1$, we let z_l^a denote the finite quadratic form $(\mathbf{Z}/l\mathbf{Z}, q)$ where $q(x) = a/l \bmod 2\mathbf{Z}$ for some generator x of $\mathbf{Z}/l\mathbf{Z}$.

The intersection form on the Todorov lattice $M_{\alpha,k}$ induces an adjoint map $\operatorname{ad} : M_{\alpha,k} \to M^*_{\alpha,k} = \operatorname{Hom}(M_{\alpha,k}, \mathbf{Z})$; we define $G_{\alpha,k}$ to be the cokernel of this map. $G_{\alpha,k}$ is a finite abelian group which inherits the structure of a finite quadratic form $(G_{\alpha,k}, q_{\alpha,k})$ (called the *discriminat-form*) from $M_{\alpha,k}$ by the following procedure: for each $x \in M^*_{\alpha,k}$ there is some $n \in \mathbf{Z}$ and some $y \in M_{\alpha,k}$ such that $\operatorname{ad}(y) = nx$; one defines $q_{\alpha,k}(x) = (1/n^2) y \cdot y \bmod 2\mathbf{Z}$.

Proposition (6.2). *Let $M_{\alpha,k}$ be a Todorov lattice.*

(i) *There is a special 2-elementary quadratic form $(G'_{\alpha,k}, q'_{\alpha,k})$ of rank $2s$ and an orthogonal direct sum decomposition $(G_{\alpha,k}, q_{\alpha,k}) \cong z_l^a \oplus (G'_{\alpha,k}, q'_{\alpha,k})$, where*

$$(2s, l, a) = \begin{cases} (k - 2\alpha - 1, k - 8, 2) & \text{if } k \text{ is odd} \\ (k - 2\alpha, 2k - 16, k - 7) & \text{if } k \text{ is even, } k < 16 \\ (4, 4, 1) & \text{if } (\alpha, k) = (5, 16) \end{cases}$$

Note that since $k \geq 9$ and (α, k) satisfies (2.1)(i), $2s \geq 4$ in all cases.

(ii) *If $(\alpha, k) \neq (5, 16)$ and $\sigma \in \operatorname{Aut}(\mathcal{C}_{\alpha,k})$, consider the automorphism of $M_{\alpha,k}$ defined by $\lambda \mapsto \lambda$, $e_p \mapsto e_{\sigma(P)}$. If the induced automorphism of $G_{\alpha,k}$ coincides with the action of -1, then $(\alpha, k) = (0, 9)$ or $(1, 10)$.*

Proof. (i) We first treat the case $(\alpha, k) = (5, 16)$. Let A_0 be the principally polarized abelian surface used to construct $\Sigma_0 = \Sigma_{5,16}$, let $f : A_0 \to \Sigma_0$ be the quotient map and let $\pi_0 : S_0 \to \Sigma_0$ be the minimal desigularization. Since $\operatorname{End}(A_0) = \mathbf{Z}$, the principal polarization Θ_0 generates $\operatorname{NS}(A_0)$, and $\Theta_0 \cdot \Theta_0 = 2$. Let $T(A_0)$ be the transcendental lattice, that is, the orthogonal complement of $\operatorname{NS}(A_0)$ in $H^2(A_0, \mathbf{Z})$. The Kneser-Nikulin uniquness theorem [31;corollary 1.13.3] immediately implies that $T(A_0) \cong U^{\oplus 2} \oplus T'$ (cf. [24;corollary 2.6]) where U is the hyperbolic plane and T' is infinite cyclic with generator y such that $y \cdot y = -2$.

f induces a map $f_* : T(A_0)(2) \to H^2(S_0, \mathbf{Z})$ preserving quadratic forms (where $L(2)$ denote multiplication of the quadratic form L by 2) whose image has finite index in $T(S_0)$. If $n = [T(S_0) : T(A_0)(2)]$ then the discriminant of $T(S_0)$ is $n^2 \cdot disc(U(2)^{\oplus 2} \oplus T'(2)) = -2^6 n^2$. On the other hand, the discriminant of $\mathrm{NS}(S_0) = M_{5,16}$ is $(1/4)disc(\tilde{M}_{5,16}) = (1/4) \cdot 4disc(L_{5,16}) = 2^6$. Since these discriminants have the same absolute value, we must have $n = 1$.

This implies that $(G_{5,16}, q_{5,16})$ is isomorphic to the discriminant-form of $T(A_0)(2)$ by an isomorphism reversing the sign on the quadratic form. Let $(G'_{5,16}, q'_{5,16})$ be the inverse image of the discriminant-form $U(2)^{\oplus 2}$ under this isomorphism, and let x be the inverse image of $\mathrm{ad}((1/4)y')$ where y' is the image of y in $T(A_0)(2)$. Then $(G'_{5,16}, q'_{5,16})$ is a special 2-elementary form of rank 4, and $q_{5,16}(x) = (-1/16)y' \cdot y' = 1/4$ so that the subgroup $< x >$ of $G_{5,16}$ generated by x is isomorphic to z_4^1. The orthogonal direct sum decomposition $T(A_0)(2) \cong U(2)^{\oplus 2} \oplus T'(2)$ induces an orthogonal direct sum decomposition $(G_{5,16}, q_{5,16}) \cong (G'_{5,16}, q'_{5,16}) \oplus < x >$, proving (i) in this case.

We assume for the rest of the proof that $k \leq 15$ so that $M_{\alpha,k} = \tilde{M}_{\alpha,k}$ by proposition (6.1). We abbreviate $q_{\alpha,k}$ by q, and let b be the associated bilinear form. Let $\hat{M}_{\alpha,k}$ be the span of μ and $\{e_P \mid P \in \mathrm{Sing}\,\Sigma_{\alpha,k}\}$, and let μ^* and $\{e_P^*\}$ be the dual basis of $\hat{M}^*_{\alpha,k} = \mathrm{Hom}(\hat{M}_{\alpha,k}, \mathbf{Z})$. We will first compute $\hat{G}_{\alpha,k} = \hat{M}^*_{\alpha,k} / \mathrm{ad}(\hat{M}_{\alpha,k})$.

Fix a point $Q \in \mathrm{Sing}\,\Sigma_{\alpha,k}$ and define $\xi_P = e_Q^* - e_P^* \in \hat{M}_{\alpha,k}$. μ^* and $\{e_P^*\}$ generate $\hat{G}_{\alpha,k}$, with relations:

$$\mathrm{ad}(\mu) = -4\mu^* - \sum_P e_P^* = -4\mu^* - ke_Q^* + \sum_P \xi_P$$
$$\mathrm{ad}(e_P) = -\mu^* - 2e_P^* = -\mu^* - 2e_Q^* + 2\xi_P.$$

We fix a second point $R \neq Q$ in $\mathrm{Sing}\,\Sigma_{\alpha,k}$, and consider the basis of $\hat{M}_{\alpha,k}$ consisting of e_Q, $\mu - 4e_Q$, $-2\mu + \sum_P e_P + (8-k)e_Q$, $\{e_P - e_Q \mid P \neq Q, R\}$. Then one easily computes (using the fact that $\xi_Q = 0$):

$$\mathrm{ad}(e_Q) = -\mu^* - 2e_Q^*$$
$$\mathrm{ad}(\mu - 4e_Q) = (8-k)e_Q^* + \sum_{P \neq Q} \xi_P$$
$$\mathrm{ad}(-2\mu + \sum_P e_P + (8-k)e_Q) = (2k-16)e_Q^*$$
$$\mathrm{ad}(e_P - e_Q) = 2\xi_P$$

This implies that $\hat{G}_{\alpha,k}$ is generated by e_Q^* and $\{\xi_P \mid P \neq Q\}$, subject to the relations $(2k-16)e_Q^* \equiv 2\xi_P \equiv 0$ and $\sum_{p \neq Q} \xi_P \equiv (k-8)e_Q^*$. This presentation also makes it easy to compute the quadratic form on $\hat{G}_{\alpha,k}$. For example, $b(\xi_{P_1}, \xi_{P_2}) = (1/4)(e_{P_1} - e_Q) \cdot (e_{P_2} - e_Q) \equiv (-1/2) \bmod \mathbf{Z}$ if $P_1 \neq P_2$, $P_i \neq Q$. The complete results of the computation are the following:

$$b(\xi_{P_1}, \xi_{P_2}) \equiv \frac{1}{2} \bmod \mathbf{Z} \quad \text{if } P_1 \neq P_2,\ P_i \neq Q$$

$$b(\xi_P, e_Q^*) \equiv \frac{1}{2} \bmod \mathbf{Z} \quad \text{if } P \neq Q$$
$$q(\xi_P) \equiv 1 \bmod 2\mathbf{Z} \quad \text{if } P \neq Q$$
$$q(e_Q^*) \equiv \frac{9-k}{2k-16} \bmod 2\mathbf{Z}.$$

Let V denote the subgroup $\mathrm{ad}(M_{\alpha,k})/\mathrm{ad}(\hat{M}_{\alpha,k})$ of $\hat{G}_{\alpha,k}$, so that $G_{\alpha,k} = V^\perp/V$. Since $M_{\alpha,k}/\hat{M}_{\alpha,k} = L_{\alpha,k}/E_{\alpha,k}$, every element of V has the form $\mathrm{ad}(c_1(\Gamma_\phi))$, where $\phi \in \mathbf{F}_2{}^{\mathrm{Sing}\,\Sigma_{\alpha,k}}$ belongs to the double point code of $\Sigma_{\alpha,k}$ and $c_1(\Gamma_\phi) = (1/2)\sum_{P\in\mathrm{Sing}\,\Sigma_{\alpha,k}} e_P$. If we let $I = I_\phi = \{P \in \mathrm{Sing}\,\Sigma_{\alpha,k} \mid \phi(P) = 1\}$ then $\#(I) = 0$ or 8 and

$$\mathrm{ad}(c_1(\Gamma_\phi)) = \mathrm{ad}(4e_Q) - \mathrm{ad}(\frac{1}{2}\sum_{P\in I}(e_Q - e_P)) \equiv \sum_{P\in I}\xi_P \bmod \mathrm{ad}(\hat{M}_{\alpha,k}).$$

For such sets I, let $\eta_I = \sum_{P\in I}\xi_P$.

Suppose that k is odd. Then the 2-Sylow subgroup $\hat{G}'_{\alpha,k}$ of $\hat{G}_{\alpha,k}$ is generated by $(k-8)e_Q^*$ and $\{\xi_P\}$, while the sum of the remaining Sylow subgroups is generated by $x = 2e_Q^*$. The Sylow decomposition then induces an orthogonal direct sum decomposition $(\hat{G}_{\alpha,k}, q_{\alpha,k}) \cong (\hat{G}'_{\alpha,k}, q'_{\alpha,k}) \oplus < x >$, where $< x >$ denotes the subgroup generated by x. Moreover, since

$$q(x) = 4q(e_Q^*) \equiv 2/(k-8) \bmod 2\mathbf{Z},$$

we have $< x > \cong z_{k-8}^2$.

Since each η_I belongs to $\hat{G}'_{\alpha,k}$ we have $V \subset \hat{G}'_{\alpha,k}$; if we define $\hat{V}^\perp$ to be the orthogonal complement of V in $\hat{G}'_{\alpha,k}$ and $G'_{\alpha,k} = \hat{V}^\perp/V$, then there is an orthogonal direct sum decomposition $(G_{\alpha,k}, q_{\alpha,k}) \cong (G'_{\alpha,k}, q'_{\alpha,k}) \oplus < x >$. From the values of q and b on the generators of $\hat{G}'_{\alpha,k}$ it follows that $(\hat{G}'_{\alpha,k}, q'_{\alpha,k})$ is a special 2-elementary form; since $G'_{\alpha,k}$ is a subquotient of $\hat{G}'_{\alpha,k}$, $(G'_{\alpha,k}, q'_{\alpha,k})$ has the same property. Finally, by computing the discriminant, we see that $G'_{\alpha,k}$ has rank $k - 2\alpha - 1$, proving (i) when k is odd.

Suppose now that k is even, $k < 16$. We choose a subset J of $\mathrm{Sing}\,\Sigma_{\alpha,k}$, not containing Q, in the following manner: if $\alpha = 0$, let J be any 3 points of $\Sigma_{\alpha,k}$ distinct from Q. If $\alpha = 1$ and $\eta_I \in V$ is the nontrivial element, let J consist of 2 points in I and 1 point not in I with $Q \notin J$; this is possible since $k \geq 10$ so that $\#(\mathrm{Sing}\,\Sigma_{\alpha,k} \setminus I) \geq 2$. If $\alpha = 2$ or 3, let W be an $\mathbf{F}_2$-vector space of dimension α let $\tau : \mathrm{Sing}\,\Sigma_{\alpha,k} \to W$ be a surjective map such that $\tau^*(\mathcal{U}_\alpha)$ is the double point code of $\Sigma_{\alpha,k}$; such maps exist by theorem (1.1). Choose J so that $Q \notin J$, $J \cap \tau^{-1}(0) = \emptyset$, and for each nonzero $w \in W$, $J \cap \tau^{-1}(w)$ consists of a single point.

The subset J has the following properties: $\#(J) \equiv 3 \bmod 4$, and for each $I \subset \mathrm{Sing}\,\Sigma_{\alpha,k}$ such that $\eta_I \in V$; the set $I \cap J$ contains an even number of points.

We let $f_J = e^*_Q + \sum_{P\in J} \xi_P$. The properties of J enable us to compute:

$$b(f_J, \xi_P) \equiv \begin{cases} \frac{1}{2} \bmod \mathbf{Z} & \text{if } P \in J \\ 0 \bmod \mathbf{Z} & \text{if } P \notin J,\ P \neq Q \end{cases}$$

$$b(f_J, (k-8)e^*_Q) \equiv \frac{1}{2} \bmod \mathbf{Z}$$

$$q(f_J) \equiv \frac{9-k}{2k-16} - 3 \equiv \frac{k-7}{2k-16} \bmod 2\mathbf{Z}.$$

The subgroup $< f_J >$ of $(G_{\alpha,k}, q_{\alpha,k})$ generated by f_J is thus isomorphic to z^{k-7}_{2k-16}.

Let $\zeta_P = (k-8)e^*_Q + \xi_P$, and let $\hat{G}'_{\alpha,k}$ be the subgroup of $\hat{G}_{\alpha,k}$ generated by $\{\zeta_P \mid P \in J\}$ and $\{\xi_P \mid P \notin J\}$. We claim there is an orthogonal direct sum decomposition $(\hat{G}_{\alpha,k}, q_{\alpha,k}) = (\hat{G}'_{\alpha,k}, q'_{\alpha,k}) \oplus < f_J >$. It is clear that $\hat{G}'_{\alpha,k} \perp f_J$; we must show that $\hat{G}'_{\alpha,k} + < f_J > = \hat{G}_{\alpha,k}$. Since k is even, $k-7$ is a unit mod $(2k-16)$ so that $e^*_Q \in \hat{G}'_{\alpha,k} + < f_J >$. But then $\xi_P \in \hat{G}'_{\alpha,k} + < f_J >$ for each P, so that $\hat{G}'_{\alpha,k} + < f_J > = \hat{G}_{\alpha,k}$.

It remains to show that $\hat{G}'_{\alpha,k}$ is a special 2-elementary form and that $V \subset \hat{G}'_{\alpha,k}$; the rest of the proof is then the same as in the case of odd k. For the first statement, we simply compute

$$q(\zeta_P) \equiv \frac{(k-8)(9-k)}{2} + 2(k-8)(\frac{1}{2}) + 1 \bmod 2\mathbf{Z}$$

so that $q(\zeta_P) \in \mathbf{Z} \bmod 2\mathbf{Z}$. For the second statement, if $\eta_I \in V$ let $\#(I \cap J) = 2m$; then

$$\eta_I \equiv \sum_{P\in I\setminus I} \xi_P + \sum_{P\in J\setminus I} +2m(k-8)e^*_Q = \sum_{P\in I\setminus J} \xi_P + \sum_{P\in J\setminus I} \zeta_P \in \hat{G}'_{\alpha,k};$$

hence $V \subset \hat{G}'_{\alpha,k}$, proving (i).

(ii) Let $\sigma \in \mathrm{Aut}(\mathbf{C}_{\alpha,k})$ be an element whose induced action on $G_{\alpha,k}$ coincides with the action of -1. The natural action of σ sends e^*_Q to $e^*_Q - \xi_{\sigma(Q)}$ and sends ξ_P to $\xi_{\sigma(P)} - \xi_{\sigma(Q)}$.

Fisrt, suppose that k is odd. Since σ acts as -1 on $G_{\alpha,k}$, $e^*_Q + e^*_Q - \xi_{\sigma(Q)} \equiv 0$ in $G_{\alpha,k}$; this implies that $2e^*_Q - \xi_{\sigma(Q)} \in V$. In particular, since every element of V has order 2, we have $4e^*_Q \equiv 0$ in $\hat{G}_{\alpha,k}$. Now the order of e^*_Q in $\hat{G}_{\alpha,k}$ is $2k-16$; hence, $2k-16$ divides 4. Since $k \geq 9$ is odd, this implies that $k = 9$; α must then be 0 since $M_{\alpha,k}$ is a Todorov lattice.

Suppose instead that k is even; by hypothesis, $k < 16$. Let $J \subset \mathrm{Sing}\,\Sigma_{\alpha,k}$ be the subset chosen in the proof of (i). Since $\#(J)$ is odd, any $\sigma \in \mathrm{Aut}(\mathbf{C}_{\alpha,k})$ sends f_J to $f_{\sigma(J)}$. If σ acts as -1 on $G_{\alpha,k}$, we see as above that $f_J + f_{\sigma(J)} = 2f_J - \sum_{P\in J} \xi_P + \sum_{P\in\sigma(J)} \xi_P \in V$. Since each ξ_P and every element of V has order 2, this implies that $4f_J \equiv 0$ in $\hat{G}_{\alpha,k}$. Now f_J has order $2k-16$, so that $2k-16$ divides 4 as before; since k is even, we find that $k = 10$.

The construction of J now implies that $\#(J) = 3$. Thus, $\#(J \cup \sigma(J) \cup \{Q\}) \leq 7$ so that $\sum_{P\in J} \xi_P - \sum_{P\in\sigma(J)} \xi_P \neq \sum_P \xi_p = 2e_Q^*$. Hence, $f_J + f_{\sigma(J)} = 2e_Q^* - \sum_{P\in J} \xi_P + \sum_{P\in\sigma(J)} \xi_P$ is a nontrivial element of V; this implies that $\alpha \geq 1$. Since $M_{\alpha,k}$ is a Todorov lattice, $(\alpha, k) = (1, 10)$. *Q.E.D.*

Let L be a nondegenerate lattice of signature (r_+, r_-), and let $O(L)$ be its orthogonal group. There are natural homomorphisms $\det : O(L) \to \{\pm 1\}$ and $\operatorname{spin} : O(L) \to \{\pm 1\}$, where $\det\gamma$ is the determinant of γ and $\operatorname{spin}\gamma$ is its real spinor norm. We define

$$O_-(L) = \{\gamma \in O(L) \mid \det\gamma = \operatorname{spin}\gamma\}.$$

This group can be given the following geometric interpretation (cf. [21]). A *(positive) sign structure* on L is a choice of one of the connected components of the set of oriented r_+-planes in $L \otimes \mathbf{R}$ on which the form is positive definite; the sign structure containing the oriented plane ν is denoted by $[\nu]$. If $[\nu]$ is a sign structure on L, then

$$O_-(L) = \{\gamma \in O(L) \mid \gamma([\nu]) = [\nu]\}.$$

Theorem (6.3). *Let $M_{\alpha,k}$ be a Todorov lattice and let Λ be the* K3 *lattice.*
(i) There is a primitive embedding $\phi : M_{\alpha,k} \hookrightarrow \Lambda$.
(ii) If ϕ_1, $\phi_2 : M_{\alpha,k} \hookrightarrow \Lambda$ are primitive emdeddings, then there is some $\gamma \in O_-(\Lambda)$ such that $\gamma \circ \phi_1 = \phi_2$.

Proof. Statement (i) follows from the definition of $M_{\alpha,k}$, for $M_{\alpha,k} \subset H^2(\Sigma_{\alpha,k}, \mathbf{Z}) \cong \Lambda$ is a primitive sublattice.

To check statement (ii), we use a variant of a theorem of Nikulin [33;theorem 1.6] which is proved as theorem (A.1) in an appendix to this paper. Let $M = \phi_1(M_{\alpha,k})$ and $K = M^\perp$. K has signature $(2, 19-k)$ and $k \leq 16$ so that K is indefinite and $\operatorname{rank}(K) \geq 3$. To check hypothesis (iii) of theorem (A.1), note that $G_M \cong G_K$ so that G_M can be generated by a set containg at most $\operatorname{rank}(K)$ elements. Moreover, by proposition (6.2), for each $p \neq 2$ the p-Sylow subgroup of G_M can be generated by a single element, and the 2-Sylow subgroup of G_M has an orthogonal direct sum decomposition of the form $G_1 \oplus G_2$ where G_2 is a special 2-elementary form of rank at least 4.

If we now let $M_1 = \phi_2(M_{\alpha,k})$, by theorem (A.1) there is some $\gamma \in O_-(\Gamma)$ such that $\gamma|_M = \phi_2 \circ \phi_1^{-1}$, in other words, $\gamma \circ \phi_1 = \phi_2$.

§7. The moduli of Todorov surfaces.

Let Z be a Todorov surface with fundamental invariants (α, k). By lemma (5.3)(iii), the RDP involution $j : Z \to Z$ whose quotient is a K3 surface is uniquely determined by Z; this implies that there is a one to one correspondence between isomorphism classes of Todorov surfaces with these invariants, and isomorphism classes of triples $(X, \mathcal{M}, B)$ satisfying the conditions in theorem (4.1) in which X is a K3 surface with rational double points, $B^2 = 2k - 16$, and the distinguished partial desingularization Σ of X has k singular points and 2-index α.

Suppose that (α, k) is one of the 11 possible values of the fundamental invariants of a Todorov surface. A *K3 surface of Todorov type* (α, k) is a triple $(X, \mathcal{L}, \Sigma)$ consisting of a K3 surface X with rational double points, an ample line bundle $\mathcal{L}$ on X with $\mathcal{L} \cdot \mathcal{L} = 2k - 16$, and a partial desingularization $\nu : \Sigma \to X$ with k ordinary double points which has 2-index α, such that if $\pi : S \to \Sigma$ and $\mu = \nu \circ \pi : S \to X$ are the minimal desingularizations then $(1/2)c_1(\mu^*(\mathcal{L}) \otimes \mathcal{O}_S(\sum_{P \in \mathrm{Sing}\,\Sigma} \pi^{-1}(P))) \in H^2(S, \mathbf{Z})$. There is a natural transformation from triples $(X, \mathcal{M}, B)$ as in the preceding paragraph to K3 surfaces of Todorov type (α, k) given by setting $\mathcal{L} = \mathcal{O}_X(B)$ and letting Σ be the distinguished partial desingularization; we will first study the moduli of these latter surfaces. Note that when $(X, \mathcal{L}, \Sigma)$ is a K3 surface of Todorov type (α, k), $(\Sigma, \nu^*\mathcal{L})$ is a pair of the type considered in section 6.

Let us fix once and for all a primitive embedding of the Todorov lattice $M_{\alpha,k}$ into the K3 lattice Λ (these exist by theorem (6.3)(i)), and identify $M_{\alpha,k}$ with its image in Λ. We let $N_{\alpha,k}$ be the orthogonal complement of $M_{\alpha,k}$ in Λ, and fix a positive sign structure $[\nu_{\alpha,k}]$ on $N_{\alpha,k}$. The *period space* $D_{\alpha,k}$ is then defined by

$$D_{\alpha,k} = \{\omega \in \mathbf{P}(N_{\alpha,k} \otimes \mathbf{C}) \mid \omega \cdot \omega = 0,\ \omega \cdot \bar{\omega} > 0,\ \mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega) \in [\nu_{\alpha,k}]\}$$

(The last condition ensures that $D_{\alpha,k}$ is connected.) The *integral automorphism group* of $D_{\alpha,k}$ is the group $\mathrm{Aut}_{\mathbf{Z}}(D_{\alpha,k}) = O_-(N_{\alpha,k})/(\pm 1)$.

Let $(X, \mathcal{L}, \Sigma)$ be a K3 surface of Todorov type (α, k), and let $\mu : S \to X$ and $\pi : S \to \Sigma$ be the minimal desingularizations. A *special marking* of $(X, \mathcal{L}, \Sigma)$ is an isometry $\phi : H^2(S, \mathbf{Z}) \to \Lambda$ together with an isomorphism of codes $\psi : C_\Sigma \to C_{\alpha,k}$ such that $c_1(\mu^*(\mathcal{L})) = \phi^{-1}(\lambda)$, $c_1(\pi^{-1}(P)) = \phi^{-1}(e_{\psi(P)})$ for each $P \in \mathrm{Sing}\,\Sigma$, and $\mathrm{Re}(\phi(\omega)) \wedge \mathrm{Im}(\phi(\omega)) \in [\nu_{\alpha,k}]$ for any nonzero holomorphic 2-form ω on S, where we have regarded $\phi(\omega)$ as an element of $N_{\alpha,k} \otimes \mathbf{C}$.

Lemma (7.1). *Every K3 surface of Todorov type (α, k) has a special marking.*

Proof. Let $M_{\Sigma,\mathcal{L}}$ be the saturation of the lattice generated by $c_1(L_\Sigma)$ and $c_1(\mu^*(\mathcal{L}))$ as in section 6; by proposition (6.1), there is an isomorphism of codes $\psi : C_\Sigma \to C_{\alpha,k}$ and an isometry $\eta : M_{\Sigma,\mathcal{L}} \to M_{\alpha,k}$ such that $\eta^{-1}(\lambda) = c_1(\mu^*(\mathcal{L}))$ and $\eta^{-1}(e_{\psi(P)}) = c_1(\pi^{-1}(P))$. If we choose an isometry $\phi' : H^2(S, \mathbf{Z}) \to \Lambda$ such that $\phi'[c_1(\mu^*(\mathcal{L})) \wedge \mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega)] = [\lambda \wedge \nu_{\alpha,k}]$ for a nonzero holomorphic 2-form ω on S, then $\phi' \circ \eta^{-1}$ is a primitive embedding of $M_{\alpha,k}$ into Λ. By theorem (6.3)(ii), there is some $\gamma \in O_-(\Lambda)$ with $\gamma \circ \phi' \circ \eta^{-1} = 1_{M_{\alpha,k}}$. Let $\phi = \gamma \circ \phi'$; then $\phi^{-1}(\lambda) = c_1(\mu^*(\mathcal{L}))$ and $\phi^{-1}(e_{\psi(P)}) = c_1(\pi^{-1}(P))$ for each $P \in \mathrm{Sing}\,\Sigma$. Since γ

preserves the sign structure $[\lambda \wedge \nu_{\alpha,k}]$ on Λ,

$$[\lambda \wedge \mathrm{Re}(\phi(\omega)) \wedge \mathrm{Im}(\phi(\omega))] = \phi[c_1(\mu^*(\mathcal{L}) \wedge \mathrm{Re}(\omega)\mathrm{Im}(\omega)] = [\lambda \wedge \nu_{\alpha,k}]$$

so that $\mathrm{Re}(\phi(\omega)) \wedge \mathrm{Im}(\phi(\omega)) \in [\nu_{\alpha,k}]$ as sign structures on $N_{\alpha,k}$. *Q.E.D.*

Note that if $\gamma \in O(\Lambda)$ satisfies $\gamma(\lambda) = \lambda$ and $\gamma(M_{\alpha,k}) = M_{\alpha,k}$ then $\gamma|_{N_{\alpha,k}}$ preserves the sign structure $[\nu_{\alpha,k}]$ if and only if $\gamma \in O_-(\lambda)$. Thus, if (ϕ, ψ) is a special marking of $(X, \mathcal{L}, \Sigma)$, it follows directly from the definitions that (ϕ', ψ') is a special marking of the same $(X, \mathcal{L}, \Sigma)$ if and only if $(\phi', \psi') = (\gamma \circ \phi, \sigma \circ \psi)$ for some $(\gamma, \sigma) \in \tilde{\Gamma}_{\alpha,k}$, where

$$\tilde{\Gamma}_{\alpha,k} = \{(\gamma, \sigma) \in O_-(\Lambda) \times \mathrm{Aut}(C_{\alpha,k}) | \gamma(\lambda) = \lambda,\ \gamma(e_P) = e_{\sigma(P)}\}.$$

If $(\gamma, \sigma) \in \tilde{\Gamma}_{\alpha,k}$ then $\gamma|_{N_{\alpha,k}}$ preserves the sign structure $[\nu_{\alpha,k}]$ so that $\gamma|_{N_{\alpha,k}}$ acts on $D_{\alpha,k}$; we define $\Gamma_{\alpha,k} = \mathrm{Image}(\tilde{\Gamma}_{\alpha,k} \to \mathrm{Aut}_{\mathbf{Z}}(D_{\alpha,k}))$ The set of special markings of $(X, \mathcal{L}, \Sigma)$ thus determines a point in $D_{\alpha,k}/\Gamma_{\alpha,k}$.

We recall some definitions from [23]. Let X be a K3 surface with rational double points and let $\mu : S \to X$ be the minimal desingularization. The *root system of* X is the root system $R(X)$ spanned by the curves on S which are contracted by μ; the Weyl group of this root system is denoted by $W(X)$, and called the *Weyl group of* X. $W(X)$ acts on $H^2(S, \mathbf{Z})$, and the group of invariants $H^2(S, \mathbf{Z})^{W(X)}$ is denoted by $I^2(X)$. $I^2(X)$ conincides with the orthogonal complement of $c_1(R(X))$, and inherits the structure of a lattice from the intersection form on $H^2(S, \mathbf{Z})$. A *marking* of X is an embedding of lattices $\phi_0 : I^2(X) \to \Lambda$ for which there exist extensions $\phi : H^2(S, \mathbf{Z}) \to \Lambda$ such that ϕ is an isometry and $\phi|_{I^2(X)} = \phi_0$.

If $(X, \mathcal{L}, \Sigma)$ is a K3 surface of Todorov type (α, k), a special marking (ϕ, ψ) of $(X, \mathcal{L}, \Sigma)$ induces a marking $\phi_0 = \phi|_{I^2(X)} : I^2(X) \to \Lambda$ of X with the property that the image of ϕ_0 is contained in the span of λ and $N_{\alpha,k}$.

Proposition (7.2). *Let X be a K3 surface with rational double points, let $\mathcal{L}$ be an ample line bundle on X, and let $\phi_0 : I^2(X) \to \Lambda$ be a marking such that $\phi_0(c_1(\mathcal{L})) = \lambda$, the image of ϕ_0 is contained in the span of λ and $N_{\alpha,k}$, and $\mathrm{Re}(\tilde{\phi}(\omega)) \wedge \mathrm{Im}(\tilde{\phi}(\omega)) \in [\nu_{\alpha,k}]$ for any nonzero holomorphic 2-form ω on the minimal desingularization S of X and any extension $\tilde{\phi} : H^2(S, \mathbf{Z}) \to \Lambda$ of ϕ_0. Then there is a partial desingularization Σ of X such that $(X, \mathcal{L}, \Sigma)$ is a K3 surface of Todorov type (α, k), and a special marking (ϕ, ψ) of $(X, \mathcal{L}, \Sigma)$ such that $\phi|_{I^2(X)} = \phi_0$. Moreover, the partial desingularization Σ is uniquely determined by $X, \mathcal{L}$ and ϕ_0.*

Proof. Let $\tilde{\phi} : H^2(S, \mathbf{Z}) \to \Lambda$ be any extension of ϕ_0. By composing $\tilde{\phi}$ with reflection in some of the classes e_P if necessary, we may assume that each $\tilde{\phi}^{-1}(e_P)$ is the first Chern class of an effective divisor E_P. Since e_P is orthogonal to both λ and $N_{\alpha,k}$, E_P belongs to the root system $R(X)$.

The structure of the Todorov lattice $M_{\alpha,k}$ guarantees that for each $I \subset \operatorname{Sing}\Sigma_{\alpha,k}$ with $|I| = 4$ we have $(1/2)c_1(\sum_{P\in I} E_P) \notin H^2(S,\mathbf{Z})$. But now by proposition (2.4), there is some $w \in W(X)$ such that $C_P = w(E_P)$ is effective and irreducible for each $P \in \operatorname{Sing}\Sigma_{\alpha,k}$. If we let $\pi : S \to \Sigma$ be the contraction of the curves C_P for $P \in \operatorname{Sing}\Sigma_{\alpha,k}$, let $\phi = \tilde{\phi} \circ w^{-1}$ and define $\psi : C_\Sigma \to C_{\alpha,k}$ by $\psi^{-1}(P) = \pi(C_P) \in \operatorname{Sing}\Sigma$, then $(X, \mathcal{L}, \Sigma)$ is a K3 surface of Todorov type (α, k) and (ϕ, ψ) is a special marking such that $\phi|_{I^2(X)} = \phi_0$.

To prove the last statement, suppose that $(X, \mathcal{L}, \Sigma_i)$ is a K3 surface of Todorov type (α, k) with a special marking (ϕ_i, ψ_i) such that $\phi_i|_{I^2(X)} = \phi_0$ for $i = 1, 2$. The proof of the weakly polarized global Torelli theorem [23; p. 319] then implies that there is some $w \in W(X)$ with $\phi_2 = \phi_1 \circ w$.

For each $Q \in \operatorname{Sing}\Sigma_2$, let $C_Q = \pi_2^{-1}(Q)$. Then $C_Q \in R(X)$, and for any $D \in R(X)$ we have

$$D \cdot \sum_{Q\in \operatorname{Sing}\Sigma_2} C_Q = D \cdot (\mu_2^*(\mathcal{L}_2) \otimes \mathcal{O}_S(\sum_{Q\in \operatorname{Sing}\Sigma_2} C_Q)) \in 2\mathbf{Z}.$$

Moreover, if $P = \psi_1^{-1}\psi_2(Q)$ then

$$c_1(\pi_1^{-1}(P)) = \phi_1^{-1}(e_{\psi_2(Q)}) = w\phi_2^{-1}(e_{\psi_2(Q)}) = wc_1(C_Q)$$

which implies that $w(C_Q) = \pi_1^{-1}(P)$ is effective and irreducible for each $Q \in \operatorname{Sing}\Sigma_2$. By lemma (2.6), there is some permuation σ of $\operatorname{Sing}\Sigma_2$ such that $w(C_Q) = C_{\sigma(Q)}$. But this implies that $\psi_2 = \sigma \circ \psi_1$; hence, Σ_1 and Σ_2 are obtained by contracting the same set of curves on S, that is, $\Sigma_1 = \Sigma_2$. *Q.E.D.*

In spite of proposition (7.2), the choice of a partial desingularization Σ and the notion of a special marking of $(X, \mathcal{L}, \Sigma)$ are by no means superfluous when studying the moduli of K3 surfaces of Todorov type (α, k). The special markings are needed to allow us to describe the group $\Gamma_{\alpha,k}$ in an efficient way; the necessity of the choice of Σ will be discussed in section 8.

We are now in a position to prove:

Theorem (7.3). *$D_{\alpha,k}/\Gamma_{\alpha,k}$ is a coarse moduli space for K3 surfaces of Todorov type (α, k).*

Proof. Suppose that $(X_i, \mathcal{L}_i, \Sigma_i)$ for $i = 1, 2$ and K3 surfaces of Todorov type (α, k) which are assigned to the same point $x \in D_{\alpha,k}/\Gamma_{\alpha,k}$, and let $\mu_i : S_i \to X_i$ and $\pi_i : S_i \to \Sigma_i$ be the minimal desingularizations. If we choose some $\omega \in D_{\alpha,k}$ which maps to x in $D_{\alpha,k}/\Gamma_{\alpha,k}$, then there are special markings (ϕ_i, ψ_i) of $(X, \mathcal{L}_i, \Sigma_i)$ such that $\phi_i^{-1}(\omega) \in H^{2,0}(S)$ and $\phi_i^{-1}(\lambda) = c_1(\mu_i^*(\mathcal{L}_i))$ for $i = 1, 2$. In particular, $\phi = \phi_2^{-1} \circ \phi_1 : H^2(S_1, \mathbf{Z}) \to H^2(S_2, \mathbf{Z})$ is an isometry preserving the Hodge structure and $\phi(c_1(\mu_1^*(\mathcal{L}_1))) = c_1(\mu_2^*(\mathcal{L}_2))$. Let $\phi_0 = \phi|_{I^2(X_1)}$. Since $\mathcal{L}_i$ is ample on X_i, by the weakly polarized global Torelli theorem [23; p. 319] (cf. also [36]) there is an isomorphism $\phi : X_2 \xrightarrow{\sim} X_1$ such that $\phi^*|_{I^2(X_1)} = \phi_0$ and $\phi^*(\mathcal{L}_1) = \mathcal{L}_2$. By proposition (7.2), ϕ also induces an isomorphism $\bar{\phi} : \Sigma_2 \xrightarrow{\sim}$

Σ_1, and hence an isomorphism between the triples $(X_i, \mathcal{L}_i, \Sigma_i)$ for $i = 1, 2$. We conclude that the natural map from the moduli space of such triples to $D_{\alpha,k}/\Gamma_{\alpha,k}$ is injective.

To prove that the map is surjective, take an arbitrary point $\omega \in D_{\alpha,k}$. We use the surjectivity of the period map for algebraic K3 surfaces in the form given in [23; p. 325] (and due essentially to Kulikov [19]): there is a K3 surface with rational double points X, an ample line bundle $\mathcal{L}$ on X and a marking $\phi_0 : I^2(X) \to \Lambda$ with $\phi_0^{-1}(\lambda) = c_1(\mathcal{L})$ such that if $\mu : S \to X$ is the minimal desingularization then $\phi^{-1}(\omega) \in H^{2,0}(S)$ for any extension $\phi : H^2(S, \mathbf{Z}) \to \Lambda$ of ϕ_0; note that the image of ϕ_0 lies in the span of λ and $N_{\alpha,k}$. By proposition (7.2), there is a (unique) partial desingularization Σ of X and a special marking (ϕ, ψ) of $(X, \mathcal{L}, \Sigma)$ such that $\phi|_{I^2(X)} = \phi_0$. $(X, \mathcal{L}, \Sigma)$ is thus assigned to the image of ω in $D_{\alpha,k}/\Gamma_{\alpha,k}$. *Q.E.D.*

Further details about the structure of the moduli space $D_{\alpha,k}/\Gamma_{\alpha,k}$, and in particular about the "period map" $D_{\alpha,k}/\Gamma_{\alpha,k} \to D_{\alpha,k}/\operatorname{Aut}_{\mathbf{Z}}(D_{\alpha,k})$, can be found in our joint paper with M.-H. Saito [27].

Let $(X, \mathcal{L}, \Sigma)$ be a K3 surface of Todorov type (α, k). The set of Todorov surfaces whose associated K3 surface is $(X, \mathcal{L}, \Sigma)$ is parametrized by the set of divisors B with $\mathcal{L} = \mathcal{O}_X(B)$ which satisfy:

Condition (7.4). *$\mu^*(B)$ is disjoint from $\pi^{-1}(\operatorname{Sing}\Sigma)$ and is a reduced divisor with neither infinitely near triple points nor points of multiplicity greater than three.*

We may thus describe the moduli space of Todorov surfaces in the following way. Using the procedure discussed in [8] and [23; pp. 320-321] of glueing together local deformations, we construct a universal marked family $f : \mathcal{X}_{\alpha,k} \to D_{\alpha,k}$ of K3 surfaces with rational double points and a relatively ample line bundle $\mathbf{L}_{\alpha,k}$ on $\mathcal{X}_{\alpha,k}$ whose first Chern class in each fiber is mapped to λ by the marking, where the markings of the fibers are required to have images in the span of λ and $N_{\alpha,k}$, and to send the sign structure induced from the Hodge structure on the fiber to $[\nu_{\alpha,k}]$. Let $\mathcal{V}_{\alpha,k}$ be the open subset of the projective bundle $\mathbf{P}(R^\circ f_*\mathbf{L}_{\alpha,k}) \to D_{\alpha,k}$ consisting of all sections whose divisor satisfies condition (7.4). The action of the group $\tilde{\Gamma}_{\alpha,k}$ on $D_{\alpha,k}$ then extends to an action on $\mathcal{V}_{\alpha,k}$.

Theorem (7.5). *$\mathcal{V}_{\alpha,k}/\tilde{\Gamma}_{\alpha,k}$ is a coarse moduli space for Todorov surfaces with fundamental invariants (α, k), and each fiber of the natural map $\mathcal{V}_{\alpha,k}/\tilde{\Gamma}_{\alpha,k} \to D_{\alpha,k}/\Gamma_{\alpha,k}$ is connected, and nonempty. In particular, since $D_{\alpha,k}/\Gamma_{\alpha,k}$ is connected, the set of Todorov surfaces with fixed fundamental invariants forms a nonempty irreducible family.*

Proof. Most of this has been proved in the discussion above; we must still consider the structure of the fibers of the map $\mathcal{V}_{\alpha,k}/\tilde{\Gamma}_{\alpha,k} \to D_{\alpha,k}/\Gamma_{\alpha,k}$. It suffices to show that each fiber of $\mathcal{V}_{\alpha,k} \to D_{\alpha,k}$ is connected and nonempty.

Let $(X, \mathcal{L}, \Sigma)$ be a K3 surface of Todorov type (α, k). By Riemann-Roch, $\mathcal{L} = \mathcal{O}_X(B)$ for some effective divisor B which moves in a linear system $|B|$ of dimension $k - 8 \geq 1$. Since $(\alpha, k) \neq (1, 9)$, lemma (5.1) guarantees that $|B|$ is free; but then by Bertini's theorem, there is some $B' \in |B|$ which is smooth and disjoint from $\operatorname{Sing} X$. B' satisfies condition (7.4), so that the set of divisors in $|B|$ satisfying (7,4) (which coincides with the fiber of $\mathcal{V}_{\alpha,k} \to D_{\alpha,k}$) is a nonempty Zariski-open subset. *Q.E.D.*

When $(\alpha, k) \neq (5, 16)$, we can give a bit more information about the map $\mathcal{V}_{\alpha,k}/\tilde{\Gamma}_{\alpha,k} \to D_{\alpha,k}/\Gamma_{\alpha,k}$ by analyzing the action of $\tilde{\Gamma}_{\alpha,k}$ more carefully.

Lemma (7.6). *Suppose that* $(\alpha, k) \neq (5, 16)$.
(i) *The natural map* $\tilde{\Gamma}_{\alpha,k} \to O_-(N_{\alpha,k})$ *is injective.*
(ii) -1 *is in the image of this map if and only if* $(\alpha, k) = (0, 9)$ *or* $(1, 10)$.

Proof. Let $(X, \mathcal{L}, \Sigma)$ be a K3 surface of Todorov type (α, k) whose minimal desingularization S has Picard number $k+1$. (This implies that $\operatorname{NS}(S) = M_{\Sigma,\mathcal{L}}$ and $\Sigma = X$.) Let (ϕ, ψ) be a special marking of $(X, \mathcal{L}, \Sigma)$, and for $(\gamma, \sigma) \in \tilde{\Gamma}_{\alpha,k}$ let $\tilde{\gamma} = \phi^{-1} \circ \gamma \circ \phi$ and $\tilde{\sigma} = \psi^{-1} \circ \sigma \circ \psi$.

(i) Suppose that $(\gamma, \sigma) \in \operatorname{Ker}(\tilde{\Gamma}_{\alpha,k} \to O_-(N_{\alpha,k}))$. Then $\tilde{\gamma}$ is an automorphism of $H^2(S, \mathbf{Z})$ preserving the intersection form and the Hodge structure, $\tilde{\gamma}$ acts as the identity on $\Gamma^2(X)$, and $\tilde{\gamma}(\pi^{-1}(P)) = \pi^{-1}(\tilde{\sigma}(P))$. Since $\tilde{\gamma}$ preserves $c_1(\mu^*(\mathcal{L}))$ and maps each $\pi^{-1}(P)$ to an effective curve $\pi^{-1}(\tilde{\sigma}(P))$, by the weakly polarized global Torelli theorem there is an automorphism $\phi : S \to S$ such that $\phi^* = \tilde{\gamma}$. Now ϕ and $\tilde{\sigma}$ satisfy the hypotheses of proposition (2.3); since $k \leq 15$, ϕ and $\tilde{\sigma}$ must be trivial, so that γ and σ are trivial as well.

(ii) Suppose that $(\gamma, \sigma) \in \tilde{\Gamma}_{\alpha,k}$ maps to -1 in $O_-(N_{\alpha,k})$. Then $\gamma|_{N_{\alpha,k}}$ acts as -1, while $\gamma|_{M_{\alpha,k}}$ fixes λ and permutes the e_P's according to σ. If $g : M^*_{\alpha,k}/M_{\alpha,k} \to N^*_{\alpha,k}/N_{\alpha,k}$ is the natural isomorphism, then $g^{-1} \circ (\gamma|_{N_{\alpha,k}})^* \circ g = (\gamma|_{M_{\alpha,k}})^* \bmod M_{\alpha,k}$; this implies that the permutation σ acts as -1 on $G_{\alpha,k} = M^*_{\alpha,k}/M_{\alpha,k}$. By proposition (6.2)(ii), $(\alpha, k) = (0, 9)$ or $(1, 10)$.

Conversely, if $(\alpha, k) = (0, 9)$ or $(1, 10)$, let $|B|$ be the linear system on X with $\mathcal{L} = \mathcal{O}_X(B)$. By lemma (5.4), $|B|$ is hyperelliptic so that there is an involution $\phi : X \to X$ with $\phi^*(\mathcal{L}) = \mathcal{L}$ and X/ϕ rational. The rationality of X/ϕ implies that $\phi^*(\omega) = -\omega$ for any holomorphic 2-form ω on S; since $\operatorname{NS}(S) = M_{\Sigma,\mathcal{L}}$, this implies that ϕ^* acts as -1 on all of $M^\perp_{\Sigma,\mathcal{L}}$. Now if we let $\gamma = \phi \circ \phi^* \circ \phi^{-1}$ and $\sigma = \psi \circ (\phi|_{\operatorname{Sing} X}) \circ \psi^{-1}$ then $(\gamma, \sigma) \in \tilde{\Gamma}_{\alpha,k}$ and γ acts as -1 on $\phi(M^\perp_{\Sigma,\mathcal{L}}) = N_{\alpha,k}$. *Q.E.D.*

Notice that the last paragraph of the proof above applies when $(\alpha, k) \neq (0, 9)$ or $(1, 10)$ to provide a partial converse to lemma (5.4).

Corollary (7.7). *Let* $(X, \mathcal{L}, \Sigma)$ *be a K3 surface of Todorov type* (α, k) *with* $(\alpha, k) \neq (0, 9)$, $(1, 10)$, *or* $(5, 16)$, *and suppose that the minimal desingularization of* X *has Picard number* $k + 1$. *Then the linear system* $|B|$ *on* X *associated to* $\mathcal{L}$ *is not hyperelliptic.*

As an application of lemma (7.6), we prove:

Corollary (7.8). *If $(\alpha, k) \neq (5, 16)$, then the natural action of $\tilde{\Gamma}_{\alpha,k}$ on $\mathcal{V}_{\alpha,k}$ factors through the projection $\tilde{\Gamma}_{\alpha,k} \to \Gamma_{\alpha,k}$. In particular, $\mathcal{V}_{\alpha,k}/\Gamma_{\alpha,k}$ is the coarse moduli space for Todorov surfaces with fundamental invariants (α, k).*

Proof. If $(\alpha, k) \neq (0, 9)$ or $(1, 10)$ then $\Gamma_{\alpha,k} = \tilde{\Gamma}_{\alpha,k}$ by lemma (7.6) and there is nothing to prove. If $(\alpha, k) = (0, 9)$ or $(1, 10)$ and $(X, \mathcal{L}, \Sigma)$ is generic, then lemma (7.6) implies that the unique element of $\operatorname{Ker}(\tilde{\Gamma}_{\alpha,k} \to \operatorname{Aut}_{\mathbf{Z}}(D_{\alpha,k}))$ induces the hyperelliptic involution on X. Since $\phi_{|B|}$ factors through the quotient by that hyperelliptic involution, the induced action on $\mathbf{P}H^{\circ}(\mathcal{L})$ is trivial. Hence, the action of the kernel on $\mathcal{V}_{\alpha,k}$ is trivial. *Q.E.D.*

Corollary (7.8) implies that for $(\alpha, k) \neq (5, 16)$, the Stein factorization of the "period map" $\mathcal{V}_{\alpha,k}/\Gamma_{\alpha,k} \to D_{\alpha,k}/\operatorname{Aut}_{\mathbf{Z}}(D_{\alpha,k})$ is nothing other than $\mathcal{V}_{\alpha,k}/\Gamma_{\alpha,k} \to D_{\alpha,k}/\Gamma_{\alpha,k} \to D_{\alpha,k}/\operatorname{Aut}_{\mathbf{Z}}(D_{\alpha,k})$. We compute the degree of the finite part of the Stein factorization in [27].

§8. Concluding remarks.

1. The reader will have notice that much of the technical difficulty in studying the moduli of Todorov surfaces derives from the necessity of choosing a partial desingularization Σ when discussing K3 surfaces of Todorov type (α, k). Of course, if we had been willing to consider only general Todorov surfaces (for example, ones for which $\Sigma = X$), this difficulty would have been eliminated, But in studying *all* Todorov surfaces, we cannot eliminate the choice of Σ, as we now give an example to show.

Let C_1 and C_2 be conics and let L_1 and L_2 be lines in $\mathbf{P}^2$, meeting transversely (pairwise). The sextic curve $C = C_1 + C_2 + L_1 + L_2$ can then be written as a sum of two cubics in two different ways: $C = (C_1 + L_1) + (C_2 + L_2)$ and $C = (C_1 + L_2) + (C_2 + L_1)$. The base points of the two associated pencils of cubics give two distinguished subsets S_1 and S_2 of $\operatorname{Sing} C$ with $\#(S_i) = 9$. Let X be the double cover of $\mathbf{P}^2$ branched along C, and let $\mathcal{L}$ be the pullback of $\mathcal{O}_{\mathbf{P}^2}(1)$. X is a K3 surface with 13 ordinary double points; if we let Σ_i be the partial desingularization of X obtained by blowing up all points of $\operatorname{Sing} X$ not in the inverse image of S_i for $i = 1, 2$, then $(X, \mathcal{L}, \Sigma_i)$ is a K3 surface of Todorov type $(0, 9)$. In particular, the set of isomorphism classes of pairs $(X, \mathcal{L})$ contains less information than the set of isomophism classes of triples $(X, \mathcal{L}, \Sigma)$.

2. Let $(X, \mathcal{L}, \Sigma)$ be a K3 surface of Todorov type (α, k) and let $|B|$ be the linear system with $\mathcal{L} = \mathcal{O}_X(B)$. Suppose that $B_t \in |B|$ is a family of divisors parametrized by the unit disk Δ such that B_t satisfies condition (7.4) for $t \neq 0$, but B_0 does not satisfy (7.4). (For example, if $\Sigma = X$ we could require that B_0 pass through a singular point of X.) The corresponding family Z_t of Todorov surfaces over the punctured disk Δ^* exhibits a phenomenon first

observed by Friedman [14]: it is a family of regular[7] surfaces of general type with trivial monodromy which can be completed to a family over the disk Δ, but *cannot* be completed to such a family with a smooth central fiber. For if the central fiber were smooth, that fiber would also be a Todorov surface with the same fundamental invariants and its branch locus would be B_0, a contradiction. (In fact, the natural central fiber contains a component birational to either an elliptic surface, or a Todorov surface with different invariants.)

3. In [26], we showed that any algebraic surface Z with geometric genus 1 has an "associated K3 surface": one whose transcendental lattice is isomorphic to that of Z by an isomorphism preserving the intersection forms and (integral) Hodge structures. We then asked whether there is an algebraic cycle on the product of Z with its associated K3 surface which realizes this isomorphism. (The existence of such a cycle is predicted by the Hodge conjecture.) Somewhat surprisingly, if Z is a Todorov surface, the K3 surface $X = Z/j$ is *not* the associated K3 surface of Z! This happens because the natural map $Z \to X$ multiplies the intersection form by 2 (cf. the proof of proposition (6.2) in the case $(\alpha, k) = (5, 16)$). To construct the desired algebraic cycle, we must find a double cover of X (or some surface birational to X) which is also a K3 surface.

Such a double cover is easily found when $\alpha > 0$: the double cover of Σ branched along a set $I \subset \operatorname{Sing}\Sigma$ such that $(1/2)c_i(\sum_{P \in I} \pi^{-1}(P)) \in \mathrm{NS}(S)$ is a K3 surface Y. If $\alpha = 0$ (so that $k = 9, 10$, or 11) we must work somewhat harder to find Y, and we will only sketch the construction when Z is generic. In that case, let $\operatorname{Sing}\Sigma = \{P_1, \ldots, P_k\}$, let Q_i be the image of P_i in $\mathbf{P}^{k-7}$ under the map $\phi_{|B|}$, and let $H \subset \mathbf{P}^{k-7}$ be the hyperplane spanned by the $k-7$ points $Q_8, Q_9, \ldots, Q_K$. When Z is generic the inverse image of H on X is an irreducible rational curve whose proper transform C on S is smooth. If $C_i = \pi^{-1}(P_i)$, then $C, C_1, \ldots, C_7$ are smooth disjoint rational curves on S such that $1/2c_1(C + C_1 + \cdots + C_7) \in \mathrm{NS}(S)$. The double cover $\tilde{Y}$ of S branched along $C + C_1 + \cdots + C_7$ is then birational to a K3 surface Y.

The graph of the pair of degree 2 rational maps $Z \to X$ and $Y \dashrightarrow X$ now gives a cycle on the product $Z \times Y$ which induces an isomorphism $T(Z) \otimes \mathbf{Q} \to T(Y) \otimes \mathbf{Q}$ preserving intersection forms and Hodge structures. Since this isomorphism might not be defined over $\mathbf{Z}$, Y might not itself be the associated K3 surface W of Z. However, by a theorem of Mukai [28] (combined with a result of Nikulin [32] in the case $(\alpha, k) = (0, 9)$ in which the Picard number of S may be 10), there is an algebraic cycle in the product $Y \times W$ inducing an isomorphism $T(Y) \times \mathbf{Q} \to T(W) \otimes \mathbf{Q}$ such that the composite isomorphism $T(Z) \times \mathbf{Q} \to T(W) \otimes \mathbf{Q}$ preserves intersection forms and Hodge structures, and is defined over $\mathbf{Z}$. We may then "compose" the algebraic cycles on $Z \times Y$ and $Y \times W$ as in [26] to get a cycle on $Z \times W$ with the desired property.

[7]We may even assume that the surfaces are simply connected, if we take $(\alpha, k) = (0, 9)$ or $(0, 10)$; cf. [9], [41].

Appendix

In this appendix we modify a theorem of Nikulin [30] concerning embeddings into an even unimodular lattice L to cover the case of equivalence of embedding under the subgroup $O_-(L)$ of the orthogonal group $O(L)$ of L (which was defined in section 6).

We denote the discrininant-form of a lattice L by (G_L, q_L). If G is a finite abelian group, G_p denotes the p-Sylow subgroup of G, and $l(G)$ denotes the minimum number of generators of G.

Theorem (A.1). *Let L be an even integral unimodular symmetric bilinear form, let M and M_1 be nondegenerate primitive sublattices of L, let $\phi : M \to M_1$ be an isometry and let K be the orthogonal complement of M. Suppose that*

(i) *the bilinear form is indefinite when restricted to K,*

(ii) $\operatorname{rank}(K) \geq 3$, *and*

(iii) *either*

(a) $l(G_M) \leq \operatorname{rank}(K) - 2$, *or*

(b) $l(G_{M_p}) \leq \operatorname{rank}(K) - 2$ *for all $p \neq 2$,* $l(G_{M_2}) = \operatorname{rank}(K)$, *and there is an orthogonal direct sum decomposition*

$$(G_{M_2}, q_{M_2}) \cong (G_1, q_1) \oplus (G_2, q_2)$$

such that (G_2, q_2) is a special 2-elementary form of rank at least 2.

Then there is some $\phi \in O_-(L)$ such that $\phi|_M = \phi$.

Proof. We modify slightly the proof of theorems 1.2 and 1.6 in [30], following the notation there. Note first that our hypothesis (iii) implies that conditions (1.5) and (1.6) of [30] hold (cf. [31]; theorem 1.14.2]; the point is that a special 2-elementary form of rank at least 2 must contain $u_+^{(2)}(2)$ or $v_+^{(2)}(2)$ as a direct summand, in the notation of [31]); hence, we may apply the argument of [30].

We begin as Nikulin does, by invoking the version of Witt's theorem over local rings due to Kneser [17] to guarantee the existence of p-adic isometries $\phi_p \in O(L \otimes \mathbf{Z}_p)$ such that $\phi_p|_{M \otimes \mathbf{Z}_p} = \phi \otimes \mathbf{Z}_p$. Suppose that $\det \phi_2 = -1$. If there is some $\alpha \in O_-(L)$ with $\det \alpha = -1$, we may proceed as at the bottom of p. 77, [8] replacing M_1, ϕ and Φ by $\alpha(M_1)$, $\alpha \circ \phi$ and $\alpha \circ \Phi$ to reduce to the case $\det \phi_2 = 1$. To see that such an α exists, note that L, being even, indefinite, and unimodular, contains elements x and y on which the bilinear form has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; α may be taken to be the reflection in $x - y$.

We may thus assume that $\det \phi_2 = 1$. We now proceed as on pp. 77-78, making the following modification of formula (1.7) on p. 78: define $O_{\mathbf{Q}}^{++} = \{\psi \in O_{\mathbf{Q}}^{+} \mid \Theta(\psi) > 0\}$. Then

$$(*) \qquad O_{\mathbf{A}}^{+} = O'_{\mathbf{A}} \cdot O_{\mathbf{Q}}^{++} \cdot H_{\mathbf{A}}^{+}(K).$$

[8] Page numbers refer to the English translation of [30].

This is proved the same way as formula (1.7): if $\mathbf{Q}^{*+}$ denotes the positive rational numbers, then

$$[O^+_{\mathbf{A}} : O'_{\mathbf{A}} \cdot O^{++}_{\mathbf{Q}} \cdot H^+_{\mathbf{A}}(K)] = [I_{\mathbf{Q}} : \mathbf{Q}^{*+} \cdot \Theta(H^+_{\mathbf{A}}(K)]$$

so that (*) follows from the fact (proved on pp. 78-79) that under our hypotheses, $\Theta(H^+(K_p)) \subset \mathbf{Z}^*_p \cdot \mathbf{Q}^{*2}_p$.

Thus, when formula (1.7) is applied in the proof (near the top of p. 80) we may assume that $\Theta(\psi) > 0$ in the decomposition $\phi_p = \phi'_p \circ \psi \circ \alpha_p$. Hence, at the end of the proof we have

$$\Theta(\phi) = \Theta(\phi' \circ \psi) = \Theta(\psi) > 0$$

while $\det \phi = 1$ so that $\phi \in O_-(L)$. *Q.E.D.*

References

[1] W. Barth, C. Peters, and A. Van de Ven, Compact Complex Surfaces, Berlin-Heidelberg-New York-Tokyo: Springer-Verlag (1984).

[2] A. Beauville, "L'application canonique pour les surfaces de type général", Invent. Math. **55** (1978) 121-140.

[3] A. Beauville, "Sur le nombre maximum de points doubles d'une surface dans $\mathbf{P}^3(\mu(5) = 31)$", Journées de géométrie algébrique (juillet 1979), Alphen aan den Rijn: Sijthoff & Noordhoff (1980) 207-215.

[4] I. F. Blake and R. C. Mullin, An Introduction to Algebraic and Combinatorial Coding Theory, New York-San Francisco-London: Academic Press (1976).

[5] E. Bombieri, "Canonical models of surfaces of general type", Publ. Math. IHES **42** (1973) 171-219.

[6] N. Bourbaki, Groupes et algèbres de Lie Chap. IV, V, VI, Paris: Hermann (1968).

[7] E. Brieskorn, "Singular elements of semi-simple groups", Proc. Int. Cong. Math. Nice, t. **2** (1970) 279-284.

[8] D. Burns, Jr. and M. Rapoport, "On the Torelli problems for Kählerian K3 surfaces", Ann. scient. Éc. Norm. Sup. (4) **8** (1975) 235-274.

[9] F. Catanese, "Surfaces with $K^2 = p_g = 1$ and their period mapping", Algebraic Geometry (Proceedings, Copenhagen 1978), Lecture Notes in Math. **732**, Berlin-Heidelberg-New York: Springer-Verlag(1979) 1-29.

[10] F. Catanese, "The moduli and the global period mapping of surfaces with $K^2 = p_g = 1$: a counterexample to the global Torelli problem:", Compositio Math. **41** (1980) 401-414.

[11] F. Catanese, "On the period map of surfaces with $K^2 = \chi = 2$", Classification of Algebraic and Analytic manifolds, Progress in Math. **39**, Boston-Basel-Stuttgart: Birkhäuser (1983) 27-43.

[12] F. Catanese and O. Debarre, "Surfaces with $K^2 = 2$, $p_g = 1$, $q = 0$", preprint.
[13] P. Francia, "The bicanonical map for surfaces of general type ", preprint.
[14] R. Friedman, "A degenerating family of quintic surfaces with trivial monodromy", Duke Math. J. **50** (1983) 203-214.
[15] P. Griffiths and J. Harris, Principles of Algebraic Geometry, New York-Chichester-Brisbane-Toronto: John Wiley & Sons (1978).
[16] E. Horikawa, "On deformations of quintic surfaces", Invent. Math. **31** (1975) 43-85.
[17] M. Keneser, "Witts Satz für quadratischer Formen über lokalen Ringen", Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1972) 195-203.
[18] K. Kodaira, "Pluricanonical systems on algebraic surfaces of general type", J. Math. Soc. Japan **20**(1968) 170-192.
[19] V. S. Kulikov, "Epimorphicity of the period mapping for surfaces of type K3" (Russian), Uspehi Mat. Nauk. **32**: 4 (1977) 257-258.
[20] V. T. Kunev, "An example of a simply-connected surface of general type for which the local Torelli theorem does not hold" (Russian), C. R. Ac. Bulg. Sc. **30** (1977) 323-325.
[21] E. Looijenga and J. Wahl, "Quadratic functions and smoothing surface singularities", Topology **25** (1986) 261-291.
[22] A. Mayer, "Families of K3 surfaces", Nagoya Math. J. **48** (1972) 1-17.
[23] D. R. Morrison, "Some remarks on the moduli of K3 surfaces", Classification of Algebraic and Analytic Manifolds, Progress in Math. **39**, Boston-Basel-Stuttgart: Birkhäuser (1983) 303-332.
[24] D. R. Morrison, "On K3 surfaces with large Picard number", Invent. Math. **75** (1984) 105-121.
[25] D. R. Morrison, "The birational geometry of surfaces with rational double points", Math. Ann. **271** (1985) 415-438.
[26] D. R. Morrison, "Isogenies between algebraic surfaces with geometric genus one", Tokyo J. Math. **10** (1987) 179-187.
[27] D. R. Morrison and M.-H. Saito, "Cremona transformations and degrees of period maps for K3 surfaces with ordinary double points", Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. **10** (1987) 477-513.
[28] S. Mukai, "On the moduli space of bundles on K3 surfaces, I", Vector Bundles on Algebraic Varieties, Tata Institute of Fundamental Reserch (1987) 341-413.
[29] V. V. Nikulin, "On Kummer surfaces", Izv. Akad. Nauk SSSR **39** (1975) 278-293; Math. USSR Izvestija **9** (1975) 261-275.
[30] V. V. Nikulin, "Finite automorphism groups of Kähler K3 surfaces", Trudy Mosk. Mat. Ob. **38** (1979) 75-137; Trans. Moscow Math. Soc. **38** (1980) 71-135.
[31] V. V. Nikulin, "Integral symmetric bilinear forms and some of their applications", Izv. Akad. Nauk. SSSR **43** (1979) 111-177; Math. USSR Izvestija **14** (1980) 103-167.

[32] V. V. Nikulin, Letter to the author, 15 November 1984.
[33] A. Ogus, "A crystalline Torelli theorem for supersingular K3 surfaces", Arithmetic and Geometry: Papers Dedicated to I. R. Shafarevich (vol. II: Geometry), Progress in Math. **36**, Boston-Basel-Stuttgart: Birkhäuser (1983) 361-394.
[34] P. Oliverio, "On the period map for surfaces with $K_S^2 - 2$, $p_g = 1$, $q = o$ and torsion $\mathbf{Z}/2\mathbf{Z}$", Duke Math. J. **50** (1983) 561-571.
[35] U. Persson, "Double coverings and surfaces of general type", Algebraic Geometry (Proceedings, Tromsø, Norway 1977), Lecture Notes in Math. **687**, Berlin-Heidelberg-New York: Springer-Verlag (1978) 168-195.
[36] I. Piateckii-Shapiro and I. R. Shafarevich, "A Torelli theorem for algebraic surfaces of type K3", Izv. Akad. Nauk. SSSR **35** (1971) 530-572; Math. USSR Izvestija **5** (1971) 547-587.
[37] C. P. Ramanujam, "Remarks on the Kodaira vanishing theorem", J. Indian Math. Soc. **36** (1972) 41-51.
[38] M. Reid, "Minimal models of canonical 3-folds", Algebraic Varieties and Analytic Varieties, Adv. Studies in Pure Math. 1, Amsterdam-New York-Oxford: North-Holland and Tokyo: Kinokuniya (1983) 131-180.
[39] F. Sakai, "Normal Gorenstein surfaces and blowing ups", Saitama Math. J. **1** (1983) 29-35.
[40] A. N. Todorov, "Surfaces of general type with $p_g = 1$ and $(K, K) = 1$, I", Ann. scient. Éc. Norm. Sup. (4) **13** (1980) 1-21.
[41] A. N. Todorov "A construction of surfaces with $p_g = 1$, $q = 0$ and $2 \leq (K^2) \leq 8$: counterexamples of the global Torelli theorem", Invent. Math. **63** (1981) 287-304.
[42] S. Usui, "Period map of surfaces with $p_g = c_1^2 = 1$ and K ample", Mem. Fac. Sci. Kochi Univ. (Math.) **3** (1981) 37-73.
[43] S. Usui, "Torelli theorem for surfaces with $p_g = c_1^2 = 1$ and K ample and with certain type of automorphism", Compositio Math. **45** (1982) 293-314.
[44] S. Usui, "Period map of surfaces with $p_g = 1$, $c_1^2 = 2$ and $\pi_1 = \mathbf{Z}/2\mathbf{Z}$", Mem. Fac. Sci. Kochi Univ. (Math.) **5** (1984) 15-26; 103-104.
[45] E. B. Vinberg, "Discrete groups generated by reflections in Lobačevskii spaces", Mat. Sbornik **72** (1967) 471-488; Math. USSR Sbornik **1** (1967) 429-444.

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