Induced Induction Inductively

1 Baroque

1.1 Little Fugue

Let X be a finite set with $|X| = n$, where $n \in \mathbb{N}$ and $|X|$ denoted the number of elements in X. Prove that $|P(X)| = 2^{|X|}$.

1.2 Well Tempered

Recall from combinatorics that n choose k is defined to be:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

1.2.1 Prelude

Given , $n, k \in \mathbb{N}$ with $n > 0$ and $0 \leq k < n$, Prove that we have the following equality:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

How does this establish that $\binom{n}{k}$ is always an integer?

1.2.2 Fugue

Prove that the following equation is true:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

1.3 Toccata and Fugue

Prove that any natural number greater than 1 is either prime or a product of primes.

2 Classical and Romantic

2.1 Tristesse

2.1.1 Odd Sums

Let $n \in \mathbb{N}$ be positive. Find a closed form for the sum of the first n **odd** integers. In otherwords, find $f(n)$ where:

$$\sum_{i=1}^{n}(2i-1) = f(n)$$

so that $f(n)$ does not contain a summation symbol. Prove that this holds for all positive n.

2.1.2 Even Sums

Prove that this holds for all positive n. Let $n \in \mathbb{N}$ be positive. Using the above, find a closed form for the sum of the first n **even** integers. In otherwords, find $f(n)$ where:

$$\sum_{i=1}^{n}(2i-2) = f(n)$$

so that $f(n)$ does not contain a summation symbol. Prove that this holds for all positive n.

2.1.3 Revisiting Old Sums

Can you use the above to find an alternate proof that the sum of the first n non-negative integers is $\frac{n(n-1)}{2}$?

2.2 Concerto No. 2

2.2.1 Divides I

Prove that $3|4^n + 5$, for all **positive** $n \in \mathbb{N}$, where $a|b$ means a divides b.

2.2.2 Divides II

Prove that $9|(n^3 + (n+1)^3 + (n+2)^3)$, for all $n \in \mathbb{N}$, where $a|b$ means a divides b.

2.2.3 Divides III

Prove that $9|(4^n(4^n + 10) + 25)$, for all **positive** $n \in \mathbb{N}$, where $a|b$ means a divides b. (Hint: This can be done without induction assuming you've done the previous part)

2.3 Sonata

Assume we are working in $\mathbb{R}^2$ the real plane. An integer polygon is an n sided figure with n vertices/corners that lie on the integer latices (or in other words both the coordinates are integer) such that their edges do not cross. Drawing a few examples might help.

For example we can have a square formed by the vertices:

$$\{(1,1), (-1,1), (-1,-1), (-1,1)\}$$

is an integer polygon. In fact, if we order the vertices we can describe the entire integer polygon where if two points are adjacent there is an edge between them. So that the square before, we can represent as:

$$((1,1), (-1,1), (-1,-1), (-1,1), (1,1))$$

which would mean, from the ordering, that there is an edge between:

$$\{(1,1),(-1,1)\}$$
$$\{(-1,1),(-1,-1)\}$$
$$\{(-1,-1),(1,-1)\}$$
$$\{(1,-1),(1,1)\}$$

2.3.1 Moonlight Examples

Check whether the following are integer polygons:

- $((1,1), (-1,1), (-1,1), (-1,-1), (1,1))$
- $((1,1), (-1,1), (0,0), (-1,-1), (1,-1), (1,1))$
- $((0,1), (1,1), (1,0), (0,1))$

2.3.2 Tempest Triangles

Let $a, b, c \in \mathbb{Z}^2 \subset \mathbb{R}^2$, prove that (a, b, c, a) is always an integer polygon. Assuming we all agree that this decribes a triangle, we call this an integer triangle.

2.3.3 Pathétique Polygons

Prove that all integer polygons can be triangulated by integer triangles. In other words, all integer polygons can be partitioned/cut into integer triangles.

2.4 From the New World

2.4.1 First: Rectangles

Given an integer rectangle, S in $\mathbb{R}^2$, whose sides are all vertical and horizontal (cf. Sonata). By vertical and horizontal sides, we mean that each sides of the rectangle have either fixed x or y coordiantes in $\mathbb{Z}^2$.

Let a equal the number of integer points in the interior of S and b equal the number of integer points on it's boundary. For example with the square from before:

$$((1,1),(-1,1),(-1,-1),(-1,1),(1,1))$$

there is only 1 interior point specifically $(0,0)$, but there are 8 boundary points specifically:

$$(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1),(1,0)$$

Let A equal the area of the integer rectangle S. Prove that:

$$A = a + \frac{b}{2} - 1$$

So for example, our square everyone can agree has area 4. If we check the equation we have that $a = 1$ and $b = 8$ so that $1 + \frac{8}{2} - 1 = 4$ which is the area of our square. This is asking for you to prove **all** integer rectangles with verticle and horizontal sides have this property.

2.4.2 Second: Right Triangles

Given an integer **right** triangle S in $\mathbb{R}^2$, whose base and height are vertical and horizontal. Prove that the above equation holds.

2.4.3 Third: Any Triangles

Given an integer triangle S in $\mathbb{R}^2$. Prove that the above equation holds.

2.4.4 Fourth: Integer Polygons

Given an integer polygon S in $\mathbb{R}^2$. Prove that the above equation holds.

3 Modern

3.1 Rhapsody in Blue

3.1.1 Two Avenues and n Blocks

Let $n \in \mathbb{N}$ and $P = \{(a, b) \in \mathbb{Z}^2 : a \in \{0, 1\}, 0 \leq b \leq n\}$. Assume that George is at $(0, 0) \in P$ and needs to get to $(1, n) \in P$, but can only travel vertically or horizontally on P.

So for example he can travel from $(0, 0)$ to $(0, 1)$ or $(1, 0)$. If we had allowed negative numbers then from $(0, 0)$ he would also be able to get to $(-1, 0)$ and $(0, -1)$. How many possible paths can George traverse without passing the same point twice?

3.1.2 Three Avenues and n Blocks

Let $n \in \mathbb{N}$ and $P = \{(a, b) \in \mathbb{Z}^2 : 0 \leq a \leq 2, 0 \leq b \leq n\}$. Assume that Gershwin is at $(0, 0) \in P$ and needs to get to $(2, n) \in P$, with the same limitations as George in terms of moving. How many possible paths can Gershwin traverse without passing the same point twice?

3.1.3 m Avenues and n Blocks

Can you generalize this to where $m, n \in \mathbb{N}$ and $P = \{(a, b) \in \mathbb{Z}^2 : 0 \leq a \leq m, 0 \leq b \leq n\}$ to find all paths George Gershwin can take from $(0, 0)$ to (m, n), so that he does not pass through the same point twice?