Introduction to Riemannian Geometry (240C) - Notes [Draft]

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1 The First Variation of Length

Let (M, g) be a Riemannian manifold of dimension n. Let $c_0 : [a, b] \to M$ be a smooth curve from p to q. A *variation* of c_0 is a smooth function $c : [a, b] \times (-\epsilon, \epsilon) \to M$ such that $c(t, 0) = c_0(t)$ for all $t \in [a, b]$. If we also have that $c(a, s) = c_0(a)$ and $c(b, s) = c_0(b)$ for all $s \in (-\epsilon, \epsilon)$, then we say that c is a *proper* variation of c_0. If, in addition, all the curves $t \mapsto c(t, s)$ are regular (i.e. have nowhere vanishing derivative) then we say that c is a *regular* variation of c_0. (Note that we consider only *smooth* curves and *smooth* variations. We refer the reader to DoCarmo for the more general treatment of piecewise differentiable curves and their variations). The *variational field* of a variation c is a smooth vector field V along c_0 defined by $V(t) = \frac{\partial c}{\partial s}(t, 0)$. Note a variation is proper if and only if its variational field vanishes on the boundary $\{a, b\}$.

Let $c_0 : [a, b] \to M$ be a smooth curve from p to q with a variation c with variational field V. For each fixed s we have a curve and we can speak of its *length*, so we have a $(-\epsilon, \epsilon) \to \mathbb{R}$ function:

$$s \mapsto L(c(\cdot, s)) = \int_a^b \left| \frac{\partial c}{\partial t}(t, s) \right| dt$$

Let us assume that c is a regular variation and compute the derivative of this function:

$$\begin{aligned}
\frac{d}{ds} L(c(\cdot, s)) &= \int_a^b \frac{\partial}{\partial s} \left| \frac{\partial c}{\partial t} \right| dt \\
&= \int_a^b \frac{1}{2} \frac{1}{\left|\frac{\partial c}{\partial t}\right|} \cdot 2 \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \rangle dt \\
&= \int_a^b \frac{1}{\left|\frac{\partial c}{\partial t}\right|} \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \rangle dt \\
&= \int_a^b \frac{1}{\left|\frac{\partial c}{\partial t}\right|} \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \rangle dt \qquad \text{(see hw1 to justify this)} \\
&= \int_a^b \frac{1}{\left|\frac{\partial c}{\partial t}\right|} \left(\frac{\partial}{\partial t} \langle \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \rangle - \langle \frac{\partial c}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial t} \rangle \right) dt
\end{aligned}$$

Evaluating at $s = 0$ and assuming c_0 is parameterized by arclength we have

$$\begin{aligned}
\frac{d}{ds}L(c(\cdot,s)) \mid_{s=0} &= \int_a^b \frac{1}{\left|\frac{dc_0}{dt}\right|}\left(\frac{d}{dt}\langle V(t), \dot{c_0}(t)\rangle - \langle V(t), \nabla_{\frac{\partial}{\partial t}}\dot{c_0}(t)\rangle\right) dt \\
&= \int_a^b \left(\frac{d}{dt}\langle V(t), \dot{c_0}(t)\rangle - \langle V(t), \nabla_{\frac{\partial}{\partial t}}\dot{c_0}(t)\rangle\right) dt \\
&= \langle V(t), \dot{c_0}(t)\rangle \mid_a^b \; - \int_a^b \langle V(t), (\nabla_{\frac{\partial}{\partial t}}\dot{c_0})(t)\rangle dt \qquad (1)
\end{aligned}$$

What we have found is that the first variation of length depends only on the variational field of a variation! So the following definition is a-okay: Let $L : \Omega_{pq} \to \mathbb{R}$, where Ω_{pq} is the set of smooth curves from p to q (Question: Can we give Ω_{pq} a differentiable structure such that the resulting notion of differentiability is compatible with the following definition?). The *first variation of L at the curve c_0 in the direction V* is defined to be

$$\delta L_{c_0}(V) = \frac{d}{ds} \mid_{s=0} L(c(\cdot,s))$$

where c is a variation of c_0 with variational field V. Given any choice of smooth vector field along a curve c_0 it is easy to construct a variation with that vector field as its variational field (for example use $c(t,s) = \exp_{c_0(t)}(sV(t))$).

We demanded that c be a *regular* variation during our computation of the first variation of length, but now that we have an expression for the first variation of length we note that it does in fact work for constant curves.

The first variation of length at c in the direction of a proper V is given by

$$\delta L_{c_0}(V) = - \int_a^b \langle V(t), (\nabla_{\frac{\partial}{\partial t}}\dot{c_0})(t)\rangle dt$$

Theorem 1: $c_0 \in \Omega_{pq}$ is a geodesic $\iff \delta L_{c_0}(V) = 0$ for all proper V.

This means that geodesics are precisely the minimizers for L in Ω_{pq} (restricted to the set of curves that are parameterized by constant speed).

2 The Second Variation

Our next task will be to compute the second variation. To simplify the calculation, we will only do it in the case that the variational field is normal to the curve (i.e. $V(t) \perp \dot{c_0}(t)$ for all $t \in [a,b]$).

Let (M,g) be a Riemannian manifold, let $c_0 : [a,b] \to M$ be a smooth curve, and let $c : [a,b] \times (-\epsilon,\epsilon) \to M$ be a variation of c_0 with variational field V. We already found that

$$\frac{d}{ds}L(c(\cdot,s)) = \int_a^b \frac{1}{\left|\frac{\partial c}{\partial t}\right|}\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\rangle dt$$

So taking another derivative, using compatibility of ∇ with g, and using a lemma proven in homework 2 we get

$$\frac{d^2}{ds^2}L(c(\cdot,s)) =$$

$$\int_a^b \frac{1}{\left|\frac{\partial c}{\partial t}\right|}\left(\langle R\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right)\frac{\partial c}{\partial s} + \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\rangle + \langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s}, \nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial t}\rangle\right) dt + \int_a^b \left(\frac{\partial}{\partial s}\frac{1}{\left|\frac{\partial c}{\partial t}\right|}\right)\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\rangle dt$$

Now let's make some assumptions: (1) V is normal to $\dot{c}_0$. (2) c_0 is a normal geodesic. (3) c is proper. (We will see later that the first assumption doesn't lose us much because tangential variational fields correspond to reparameterizations (although they can correspond to other kinds of variations as well)) Now if we evaluate at $s = 0$ the term

$$\int_a^b \left(\frac{\partial}{\partial s} \frac{1}{|\frac{\partial c}{\partial t}|} \right) \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \rangle dt$$

vanishes because

$$\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \rangle \ |_{s=0} \ = \frac{d}{dt} \langle V(t), \dot{c}_0(t) \rangle - \langle V(t), \nabla_{\frac{\partial}{\partial t}} \dot{c}_0 \rangle = 0$$

and c_0 is a geodesic and V is normal to it. And the term

$$\int_a^b \frac{1}{|\frac{\partial c}{\partial t}|} \langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \rangle dt$$

also vanishes because

$$\begin{aligned}
\int_a^b \frac{1}{|\frac{\partial c}{\partial t}|} \langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \rangle dt \ \ |_{s=0} &= \int_a^b \left(\frac{d}{dt} \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s} \ |_{s=0} \ , \ \dot{c}_0(t) \rangle - \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s} \ |_{s=0} \ , \ \nabla_{\frac{\partial}{\partial t}} \dot{c}_0(t) \rangle \right) dt \\
&= \int_a^b \frac{d}{dt} \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s} \ |_{s=0} \ , \ \dot{c}_0(t) \rangle dt \\
&= \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s} \ , \ \dot{c}_0(t) \rangle \ \ |_{(a,0)}^{(b,0)}
\end{aligned}$$

which vanishes because c is proper (so $s \mapsto c(a,s)$ and $s \mapsto c(b,s)$ are constant curves, which means that $\frac{\partial c}{\partial s}$ vanishes along the integral curve of $\frac{\partial}{\partial s}$ that passes through $(a,0)$ and along the integral curve of $\frac{\partial}{\partial s}$ that passes through $(b,0)$). So at $s = 0$ we have

$$\frac{d^2}{ds^2} L(c(\cdot, s)) \ |_{s=0} \ = \ \int_a^b \left(\langle R(\dot{c}_0(t), V(t))V(t) \ , \ \dot{c}_0(t) \rangle + \left| \nabla_{\frac{\partial}{\partial t}} V(t) \right|^2 \right) dt$$

Since $\dot{c}_0(t)$ and $\frac{1}{|V(t)|} V(t)$ are an orthonormal basis for the two-plane they span in $T_{c_0(t)}M$, the first term in the integral is $-K(\dot{c}_0(t), V(t))|V(t)|^2$ (okay unless $V(t)$ vanishes in which case we make the convention that K vanishes, so the equality does really hold for all t). So we finally have the *second variation of L at c_0 in the direction V (which is proper and $\perp$ to c_0 where c_0 is a normal nontrivial geodesic)*:

$$\delta^2 L_{c_0}(V) = \frac{d^2}{ds^2} L(c(\cdot, s)) \ |_{s=0} \ = \ \int_a^b \left(\left| \nabla_{\frac{\partial}{\partial t}} V(t) \right|^2 - K(\dot{c}_0(t), V(t)) \ |V(t)|^2 \right) dt$$

Observation: If the sectional curvature of M is nonpositive, then the second variation is nonnegative for all proper normal variational fields V and all normal nontrivial geodesics c_0.

3 The Bonnet-Myers Theorem

Let us first observe that the Ricci curvature of S^n is $(n-1)$ (use the fact that it has a constant sectional curvature of 1). And observe also that the diameter of S^n is π. It turns out that spheres are the extreme case of the following theorem, which bounds diameter above when Ricci curvature is bounded below.

Theorem 2: (B-M) Let (M, g) be a complete, connected Riemannian manifold. Suppose $\mathrm{Ric}_M \geq (n-1)\kappa$, with $\kappa > 0$. Then the diameter of M is at most $\pi/\sqrt{\kappa}$.

(An interesting theorem which we will not prove is that when the second inequality is an equality, M is isometric to S^n_ρ where $\rho = \frac{1}{\sqrt{\kappa}}$, the n-sphere of radius ρ.)

Proof: Assume the hypotheses. Let $p, q \in M$. By Hopf-Rinow, there is a minimizing geodesic c from p to q. Let us parameterize it by arclength, so that $c : [0, L] \to M$ and $|\dot{c}(t)| = 1$, where $L = d(p, q)$. Since c is minimizing, we have nonnegative second variation of length for all proper and normal variational fields V. The second variation is

$$\int_0^L \left(\left| \nabla_{\frac{\partial}{\partial t}} V(t) \right|^2 - K(\dot{c}(t), V(t)) \ |V(t)|^2 \right) dt \geq 0$$

We will show that if $L > \frac{\pi}{\sqrt{\kappa}}$ then the above inequality would not hold. Let $e_0 = \dot{c}$, and note that this is a parallel vector field along c. Let $e_1, \cdots, e_{n-1}$ be an extension to an orthonormal parallel basis along c. Let $\eta : [0, L] \to \mathbb{R}$ be an arbitrary smooth function which vanishes on $\{0, L\}$. Let $V_i = \eta e_i$ for $i \in n$, and note that $(\nabla_{\frac{\partial}{\partial t}} V_i)(t) = \eta'(t) e_i(t)$ since e_i is parallel. Now the second variation along V_i is

$$\delta^2 L_c(V_i) = \int_0^L \left(\eta'(t)^2 - K(e_0(t), e_i(t)) \ \eta(t)^2 \right) dt \geq 0$$

And since $\sum \{K(e_0, e_i) \mid i \in n\} = \mathrm{Ric}(e_0)$ we find ourselves with the inequality

$$\int_0^L \left((n-1)\eta'(t)^2 - \mathrm{Ric}(e_0) \ \eta(t)^2 \right) dt \geq 0$$

$$\Rightarrow (n-1) \int_0^L \eta'(t)^2 dt \geq \int_0^L \mathrm{Ric}(e_0) \eta(t)^2 dt \geq (n-1)\kappa \int_0^L \eta(t)^2 dt$$

$$\Rightarrow \int_0^L \eta'(t)^2 dt \geq \kappa \int_0^L \eta(t)^2 dt$$

Now let's try $\eta(t) = \sin(\frac{\pi t}{L})$. This gives

$$\frac{\pi^2}{L^2} \int_0^L \cos^2(t) dt \geq \kappa \int_0^L \sin^2(t) dt$$

$$\Rightarrow \frac{\pi^2}{L^2} \geq \kappa$$

$$\Rightarrow L \leq \frac{\pi}{\sqrt{\kappa}}$$

And so the diameter of M is $\leq \frac{\pi}{\kappa}$

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One interesting corollary is that if (M^n, g) is connected and complete with $\mathrm{Ric}_M \geq (n-1)\kappa$ and $\kappa > 0$ then $\pi_1(M)$ is finite. A completely geometric hypothesis with a topological conclusion! (For example this tells us that the torus does not admit a positive-everywhere Riemannian metric) Proving this is not difficult: Note that the universal cover of M has Ric satisfying the same lower bound (assuming we pull back the differentiable structure and the metric of M). So by the Bonnet-Myers theorem the universal cover is bounded and so since it is complete (also a result of pulling back the metric) it is compact (see Hopf-Rinow). We prove in a homework that a compact universal cover is finitely-sheeted and as a consequence $\pi_1(M)$ is finite.

4 Weinstein-Synge Theorem

Theorem 3: (W-S) Let $f : M \to M$ be an isometry of a compact oriented Riemannian manifold of dimension n. Suppose M has positive sectional curvature and that f is orientation preserving if n is even and orientation reversing if n is odd. Then f has a fixed point.

We will first prove the following consequence of (W-S):

Theorem 4: (Synge) If M^n is a compact manifold with positive sectional curvature then (1) if it is orientable and n is even then it is simply connected, and (2) if n is odd then it is orientable

Proof: Even case: Suppose M is orientable and n is even. Consider the universal cover $p : (\tilde{M}, \tilde{g}) \to (M, g)$ (We get the universal cover as a topological space, then we give it a differentiable structure by pulling back the differentiable structure of M, and we give it the metric p^*g). Let $f : \tilde{M} \to \tilde{M}$ be a deck transformation. It is an isometry (this is an easy homework: $\tilde{g} = p^*\tilde{g} = (f \circ p)^*g = f^*p^*g = f^*\tilde{g}$, and of course f is a diffeomorphism). Does f preserve the orientation of M? Yes, because locally the deck transformation has the form $(\pi|_{V_2})^{-1} \circ (\pi|_{V_1})$ for some pancakes V_1, V_2 of an appropriate plate neighborhood V; and π is itself orientation preserving because we used it to pullback the oriented atlas of M to give $\tilde{M}$ its atlas and therefore its orientation. Therefore we can apply (W-S) to f and see that it has a fixed point. And so f is the identity map. So the deck transformation group of M is trivial; that is, M is simply connected! Proof of the odd case: Suppose n is odd. Suppose, by way of contradiction, that (M, g) is nonorientable. Consider the orientable double-cover $\pi : \tilde{M} \to M$ of M, and of course give it the metric π^*g. Let $F : \tilde{M} \to \tilde{M}$ be a nontrivial deck transformation which reverses orientation (one exists because $M \cong \tilde{M}/\mathrm{deck}(\pi)$. Then by $W - S$ F has a fixed point, so it's the identity (which preserves orientation).

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Let us try to prove W-S theorem. First we need to go back and look at the first and second variations of length in the more general case of a non-proper variation. Let $c : [a, b] \times (-\epsilon, \epsilon) \to M$ be a variation of a normal-speed curve c_0, with variational field V (not necessarily proper). Then

$$\delta L_{c_0}(V) = \langle V(t), \dot{c}_0(t)\rangle \;|_a^b \; - \int_a^b \langle V(t), (\nabla_{\frac{\partial}{\partial t}} \dot{c}_0)(t)\rangle dt$$

This is what we had when we were computing the first variation before; the only difference is that this time the boundary term does not necessarily vanish. For the second variation we had:

$$\frac{d^2}{ds^2} L(c(\cdot, s)) =$$
$$\int_a^b \frac{1}{|\frac{\partial c}{\partial t}|} \left(\langle R\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right)\frac{\partial c}{\partial s} + \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\rangle + \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial t}\rangle \right) dt + \int_a^b \left(\frac{\partial}{\partial s} \frac{1}{|\frac{\partial c}{\partial t}|} \right) \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t}\rangle dt$$

If we assume again that c_0 is a normal *geodesic* and that V is normal to c_0 (without assuming that V is proper, then after evaluating at $s = 0$ we compute:

$$\frac{d^2}{ds^2} L(c(\cdot, s)) \;|_{s=0} \; = \int_a^b \left(\left|\nabla_{\frac{\partial}{\partial t}} V(t)\right|^2 - K(\dot{c}_0(t), V(t)) \; |V(t)|^2 \right) dt + \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s} \; , \; \dot{c}_0(t)\rangle \;|_{(a,0)}^{(b,0)} \qquad (2)$$

Note that because this expression contains the term $\nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s}$, it actually depends on the variation c itself and not only on its variational field V! So a notation like "$\delta^2 L_{c_0}(V)$" for the second variation would be incorrect. Notice also that the boundary term that appears at the end vanishes when the transversal curves $s \mapsto c(t, s)$

are themselves geodesics. We are now ready to proceed with the proof of W-S. We will use the following lemma in the proof:

Theorem 5: Let A be an orthonormal linear operator on $\mathbb{R}^{n-1}$ and suppose $\det(A) = (-1)^n$. Then 1 is an eigenvalue of A.

The proof of the lemma appears in DoCarmo, and will be left out here. Finally:

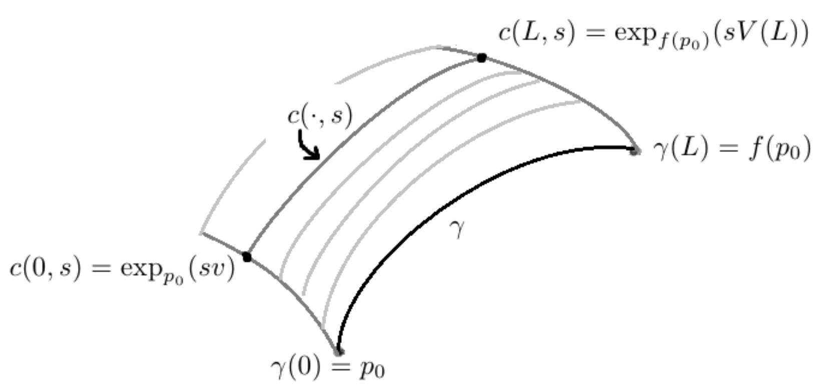
Theorem 6: (W-S) Let $f : M \to M$ be an isometry of a compact oriented Riemannian manifold of dimension n. Suppose M has positive sectional curvature and that f is orientation preserving if n is even and orientation reversing if n is odd. Then f has a fixed point.

Proof: Assume the hypotheses. Suppose, by way of contradiction, that f has no fixed point. Let p_0 be a point in M at which the mapping $p \mapsto d(p, f(p))$ attains its minimum (M is compact). Let $\gamma : [0, L] \to M$ be a normalized minimizing geodesic from p_0 to $f(p_0)$, where $L = d(p_0, f(p_0)) > 0$. (Note: The second variation of length will of course be non-negative if we consider *proper* variational fields. This just follows from the fact that γ is a minimizing geodesic. If we want to squeeze any more information out of the second variation and the hypotheses above we will have to consider nonproper variations, and this is what motivates the rest of the proof).

Pick some $v \in \dot{\gamma}(0)^\perp \subset T_{p_0}M$ (the orthogonal complement of $\dot{\gamma}(0)$). This choice is arbitrary for now, but later in the proof we will go back and make sure v was chosen in a special way. Now parallel transport v along γ to produce a *parallel* vector field V along γ. Define $c : [0, L] \times (-\epsilon, \epsilon) \to M$ by $c(t, s) = \exp_{\gamma(t)}(sV(t))$, a variation of γ with variational field V in which every transversal curve is a geodesic. (Question: How was ϵ chosen? Is it such that exp is a diffeo on ϵ-balls in all the tangent spaces along γ?) Because we chose V to be parallel and the transversal curves to be geodesics, the second variation of length is:

$$-|v|^2 \int_a^b K(\gamma(t), V(t))dt$$

which is clearly strictly negative. (Note: We would get the desired contradiction if we knew for sure that $s \mapsto L(c(\cdot, s))$ achieves its minimum at $s = 0$. However this does *not* follow from γ being a minimizing geodesic, because the curves in the family c have *varying endpoints*.) This is the current picture:



Here is how the proof will work: If we had that $c(L, s) = f(c(0, s))$ then we could get the contradiction we need from the following inequality for all s:

$$\begin{aligned}L(c(\cdot,s)) &\geq d(c(0,s),f(c(0,s)))\\ &\geq d(p_0,f(p_0))=L(\gamma)\end{aligned}$$

The trouble is that we do not have $c(L,s)=f(c(0,s))$... unless we choose v very cleverly. f is an isometry, so it carries geodesics to geodesics. And $s\mapsto c(L,s)$ is a geodesic. And $c(L,0)=f(p_0)=f(c(0,0))$. So if we could just get $s\mapsto c(L,s)$ and $s\mapsto f(c(0,s))$ to agree on their initial velocity then we would have the desired equality by uniqueness of geodesics and the proof would be complete. The initial velocity of $s\mapsto c(L,s)$ is $V(L)$ and the initial velocity of $s\mapsto f(c(0,s))$ is $\frac{d}{ds}f(c(0,s))|_{s=0}=df_{p_0}(\frac{d}{ds}c(0,s)|_{s=0})=df_{p_0}(v)$. So the entire proof now hinges upon finding a $v\in\dot\gamma(0)^\perp\subset T_{p_0}M$ such that when V is parallel along γ and $V(0)=v$, we have $V(L)=df_{p_0}(v)$.

Consider the linear map $\tilde A=P\circ df_{p_0}:T_{p_0}M\to T_{p_0}M$, where $P:T_{f(p_0)}M\to T_{p_0}M$ is parallel transport along γ (for example $P(V(L))=v$). Consider the restriction of $\tilde A$ to $\dot\gamma(0)^\perp$, and call that A.

Claim: $A:\dot\gamma(0)^\perp\to\dot\gamma(0)^\perp$ (i.e. $\dot\gamma(0)^\perp$ contains the range of A)
Pf: df_{p_0} and P are orthogonal, so $\tilde A$ is orthogonal. The claim is proven if we can show in addition that $\tilde A(\dot\gamma(0))=\dot\gamma(0)$. That is, we need to show that $df_{p_0}(\dot\gamma(0))=P^{-1}(\dot\gamma(0))=\dot\gamma(L)$. Consider an arbitrary $t\in(0,L)$, and let $p'=\gamma(t)$. Then

$$\begin{aligned}d(p',f(p'))&\leq d(p',f(p_0))+d(f(p_0),f(p'))\\ &=d(p',f(p_0))+d(p_0,p') && \text{(because } f \text{ is an isometry)}\\ &=d(p_0,f(p_0)) && \text{(because those were portions of the same minimizing geodesic)}\\ &\leq d(p',f(p')) && \text{(because } d(p_0,f(p_0)) \text{ was minimal)}\end{aligned}$$

and so the above is string of equalities. In particular:

$$d(p',f(p'))=d(p',f(p_0))+d(f(p_0),f(p'))$$

Consider the concatenation of the curves γ and $f\circ\gamma$ (which would have domain $[0,2L]$) restricted to $[t,t+L]$. The above equality shows that this curve (let's call it c_0) is the shortest between its endpoints, p' and $f(p')$. This implies that the first variation of this curve must vanish for all proper variational fields. Computing the first variation of length for a proper variational field X along c_0 requires dealing with *piecewise smooth* curves, which we have not yet done. But the calculation is not very different; we simply perform the same calculation as before on the two smooth pieces of c_0. The calculation gives the following for the first variation of length at c_0 along proper X:

$$-\int_t^{t+L}\langle X(t'),\nabla_{\frac{\partial}{\partial t}}\dot{c_0}(t')\rangle dt'+\langle\dot{c_0}(L)_- - \dot{c_0}(L)_+,X(L)\rangle$$

Since c_0 is minimizing it must be minimizing on its pieces and so it satisfies the geodesic equation. Therefore the integral vanishes and we are left with the other term. If we then choose some proper variation in such a way that $X(L)=\dot{c_0}(L)_- - \dot{c_0}(L)_+$, then we get $|\dot{c_0}(L)_- - \dot{c_0}(L)_+|^2$ as the first variation of length. But we found above that the first variation of length vanishes for c_0, and so $\dot{c_0}(L)_-=\dot{c_0}(L)_+$. In other words, $\dot\gamma(L)=df(\dot\gamma(0))$, completing the proof of the claim.

We apply theorem 5 to A. $\dot\gamma(0)^\perp$ is $(n-1)$-dimensional. We just need to make sure the determinant condition is satisfied. We have $\det A=\det\tilde A$ because $\tilde A(\dot\gamma(0))=\dot\gamma(0)$. We have $\det P=+1$ because M is oriented (showing this is a homework). And we have $\det df_{p_0}=(-1)^n$ from the hypotheses on f. So $\det A=\det\tilde A=(\det P)(\det df_{p_0})=(-1)^n$. Therefore A has 1 as an eigenvalue, and we can finally choose v! Get $v\in\dot\gamma(0)^\perp$ such that $A(v)=v$. Then $V(L)=P^{-1}(v)=df_{p_0}(v)$, and the proof is complete.

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5 Some Homeworks

Theorem 7: (doCarmo problem 9.4) Let M^n be an orientable Riemannian manifold with positive sectional curvature and even dimension. Let $\gamma : [0, L] \to$ be a normal geodesic in M such that $\dot{\gamma}(0) = \dot{\gamma}(L)$ (so obviously also $\gamma(0) = \gamma(L)$). Then there is some closed curve $\gamma' : [0, L] \to M$ smoothly homotopic to γ such that γ' is strictly shorter than γ.

Proof: Assume the hypotheses. Let $p = \gamma(0)$. Let $P : T_pM \to T_pM$ denote parallel transport along γ from 0 to L. Note that since $\dot{\gamma}$ is a parallel vector field along γ we have $P(\dot{\gamma}(0)) = \dot{\gamma}(L) = \dot{\gamma}(0)$. Therefore, since P is orthogonal P restricts to a map $P' : \dot{\gamma}(0)^\perp \to \dot{\gamma}(0)^\perp$ on the orthogonal complement of $\dot{\gamma}(0)$ in T_pM. Since P preserves orientation (by a previous homework) and M is of even dimension we have

$$\det P' = \det P = 1 = (-1)^n$$

so P' satisfies the conditions of Lemma 3.8 in doCarmo. Get a vector $v \in \dot{\gamma}(0)^\perp$ such that $P(v) = v$. Let V be the vector field along γ obtained by parallel transport of v along γ. Note that $V(0) = V(L) = v$. Since γ is a geodesic the first variation of length at γ along any proper variational field is 0. However we cannot use this result on V, because it is not proper. But looking to the expression for the first variation in the first section of these notes (eqn 1), we see that the boundary term

$$\langle V(L), \dot{\gamma}(L)\rangle - \langle V(0), \dot{\gamma}(0)\rangle$$

still vanishes! Therefore the first variation of length at γ along V is indeed 0. If we can now show that the second variation of length at γ along V is negative, then we will know that there is a curve near γ with strictly shorter length.

More explicitly: We know how to use the Riemannian exp to construct a smooth variation $c : [0, L]\times(-\epsilon, \epsilon) \to M$ which has variational field V (this is seen on p193 in doCarmo). By applying basic analysis to the function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ defined by $s \mapsto L(c(\cdot, s))$ using the fact that $f'(0) = 0$ and $f''(0) < 0$, we can find an $s_0 \in (-\epsilon, \epsilon)$ such that the curve $t \mapsto c(t, s_0)$ is the desired γ', i.e. it is strictly shorter than γ. The smooth homotopy is $c|_{[0,L]\times[0,s_0]}$. Notice that the construction of c as $c(t, s) = \exp_{\gamma(t)}(sV(t))$ automatically ensures that $c(0, s) = c(L, s)$ for all s. Therefore γ' is a *closed* curve. So if we can show that $f''(0) < 0$ this proof will be complete.

Let c be a variation of γ as described in the paragraph above. From equation 2 in these notes, we have the second variation of length:

$$\frac{d^2}{ds^2}L(c(\cdot, s)) \;|_{s=0} \;=\; \int_0^L \left(\left|\nabla_{\frac{\partial}{\partial t}}V(t)\right|^2 - K(\dot{\gamma}(t), V(t))\; |V(t)|^2\right) dt + \langle\nabla_{\frac{\partial}{\partial s}}\frac{\partial c}{\partial s}\;,\; \dot{\gamma}(t)\rangle\;\; |_{(0,0)}^{(L,0)}$$

Now two annoying terms conveniently vanish: The first term in the integral vanishes because V is parallel along γ, and the final boundary term vanishes because the transversal curves of c are geodesics (remember that we defined c using exp). We are left with

$$\frac{d^2}{ds^2}L(c(\cdot, s)) \;|_{s=0} \;=\; -\int_0^L K(\dot{\gamma}(t), V(t))\; |V(t)|^2 dt$$

which is strictly negative because the sectional curvature of M is strictly positive.

∎

Theorem 8: Let N_1, N_2 be closed disjoint submanifolds of a compact Riemannian manifold M. Then the distance between N_1 and N_2 is assumed by a geodesic perpendicular to N_1, N_2.

Proof: N_1, N_2 are compact so their distance is positive and there are two points $p \in N_1$ and $q \in N_2$ whose distance is precisely that distance. Let γ be a geodesic in M from p to q (M is complete). It remains to check that γ is perpendicular to both N_1 and N_2. Now since the distance between p and q is a minimal distance for pairs of points from $N_1 \times N_2$, the first variation of length at γ along any family of curves with endpoints in N_1, N_2 must vanish. Let's assume $\gamma : [0, L] \to M$ is parameterized by arclength. Then equation 1 and the fact that γ is a normal geodesic tells us that for any variational field V which corresponds to some variation with endpoints in N_1, N_2 we have

$$0 = \langle V(t), \dot{\gamma}(t)\rangle|_0^L$$

So the proof will be complete if we can show that given any $v \in T_pN_1 \subset T_pM$ we can find a variation c of γ with variational field V such that $V(0) = v$ and $V(L) = 0$, and such that $c(s, 0) \in N_1$ and $c(s, L) \in N_2$ for all $s \in (-\epsilon, \epsilon)$. (Because then the equation above holds and gives exactly the desired result, and a symmetric argument could be applies to q and T_qN_2 to the give the other half of the result. Or we could just flip γ).

Consider any $v \in T_pN_1 \subset T_pM$. Get a path $\sigma : (-e', e') \to N_1$ such that $\sigma'(0) = v$. Choose $e'' > 0$ such that $\exp_p$ is a diffeo on a ball of radius e'' at the origin in T_pM. Get $e > 0$ such that $\sigma((-e, e)) \subseteq \exp_p(B_{e''}(0))$. Define $\widetilde{\sigma} : (-e, e) \to T_pM$ to be $\exp_p^{-1} \circ \sigma|_{(-e,e)}$. Define $c' : (-e, e) \times [0, L] \to M$ by

$$c'(s, t) = \exp_{\gamma(t)}(P_{\gamma,0,t}(\eta(t)\widetilde{\sigma}(s)))$$

where $P_{\gamma,0,t} : T_pM \to T_{\gamma(t)}M$ denotes parallel transport along γ, and $\eta : [0, L] \to \mathbb{R}$ is a smooth function such that $\eta(0) = 1$ and $\eta(L) = 0$. Then it is easily checked that c has the desired properties, and the proof is complete.

■

6 Intro to Spaces of Constant Curvature

This section will be an attempt to classify spaces of constant curvature which are connected, simply connected, and complete. Since we can always rescale curvatures by rescaling metrics, we only really need to think about curvatures of 1, -1, and 0.

We begin the discussion with three model spaces: $\mathbb{R}^n, S^n$, and H^n. We already know how to construct $\mathbb{R}^n$. H^n, hyperbolic space, is the upper half plane $\{(x^1, x^2, \cdots, x^{n-1}, y) \mid x^1, \cdots, x^{n-1} \in \mathbb{R} \,;\, y > 0\}$ with topology and differentiable structure inherted from $\mathbb{R}^n$ and with with metric

$$g_{H^n} = \frac{1}{y^2}((dx^1)^2 + \cdots + (dx^{n-1})^2 + dy^2)$$

Notice that H^n is *diffeomorphic* to $\mathbb{R}^n$.

We can try to study these model spaces by using so-called "generalized coordinates." For example, we can study $\mathbb{R}^n$ by defining the following diffeomorphism to $\mathbb{R}^n \setminus \{0\}$:

$$\psi : S^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n \setminus \{0\}, \quad \psi(w, r) = rw$$

We can use ψ to pull back the euclidian metric on $\mathbb{R}^n \setminus \{0\}$: $\psi^* g_{\mathbb{R}^n}$. It is true that $S^{n-1} \times \mathbb{R}^+$ already has the product metric $dr^2 + g_{S^{n-1}}$, but pulling back the metric on $\mathbb{R}^n \setminus \{0\}$ gives a slightly different metric:

$$\psi^* g_{\mathbb{R}^n} = dr^2 + r^2 g_{S^{n-1}}$$

(This is not too difficult to compute, just evaluate $\psi^* g_{\mathbb{R}^n}$ at (w, r) at different combinations of $\frac{\partial}{\partial r}|_{w,r}$ and arbitrary elements of T_wS^{n-1}).

We can apply the same method on S^{n-1}. Choose a north and south pole: say $N = (0, \cdots, 0, 1)$ and $S = -N$. Then define a diffeomorphism to $S^n \setminus \{N, S\}$ as follows:

$$\psi : S^{n-1} \times (0, \pi) \to S^n \setminus \{N, S\}, \qquad \psi(w, r) = (w \sin r, \cos r)$$

($w \sin r$ is the first $n-1$ components and $\cos r$ is the last component). This in a sense "coordinatizes" the n-sphere minus the poles by naming each point in terms of some point w on the equator of S^n and some angle r from the north pole. Pulling back the metric on S^n yields:

$$\psi^* g_{S^n} = dr^2 + \sin^2 r g_{S^{n-1}}$$

Now we're not quite going to repeat the process for H^n. Instead of coming up with generalized coordinates and pulling back the metric, we will invent a metric directly: We will consider $S^{n-1} \times \mathbb{R}^+$ with the metric $g_H = dr^2 + \sinh^2 r g_{S^{n-1}}$, which is inspired by the paragraph above. Let $\psi : S^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n \setminus \{0\}$ with $\psi(w, r) = wr$ as before. Let $\phi = \psi^{-1}$. It turns out that $\phi^* g_H$ extends smoothly across the origin. (Why? We showed this in class. Note that in the pulled back r coordinate we have $g_{\mathbb{R}^n} = dr^2 + r^2 g_{S^{n-1}}$ and write $\phi^* g_H$ as $g_{\mathbb{R}^n} + \frac{\sinh^r - r^2}{r^2}(g_{\mathbb{R}^n} - dr^2)$ then think about it).

Our next task is to prove some things about g_H: That it has constant sectional curvature -1, and that the extended metric on $\mathbb{R}^n$ is complete!

Before we continue studying spaces of constant curvature, we will develop some more advanced techniques to compute curvature...

7 Connection Forms and the Curvature Form

Let (M^n, g) be a Riemannian manifold with local orthonormal frame $\{e_1, \cdots, e_n\}$ defined on U and with dual frame $\omega_1, \cdots, \omega_n$. Note that since the covariant derivative ∇ is tensorial in its first variable, upon omitting the first variable we find ourselves left with a (1,1) tensor. The *connection forms* of ∇ associated to the local orthonormal frame $\{e_i\}$ are 1-forms ω_j^i such that

$$\nabla_v e_i = \omega_i^j(v) e_j \qquad\qquad \text{i.e., } \nabla e_i = \omega_i^j \otimes e_j$$

Now since the basis is orthonormal we have for $v \in T_pM$, $p \in U$:

$$\begin{aligned} 0 = v(\langle e_i, e_j \rangle) &= \langle \nabla_v e_i, e_j \rangle + \langle e_i, \nabla_v e_j \rangle \\ &= \langle \omega_i^k(v) e_k, e_j \rangle + \langle e_i, \omega_j^k(v) e_k \rangle \end{aligned}$$

Or simply:

$$\langle \omega_i^k \otimes e_k, e_j \rangle + \langle e_i, \omega_j^k \otimes e_k \rangle = 0$$

where it is understood that the inner product applies to the vector component of the tensor product. This gives the result

$$\omega_j^i + \omega_i^j = 0 \tag{3}$$

That is, the matrix of 1-forms ω is skew-symmetric.

(Side note on taking the covariant derivative of tensor fields: We define the covariant derivative of a function f to simply be df. We then start with a definition of covariant derivative for vector fields, which is pinned down by the requirement that it be linear/tensorial, leibniz, metric-compatible, and torsion-free. And we extend to tensors of other rank by demanding two properties: commutativity of the connection with *contractions*, and leibniz rule of the connection over tensor products. In particular the derivative of a 1-form ends up being defined by $(\nabla_v \omega)(X) = v(\omega(X)) - \omega(p)(\nabla_v X)$ for X a smooth vector field)

Consider applying a derivative ∇ to the (1,1) tensor $\nabla e_i = \omega_i^k \otimes e_k$. We get the (1,2) tensor $\nabla^2 e_i$ whose value at smooth vector fields X, Y is:

$$\begin{aligned}(\nabla^2 e_i)(X,Y) &= ((\nabla \omega_i^j) \otimes e_j + \omega_i^j \otimes (\nabla e_j))(X,Y)\\ &= (\nabla_X \omega_i^j)(Y) e_j + \omega_i^j(Y)(\nabla_X e_j)\\ &= (\nabla_X \omega_i^j)(Y) e_j + \omega_i^j(Y)\omega_j^k(X) e_k\end{aligned}$$

The interesting thing about $\nabla^2 e_i$ is its close relationship with curvature. We can evaluate its commutator using the rules for ∇ on (1,2) tensors:

$$\begin{aligned}-(\nabla^2 e_i)(X,Y) + (\nabla^2 e_i)(Y,X) &= -(\nabla_X(\nabla e_i))(Y) + (\nabla_Y(\nabla e_i))(X)\\ &= -\nabla_X \nabla_Y e_i + \nabla_{\nabla_X Y} e_i + \nabla_Y \nabla_X e_i - \nabla_{\nabla_Y X} e_i\\ &= \nabla_Y \nabla_X e_i - \nabla_X \nabla_Y e_i + \nabla_{\nabla_X Y - \nabla_Y X} e_i\\ &= \nabla_Y \nabla_X e_i - \nabla_X \nabla_Y e_i + \nabla_{[X,Y]} e_i\\ &= R(X,Y) e_i\end{aligned}$$

So let us compute the commutator of $\nabla^2 e_i$ in terms of the connection one-forms:

$$\begin{aligned}-(\nabla^2 e_i)(X,Y) + (\nabla^2 e_i)(Y,X) &= [(\nabla_Y \omega_i^j)(X) - (\nabla_X \omega_i^j)(Y)] e_j + (\omega_i^j(X)\omega_j^k(Y) - \omega_i^j(Y)\omega_j^k(X)) e_k\\ &= [Y(\omega_i^j(X)) - \omega_i^j(\nabla_Y X) - X(\omega_i^j(Y)) + \omega_i^j(\nabla_X Y)] e_j + (\omega_i^j \wedge \omega_j^k)(X,Y) e_k\\ &= [Y(\omega_i^j(X)) - X(\omega_i^j(Y)) - \omega_i^j([X,Y])] e_j + (\omega_i^j \wedge \omega_j^k)(X,Y) e_k\\ &= d\omega_i^j(Y,X) e_j + (\omega_i^j \wedge \omega_j^k)(X,Y) e_k\\ &= -d\omega_i^j(X,Y) e_j + (\omega_i^k \wedge \omega_k^j)(X,Y) e_j\\ &= -(d\omega_i^j + (\omega_k^j \wedge \omega_i^k))(X,Y) e_j\end{aligned}$$

We define the *curvature forms* Ω_i^j associated with the local orthonormal fram $\{e_i\}$ to be the two-forms:

$$\Omega_i^j = d\omega_i^j + \omega_k^j \wedge \omega_i^k$$

Note that the matrix of curvature forms is skew-symmetric. Here is the neat result from all this work:

$$R(X,Y)e_i = -\Omega_i^j(X,Y) e_j$$

And

$$\begin{aligned}K(e_i, e_j) = \langle R(e_i, e_j) e_i, e_j \rangle &= \langle -\Omega_i^k(e_i, e_j) e_k, e_j \rangle\\ &= \Omega_j^i(e_i, e_j) \qquad \text{(with no sum over } i\text{)}\end{aligned}$$

Okay we almost have everything we need to start using this to compute curvature. The missing piece is having a way to *find* the connection 1-forms. What remains is to compute $d\omega^i$ and then use that and skew-symmetry to characterize connection 1-forms. Let's begin with $d\omega^i$:

$$\begin{aligned}d\omega^i(e_k, e_l) &= e_k(\omega^i(e_l)) - e_l(\omega^i(e_k)) - \omega^i([e_k, e_l])\\ &= -\omega^i([e_k, e_l])\\ &= -\omega^i(\nabla_{e_k} e_l - \nabla_{e_l} e_k)\\ &= -\omega \qquad\qquad i(\omega_l^m(e_k) e_m - \omega_k^m(e_l) e_m)\\ &= -\omega_l^i(e_k) + \omega_k^i(e_l)\\ &= (\omega^j \wedge \omega_j^i)(e_k, e_l)\end{aligned}$$

So we have

$$d\omega^i = \omega^j \wedge \omega^i_j \tag{4}$$

How do we actually calculate the connection one-forms in practice? Well we *could* use a coordinate basis which has been gramm-schmidt orthonormalized and compute the connection one-forms from the corresponding christoffel symbols. But that is absolutely disgusting, and the point of this technique was to avoid such a calculation. The way we compute the connection one-forms is by cleverly finding a matrix of one-forms ω^i_j that satisfies equations 3 and 4! Check it out:

Theorem 9: The connection one-forms $\{\omega^i_j\}$ corresponding to (M, g) and a local orthonormal coframe $\{\omega^1, \cdots, \omega^n\}$ are completely characterized by equations 3 and 4.

Proof: This proof is a homeowork. Suppose (M, g) is a Riemannian manifold and $\{\omega^1, \cdots, \omega^n\}$ is a local orthonormal coframe with dual frame $\{e_1, \cdots, e_n\}$. Suppose we have a collection of n^2 one-forms $\{\omega^i_j\}$ such that:

$$\omega^i_j + \omega^j_i = 0$$

and:

$$d\omega^i = \omega^j \wedge \omega^i_j$$

Then from the above identity we have

$$\begin{aligned} d\omega^i(e_j, e_l) &= (\omega^k \wedge \omega^i_k)(e_j, e_l) \\ &= \omega^i_j(e_l) - \omega^i_l(e_j) \end{aligned}$$

And from the definition of the exterior derivative (or an identity if you prefer):

$$\begin{aligned} d\omega^i_(e_j, e_l) &= e_j(\omega^i(e_l)) - e_l(\omega^i(e_j)) - \omega^i([e_j, e_l]) \\ &= -\omega^i([e_j, e_l]) \\ &= \omega^i(\nabla_{e_l} e_j - \nabla_{e_j} e_l) \end{aligned}$$

So we have

$$\omega^i_j(e_l) - \omega^i_l(e_j) = \omega^i(\nabla_{e_l} e_j) - \omega^i(\nabla_{e_j} e_l)$$

The first term on each side of the equality is skew-symmetric in the indices i, j, and the second term on each side of the equality is skew-symmetric in the indices i, l. The skew-symmetries on the left-hand-side are hypotheses of the theorem. The skew-symmetries on the right-hand-side follow from the fact that "$\omega^i(\nabla_{e_l} e_j)$" is a fancy way of saying "the i^{th} component function of the vector field $\nabla_{e_l} e_j$." In other words it is the true $(i, j)^{\text{th}}$ connection 1-form associated with the coframe $\{\omega^k\}$ evaluated at e_l. Keeping this in mind we can pull the following trick: add the equation to itself with indices permuted cyclically, and then subtract a copy of the equation with a further cyclic permutation of indices. The result is a lot of nice cancellations and a final equality:

$$2\omega^i_j(e_l) = 2\omega^i(\nabla_{e_l} e_j)$$

■

We can use the curvature forms to see that under the assumption of constant sectional curvature (i.e. $\Omega^i_j = c\omega^u \wedge \omega^j$) we have the following nice equality for curvature:

$$(X, Y, Z, W) = c(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle)$$

We are now ready to use the connection one-forms to calculate the sectional curvature of our model space for (hopefully) hyperbolic space. But first, just for fun let's use this machinery to give an alternate proof of the fundamental theorem of Riemannian geometry:

Theorem 10: Let (M, g) be a Riemannian manifold. Then there exists a unique $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which is tensorial in the first variable, linear and leibniz in the second variable, torsion free, and compatible with g.

Proof: Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame with dual frame $\{\omega^1, \cdots, \omega^n\}$. Define a collection of one-forms $\{\omega_j^i\}$ by:

$$\omega_k^i(e_l) = \frac{1}{2}(d\omega^i(e_k, e_l) + d\omega^k(e_l, e_i) - d\omega^l(e_i, e_k)) \tag{5}$$

Note that we now automatically have $\omega_j^i = -\omega_i^j$ and $d\omega^i = \omega^k \wedge \omega_k^i$. Now define

$$\nabla(X, e_i) = \omega_i^j(X)e_j$$

and extend this definition for $Y = f^i e_i$ by

$$\nabla(X, Y) = X(f^i)e_i + f^i\nabla(X, e_i)$$

Note that this definition is by construction tensorial in the first variable and linear and leibniz in the second. It remains to check that ∇ is torsion-free, metric-compatible, unique, and that it is well-defined (independent of choice of orthonormal frame, so we can globally define ∇ using local frames). If we fix a local orthonormal frame beforehand then uniqueness and well-definedness follow from theorem 9. So we really just need to ensure that choosing a different orthonormal coframe $\{\omega'^1, \cdots, \omega'^n\}$ gives the same ∇. I'm not going to do this explicitly because it is tedious, but it is not too hard to see. Consider some other connection ∇' defined similarly to ∇ except using a different local orthonormal coframe $\{\omega'^1, \cdots, \omega'^n\}$ with dual frame $\{e'_1, \cdots, e'_n\}$. Then it is easily checked that $\nabla(X, Y)$ and $\nabla'(X, Y)$ are two ways of writing the same thing in different bases.

Let's check that ∇ is torsion-free. Note that

$$\begin{aligned}
\omega^k([e_i, e_j]) &= -d\omega^k(e_i, e_j) + e_i(\omega^k(e_j)) - e_j(\omega^k(e_i)) \\
&= \omega_l^k \wedge \omega^l(e_i, e_j) + e_i(\omega^k(e_j)) - e_j(\omega^k(e_i)) \\
&= \omega_j^k(e_i) - \omega_i^k(e_j) + e_i(\omega^k(e_j)) - e_j(\omega^k(e_i)) \\
&= \omega_j^k(e_i) - \omega_i^k(e_j)
\end{aligned}$$

For vector fields $X = X^i e_i,\ Y = Y^i e_i$ we have

$$\begin{aligned}
\nabla(X, Y) - \nabla(Y, X) &= X(Y^i)e_i - Y(X^i)e_i + Y^i\nabla(X, e_i) - X^i\nabla(Y, e_i) \\
&= (X(Y^i) - Y(X^i))e_i + Y^i\omega_i^j(X)e_j - X^i\omega_i^j(Y)e_j \\
&= (X(Y^i) - Y(X^i))e_i + \omega_i^j(Y^iX - X^iY)e_j \\
&= (X(Y^i) - Y(X^i) + \omega_j^i(Y^jX - X^jY))e_i
\end{aligned}$$

and we have

$$\begin{aligned}
[X,Y] &= [X^i e_i, Y] \\
&= X^i[e_i, Y] - Y(X^i)e_i \\
&= X^iY^j[e_i,e_j] + X^ie_i(Y^j)e_j - Y(X^i)e_i \\
&= X^iY^j[e_i,e_j] + X(Y^i)e_i - Y(X^i)e_i \\
&= X^iY^j[\omega_j^k(e_i) - \omega_i^k(e_j)]e_k + (X(Y^i) - Y(X^i))e_i \\
&= [X^iY^j\omega_j^k(e_i) - X^jY^i\omega_j^k(e_i)]e_k + (X(Y^i) - Y(X^i))e_i \\
&= [Y^j\omega_j^k(X) - X^j\omega_j^k(Y)]e_k + (X(Y^i) - Y(X^i))e_i \\
&= [\omega_j^k(Y^jX - X^jY)]e_k + (X(Y^i) - Y(X^i))e_i \\
&= [\omega_j^i(Y^jX - X^jY) + X(Y^i) - Y(X^i)]e_i
\end{aligned}$$

So

$$\nabla(X,Y) - \nabla(Y,X) = [X,Y]$$

and ∇ is indeed torsion-free.

It remains only to check compatibility of ∇ with g. For vector fields $X = X^ie_i,\ Y = Y^ie_i,\ Z = Z^ie_i$ we have

$$\begin{aligned}
Z(\langle X,Y\rangle) &= Z(\sum_i X^iY^i) \\
&= \sum_i (Y^iZ(X^i) + X^iZ(Y^i))
\end{aligned}$$

and we have

$$\begin{aligned}
\langle\nabla(Z,X),Y\rangle + \langle X,\nabla(Z,Y)\rangle &= \langle Z(X^i)e_i + X^i\nabla(Z,e_i),Y\rangle + \langle X, Z(Y^i)e_i + Y^i\nabla(Z,e_i)\rangle \\
&= \langle Z(X^i)e_i + X^i\omega_i^j(Z)e_j, Y\rangle + \langle X, Z(Y^i)e_i + Y^i\omega_i^j(Z)e_j\rangle \\
&= \langle Z(X^i)e_i + X^j\omega_j^i(Z)e_i, Y\rangle + \langle X, Z(Y^i)e_i + Y^j\omega_j^i(Z)e_i\rangle \\
&= \sum_i [Y^i(Z(X^i) + X^j\omega_j^i(Z)) + X^i(Z(Y^i) + Y^j\omega_j^i(Z))] \\
&= Z(\langle X,Y\rangle) + \sum_i [X^jY^i + X^iY^j]\omega_j^i(Z) \\
&= Z(\langle X,Y\rangle)
\end{aligned}$$

where in the last line we have used antisymmetry of $\{\omega_j^i\}$

■

8 Back to Spaces of Constant Curvature

Let's try to apply the results of the previous section to $S^{n-1} \times \mathbb{R}^+$ with the metric

$$g_f = dr^2 + f^2(r) g_{S^{n-1}}$$

(where r is the name of the coordinate on the $\mathbb{R}^+$ factor of the space.) Let $\{\theta^i\}$ be a local orthonormal coframe for S^{n-1} with a corresponding local orthonormal frame $\{\epsilon_i\}$. Define the following local orthonormal frame for our manifold:

$$e_i = \frac{1}{f}\epsilon_i$$

for $1 \leq i \leq n-1$ and

$$e_n = \frac{\partial}{\partial r}$$

The corresponding dual frame is:

$$\omega^i = f\theta^i \text{ for } 1 \leq i \leq n-1 \text{ and } \quad \omega^n = dr$$

Let's check out the derivative so we can get a hint as to what would be a good guess for the connection one-forms. First note that $d\omega^n$ clearly vanishes. For $1 \leq i \leq n-1$ we have:

$$d\omega^i = f'(r)dr \wedge \omega^i + f(r)d\theta^i = \frac{f'(r)}{f(r)}\omega^n \wedge \omega^i + f(r)d\theta^i$$

Let θ^i_j denote the connection one-forms for S^{n-1}. Then:

$$d\omega^i = \frac{f'(r)}{f(r)}\omega^n \wedge \omega^i + f(r)\theta^j \wedge \theta^i_j = \frac{f'(r)}{f(r)}\omega^n \wedge \omega^i + \omega^j \wedge \theta^i_j$$

(keep in mind that the sum over j is from 1 to $n-1$). We want to choose w^i_j so that this is $\omega^j \wedge \omega^i_j$ summed from 1 to n. The obvious choice is $\omega^i_n = \frac{f'(r)}{f(r)}\omega^i$, $\omega^i_j = \theta^i_j$, $\omega^n_i = -\frac{f'(r)}{f(r)}\omega^i$, and $\omega^n_n = 0$ for all $1 \leq i, j \leq n-1$. Check that this is a skew-symmetric matrix of one-forms and that it satisfies the derivative condition! Then by the last theorem in the previous section we know that $\{\omega^i_j\}$ is the set of connection 1-forms corresponding to $\{e_i\}$.

Let's get the curvature forms! For $1 \leq i, j \leq n-1$ we have

$$\begin{aligned}
\Omega^i_j &= d\omega^i_j + \sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j \\
&= d\theta^i_j + \sum_{k=1}^{n-1} \theta^i_k \wedge \theta^k_j - \left(\frac{f'(r)}{f(r)}\right)^2 \omega^i \wedge \omega^j \\
&= \hat{\Omega}^i_j - \left(\frac{f'}{f}\right)^2 \omega^i \wedge \omega^j
\end{aligned}$$

where $\hat{\Omega}$ denotes the curvature forms of S^{n-1} corresponding to $\{\theta^i\}$. We can show (figure out how) that since S^{n-1} has constant section curvature $+1$, $\hat{\Omega}^i_j = \theta^i \wedge \theta^j$. So

$$\begin{aligned}
\Omega^i_j &= \theta^i \wedge \theta^j - \left(\frac{f'}{f}\right)^2 \omega^i \wedge \omega^j \\
&= \frac{1}{f^2}\omega^i \wedge \omega^j - \left(\frac{f'}{f}\right)^2 \omega^i \wedge \omega^j \\
&= \frac{1 - f'^2}{f^2}\omega^i \wedge \omega^j
\end{aligned}$$

So this gives us the sectional curvature for $1 \leq i, j \leq n-1$, $i \neq j$:

$$K(e_i, e_j) = \frac{1 - f'(r)^2}{f(r)^2}$$

Notice that if $f(r) = \sinh(r)$ then this is just -1, as desired! Now suppose $1 \leq i \leq n-1$, let's compute what is left of sectional curvature, $K(e_i, e_n)$.

$$\begin{aligned}
\Omega^i_n &= d\omega^i_n + \sum_{k=1}^{n} \omega^i_k \wedge \omega^k_n \\
&= d\left(\frac{f'}{f}\omega^i\right) + \sum_{k=1}^{n-1} \theta^i_k \wedge \frac{f'}{f}\omega^k \\
&= \left(\frac{f'}{f}\right)' dr \wedge \omega^i + \frac{f'}{f} d\omega^i + \frac{f'}{f} \sum_{k=1}^{n-1} \theta^i_k \wedge \omega^k \\
&= \left(\frac{f'}{f}\right)' dr \wedge \omega^i + \frac{f'}{f} (\sum_{k=1}^{n} \omega^k \wedge \omega^i_k) + \frac{f'}{f} \sum_{k=1}^{n-1} \theta^i_k \wedge \omega^k \\
&= \left(\frac{f'}{f}\right)' dr \wedge \omega^i + \left(\frac{f'}{f}\right)^2 \omega^n \wedge \omega^i + \frac{f'}{f} \left(\sum_{k=1}^{n-1} \omega^k \wedge \omega^i_k + \theta^i_k \wedge \omega^k\right) \\
&= \left(\frac{f'}{f}\right)' dr \wedge \omega^i + \left(\frac{f'}{f}\right)^2 \omega^n \wedge \omega^i \\
&= \left(\left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2\right) \omega^n \wedge \omega^i
\end{aligned}$$

and now it is easily checked that in the case $f(r) = \sinh(r)$ we have sectional curvature -1 for e_i, e_n as desired!

Now let's go back to the map $F : S^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n \setminus \{0\}$ defined by $(\omega, r) \mapsto \omega r$. Apparently $(F^{-1})^* g_H$ extends smoothly across the origin. Let $\tilde{g}_H$ denote this extended metric on $\mathbb{R}^n$. Then the sectional curvature of $(\mathbb{R}^n, \tilde{g}_H)$ is -1 and it is a complete manifold. One way to check this is to check that radial lines from the origin are geodesics. (Their derivatives correspond via F to $\frac{\partial}{\partial r}$ so one simply needs to check that $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0$. This is not so hard because we already got the connection 1-forms!)

Let's go back to the upper half-space model and also compute the curvature there. This is

$$\mathrm{H}^n = (\mathbb{R}^n_+ \ , \ g_{\mathrm{H}^n} = \frac{g_{\mathbb{R}^n}|_{\mathbb{R}^n_+}}{y^2})$$

Choose the orthonormal frame $\epsilon_i = y\frac{\partial}{\partial x^i}$, where $x^1, \cdots x^n$ are the identity chart coordinates. The corresponding coframe is $\theta^i = \frac{1}{y}dx^i$. We have

$$d\theta^i = -\frac{1}{y^2}dy \wedge dx^i = -\theta^n\theta^i$$

We'd like to guess at θ^i_j such that $d\theta^i = \sum \theta^j \wedge \theta^i_j$. That is, we want $\sum \theta^j \wedge \theta^i_j = -\theta^n\theta^i$. Let's try guessing $\theta^i_j = 0$ for $j < n$ and $\theta^i_n = -\theta^n_i = -\theta^i$. Certainly this is an anti-symmetric indexed collection of 1-forms and it is readily checked that we do indeed get $d\theta^i = \sum \theta^j \wedge \theta^i_j$. And so these are the curvature 1-forms!

Alright let's compute $\Omega^i_j = d\theta^i_j + \sum \theta^i_k\theta^k_j$. For $i, j < n$ the $d\theta^i_j$ term obviously vanishes and the sum term leaves us with just $\Omega^i_j = -\theta^i \wedge \theta^j$. And $\Omega^i_n = d\theta^i_n + \sum \theta^i_k \wedge \theta^k_n = -d\theta^i = -\theta^i \wedge \theta^n$. So for all i, j we have

$$\Omega^i_j = -\theta^i \wedge \theta^j$$

So H^n has constant sectional curvature -1. Yay!

Showing that g_{H^n} is a complete metric... I did not completely follow this discussion so I will leave this out.

9 Classifying Space Forms

Theorem 11: Let M be a connected complete n-dimensional Riemannian manifold with constant sectional curvature K. Then the universal cover $\widetilde{M}$ of M with the covering metric is isometric to H^n if $K = -1$, $\mathbb{R}^n$ if $K = 0$, and S^n if $K = 1$.

General Strategy of Proof:
Let (M^n, g) be a complete connected simply connected Riemannian manifold of constant sectional curvature $K \in \{0, \pm 1\}$. How can we produce an isometry from M to one of our model spaces? Well if $K = -1$ then the Hadamard theorem tells us that $\exp_q$ is a global diffeomorphism from T_qM to M for $q \in M$. So pick a linear isometry $I : T_qM \to T_p\mathrm{H}^n$ for some choice of $q \in M, p \in \mathrm{H}^n$. The consider the map

$$\exp_p^{\mathrm{H}^n} \circ I \circ (\exp_q^M)^{-1} : M \to \mathrm{H}^n$$

The task is show that this map is an isometry. What about the case $K = 1$? Then the big difference is that exp is not a global diffeomorphism, but rather a diffeo on a ball of radius π in the tangent space. So we must consider something like

$$(\exp_p^{S^n}|_{B_\pi(0)}) \circ I \circ (\exp_q^M)^{-1} : M \to S^n \setminus \{-p\}$$

Then we will have to define another one of those cenetered at a different point, show that they are both isometries, and finally show that they agree on the common part of their domains.

Dealing with $\exp_p^* g$ on T_pM:
This is not quite a metric on $T_pM \setminus \{0\}$, unless we stick to a ball on which exp is a diffeomorphism. Consider the vector field $\frac{\partial}{\partial r}$ on the manifold T_pM, defined by $\frac{\partial}{\partial r}(v) = v/|v| \in T_v(T_pM)$. Then $(\exp_p^* g)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 1$,

because

$$
\begin{aligned}
|(d\exp_p)_v(v)| &= |(d\exp_p)_v(\frac{d}{dt}tv|_{t=1})| \\
&= |(\frac{d}{dt}\exp_p(tv)|_{t=1})| \\
&= |(\frac{d}{dt}\exp_p(tv)|_{t=0})| \\
&= |v|
\end{aligned}
$$

and

$$(\exp_p^* g)(v)(v,v) = g((d\exp_p)_v(v),(d\exp_p)_v(v)) = |v|^2$$

So we have that $\exp_p$ is kind of like an isometry *in the radial direction*. Next consider $w \perp v$ in T_pM. Then $(\exp_p^* g)(v)(v,w)$ vanishes by the Gauss lemma. Another way to see that it vanishes is to use the first variation of length along a family of radial geodesics (exponentiation of radial lines from the origin in T_pM with transverse curves being circles such that at $v \in T_pM$ the velocity along a transverse curve is $w \in T_v(T_pM)$.

Now let's handle $(\exp_p^* g)(v)(w,w)$ for $w \perp v$. To do this we need to compute $(d\exp_p)_v(w)$. Consider the family of curves $(t,s) \mapsto t(v+sw)$ in T_pM. Let $c(t)$ be the geodesic $\exp_p(tv)$. Define a vector field Y along c by

$$Y(t) = \frac{\partial}{\partial s}\exp_p(t(v+sw))|_{s=0}$$

and note that $Y(1)$ is precisely what we're trying to compute. Y is a Jacobi field (as is the variational field corresponding to *any* family of geodesics. It is clear that $Y(0)=0$. For $\dot{(Y)}(0)$ we compute:

$$
\begin{aligned}
\dot{Y}(0) &= (\nabla_{\frac{\partial}{\partial t}} Y)(0) \\
&= (\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\exp_p(t(v+sw)))(0,0) \\
&= (\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t}\exp_p(t(v+sw)))(0,0) \\
&= (\nabla_{\frac{\partial}{\partial s}} (v+sw))(0,0) \\
&= w
\end{aligned}
$$

where the second to last line involves the covariant derivative of a vector field along a constant curve, and the last line follows from the following short theorem (which is a homework):

Theorem 12: Let $c : [a,b] \to M$ be a constant curve at p on the Riemannian manifold (M,g). Let $V : [a,b] \to TM$ be a vector field along c. Then $(\nabla_{\frac{\partial}{\partial t}} V)(t)$ is the derivative at t of the curve V defines in T_pM, moved from $T_{V(t)}(T_pM)$ to T_pM.

Proof: Get a coordinate neighborhood $(U, x^1, \cdots, x^n)$ of p. Define $\xi^i(t) = V(t)(x^i)$ so that we have $V(t) = \xi^i(t)\frac{\partial}{\partial x^i}|_p$. Then for $t \in [a,b]$ we have:

$$
\begin{aligned}
(\nabla_{\frac{\partial}{\partial t}} V)(t) &= (\xi^i)'(t)\frac{\partial}{\partial x^i}|_p + \xi^j(t)\left(\nabla_{dc_t(\frac{\partial}{\partial t}|_t)}\frac{\partial}{\partial x^j}|_p\right) \\
&= (\xi^i)'(t)\frac{\partial}{\partial x^i}|_p
\end{aligned}
$$

because $dc_t(\frac{\partial}{\partial t}|_t) = 0$.

■

So we've got that Y satisfies

$$\begin{aligned}
&\ddot{Y} + R(\dot{c}, Y)\dot{c} = 0 \\
&Y(0) = 0 \\
&\dot{Y}(0) = w
\end{aligned}$$

This is a second order ODE system which completely determines $Y(1)$! Note that Y is *normal* to c:

$$\begin{aligned}
&\frac{d}{dt}\langle Y(t)\ ,\ \dot{c}(t)\rangle = \langle \dot{Y}(t)\ ,\ \dot{c}(t)\rangle \quad \text{which vanishes at } 0 \\
&\frac{d^2}{dt^2}\langle Y(t)\ ,\ \dot{c}(t)\rangle = \langle \ddot{Y}(t)\ ,\ \dot{c}(t)\rangle = -\langle R(\dot{c}, Y)\dot{c}\ ,\ \dot{c}\rangle \quad \text{which vanishes by antisymmetry of } R \\
&\Longrightarrow \langle Y\ ,\ \dot{c}\rangle = 0
\end{aligned}$$

Now let's consider the constant curvature 0 case...

Introduce a parallel orthonormal frame along c, $\{e_i\}$. Let's make $e_1 = \frac{1}{|v|}\dot{c}(t)$. Suppose in this frame we have $Y(t) = \xi^i(t)e_i(t)$. Then clearly ξ^1 vanishes because Y is normal to c. Suppose w in this basis is $w = w^i e_i(0)$. The ODE system for Y in this basis looks like:

$$\begin{aligned}
&\ddot{\xi} = 0 \\
&\xi^i(0) = 0 \\
&\dot{\xi}^i(0) = w^i
\end{aligned}$$

The solution is $Y(t) = tw^i e_i(t)$. So $|Y(t)|^2 = t^2|w|^2$. So

$$(\exp_p^* g)(v)(w, w) = |Y(1)|^2 = |w|^2$$

Now let's make use of the facts we've established:

$(T_pM, g(p))$ is an inner product space. But inner product spaces have a very natural metric given by the inner product, because the space itself can be identified with its tangent space at any point. So what we have shown via this identification is that $\exp_p$ is in fact an *isometry* from the Riemannian manifold T_pM, to (M, g). The three facts that show this are that $\exp_p$ preserves radial lengths, preserves orthogonality with the radial direction, and preserves lengths orthogonal to the radial direction. Actually we showed that $\exp_p$ was an isometry at all $v \in T_pM$ away from the origin, but the origin is very easy to deal with ($d(\exp_p)_0$ is the identity map on T_pM). Let us now find a convenient way to write $\exp_p^* g$:

Define a function $r : T_pM \setminus \{0\} \to \mathbb{R}$ by $r(v) = |v|$. And define a function $\mathrm{pr} : T_pM \setminus \{0\} \to \S(T_pM)$ by $\mathrm{pr}(v) = \frac{v}{|v|}$. Let $g_{S(T_pM)} = \mathrm{pr}^*$[the induced metric on the unit sphere in T_pM]. Finally, define:

$$\hat{g} = dr^2 + r^2 g_{S(T_pM)}$$

Claim: $\exp_p^* g = \hat{g}$ (proving some of this was a homework)
Proof: Consider any $v \in T_pM$ and any $w \in T_v(T_pM) \cong T_pM$, with $w \perp v$ wrt $\exp_p^* g$ (so also wrt $g(p)$ by what we established).

In order to do this calculation carefully, let's make the identification $T_v(T_pM) \cong T_pM$ explicit. Choose some orthonormal (wrt g(p)) basis $\{e_i\}$ for T_pM. Let $\phi : T_pM \to \mathbb{R}^n$ denote the coordinates given by that basis (so $\phi : u^i e_i \mapsto (u^1, \cdots, u^n)$). Let $(x^1, \cdots, x^n)$ denote the coordinate functions of ϕ. Then the identification is precisely $u^i e_i \leftrightarrow u^i \frac{\partial}{\partial x^i}|_v$. Suppose $v = v^i e_i$ and $w = w^i e_i$.

Let's now establish the claim, keeping in mind how the correspondence works:

$$\hat{g}(v)(v,v) = dr^2(v,v) + |v|^2 g_{S(T_pM)}(v)(v,v) = (dr \otimes dr)(v,v) = (dr(v))^2 = v(r)^2 = \left(v^i \frac{\partial r}{\partial x^i}|_v\right)^2 = |v|^2$$

$$\hat{g}(v)(w,v) = dr^2(w,v) + |v|^2 g_{S(T_pM)}(v)(w,v) = dr(w)dr(v) = |v|dr(w) = |v|w^i \frac{\partial r}{\partial x^i}|_v = |v| \sum_i w^i \frac{v^i}{|v|} = 0$$

$$\begin{aligned}
\hat{g}(v)(w,w) &= w(r)^2 + |v|^2 g_{S(T_pM)}(v)(w,w) = \left(w^i \frac{\partial r}{\partial x^i}|_v\right)^2 + |v|^2 g_{S(T_pM)}(v/|v|)(d\mathrm{pr}_v(w), d\mathrm{pr}_v(w)) \\
&= \left(\sum_i w^i \frac{v^i}{|v|}\right)^2 + |v|^2 g_{S(T_pM)}(v/|v|)(d\mathrm{pr}_v(w), d\mathrm{pr}_v(w)) = |v|^2 g_{S(T_pM)}(v/|v|)(d\mathrm{pr}_v(w), d\mathrm{pr}_v(w)) \\
&= |v|^2 g_{S(T_pM)}(v/|v|)(w/|v|, w/|v|) = g_{S(T_pM)}(v/|v|)(w,w) = |w|^2
\end{aligned}$$

This completes the proof of the claim. We are of course already finished with the case $K = 0$ because $(T_pM, g(p))$ is isometric to euclidian space.

Case $K = -1$:
The theorem of Hadamard tells us that $\exp_p$ is a diffeomorphism. Again we need to consider $(\exp_p^* g)_v(v,v)$, $(\exp_p^* g)_v(v,w)$, and $(\exp_p^* g)_v(w,w)$ where $v, w \in T_vT_pM$ and $w \perp v$ wrt $g(p)$. Our proof for the previous case works here for the first two and gives the same result. (We of course do not expect the same result for $(\exp_p^* g)_v(w,w)$ because $\exp_p$ is not supposed to be an isometry lol). In order to compute $(\exp_p^* g)_v(w,w)$ we can define the Jacobi field Y as before to try and compute $d(\exp_p)_v(w) = Y(1)$. So we've got that Y satisfies

$$\begin{aligned}
&\ddot{Y} + R(\dot{c}, Y)\dot{c} = 0 \\
&Y(0) = 0 \\
&\dot{Y}(0) = w
\end{aligned}$$

And as we've already shown Y is normal to c. So:

$$R(\dot{c}, Y)\dot{c} = \langle \dot{c}\ ,\ Y\rangle \dot{c} - \langle \dot{c}\ ,\ \dot{c}\rangle Y = -|\dot{c}|^2 Y = -|v|^2 Y$$

So:

$$\begin{aligned}
&\ddot{Y} - |v|^2 Y = 0 \\
&Y(0) = 0 \\
&\dot{Y}(0) = w
\end{aligned}$$

Let $\tilde{w}$ be the parallel transport of w along c. To solve for Y, make the guess $Y(t) = f(t)\tilde{w}(t)$. This leaves us with the system

$$\begin{aligned}
&\ddot{f} - |v|^2 f = 0 \\
&f(0) = 0 \\
&\dot{f}(0) = 1
\end{aligned}$$

which has solution $f(t) = 1/|v| \ \sinh(|v|t)$. So

$$Y(t) = \frac{1}{|v|}\sinh(|v|t)\tilde{w}(t)$$

Notice that $\tilde{w}$ has constant norm $|w|$ because it is a parallel vector field. We are of course interested in $Y(1)$ so we can compute:

$$(\exp_p^* g)_v(w,w) = \frac{|w|^2}{|v|^2}\sinh^2(|v|) = \frac{g(p)(w,w)}{r^2}\sinh^2(r) \quad \text{(where } r = |v|)$$

We are now ready to wrap up the proof for this case. Let $\hat{g} = dr^2 + \sinh^2(r) g_{S(T_pM)}$

Claim: $\exp_p^* g = \hat{g}$ (by the way this claim is a homework)
Proof: Consider any $v \in T_pM$ and $w \in T_v(T_pM)$ with $w \perp v$ wrt $g(p)$.

$$\hat{g}(v)(v,v) = dr^2(v,v) + \sinh^2(|v|)g_{S(T_pM)}(v)(v,v) = (dr\otimes dr)(v,v) = (dr(v))^2 = v(r)^2 = \left(v^i \frac{\partial r}{\partial x^i}|_v\right)^2 = |v|^2$$

$$\hat{g}(v)(w,v) = dr^2(w,v) + \sinh^2(|v|)g_{S(T_pM)}(v)(w,v) = dr(w)dr(v) = |v|dr(w) = |v|w^i\frac{\partial r}{\partial x^i}|_v = |v|\sum_i w^i \frac{v^i}{|v|} = 0$$

$$\begin{aligned}\hat{g}(v)(w,w) &= w(r)^2 + \sinh^2(|v|)g_{S(T_pM)}(v)(w,w) = \left(w^i\frac{\partial r}{\partial x^i}|_v\right)^2 + \sinh^2(|v|)g_{S(T_pM)}(v/|v|)(d\mathrm{pr}_v(w), d\mathrm{pr}_v(w)) \\ &= \left(\sum_i w^i\frac{v^i}{|v|}\right)^2 + \sinh^2(|v|)g_{S(T_pM)}(v/|v|)(d\mathrm{pr}_v(w), d\mathrm{pr}_v(w)) \\ &= \sinh^2(|v|)g_{S(T_pM)}(v/|v|)(d\mathrm{pr}_v(w), d\mathrm{pr}_v(w)) = \frac{|w|^2}{|v|^2}\sinh^2(|v|)\end{aligned}$$

This proves the claim. Now consider some other simply connected complete n-dim'l manifold (M', g') with constant section curvature -1. We obtain a similar form for $(\exp_{p'}^{M'})^*(g')$. Choose some linear isometry

$$L : (T_pM, g(p)) \to (T_{p'}M', g'(p'))$$

Then given that both $(\exp_p^M)^*(g)$ and $(\exp_{p'}^{M'})^*(g')$ can be written like "$dr^2 + \sinh^2(r)g_{S(T_pM)}$" (with things defined on the right tangent space), it easy to check that L is in fact automatically a linear isometry:

$$L : (T_pM,\ (\exp_p^M)^*(g)) \to (T_{p'}M',\ (\exp_{p'}^{M'})^*(g'))$$

This implies that

$$L^*(\exp_{p'}^{M'})^*(g') = (\exp_p^M)^*(g)$$

In other words:

$$(\exp_{p'}^{M'} \circ L \circ (\exp_p^M)^{-1})^*(g') = g$$

So we have an isometry

$$\exp_{p'}^{M'} \circ L \circ (\exp_p^M)^{-1} : M \to M'$$

and we have proven that up to isometry, there is a unique simply connected complete curvature -1 space!

Case $K = +1$:
This time we cannot use the theorem of Hadamard to get that $\exp_p$ is a diffeomorphism. We do pretty much the same stuff that we did in the $K = -1$ case, except this time we get

$$\hat{g} = dr^2 + \sin^2(r)g_{S(T_pM)}$$

as the form of $\exp_p^* g$. And if we choose a point $q \in S^n$ and a linear isometry $L : T_q S^n \to T_p M$ then we obtain a *local* isometry

$$F = \exp_p \circ L \circ (\exp_q^{S^n}|_{B_\pi(0)})^{-1} : S^n \setminus \{-q\} \to M$$

There are a few issues: (1) F is not defined on all of S^n so it needs to be extended, (2) it is not clear that F is surjective, and (3) it is not clear that F is a diffeomorphism.

To extend F we pick some other point $q' \in S^n \setminus \{q, -q\}$, let $p' = F(q')$, and define

$$G = \exp_{p'} \circ H \circ (\exp_{q'}^{S^n}|_{B_\pi(0)})^{-1}$$

where H is a linear isometry $H : T_{q'} S^n \to T_{p'} M$ defined by:

$$H = dF_{q'}$$

No matter what our choice of linear isometry H was, we would have that G is a local isometry just as F is. But the way we chose H is exactly so that F and G actually agree on the common part of their domain: $S^n \setminus \{q, -q\}$. For we have $G(q') = p' = F(q')$ and we have

$$dG_{q'} = d(\exp_{p'})_0 \circ H \circ d((\exp_{q'}^{S^n}|_{B_\pi(0)})^{-1})_{q'} = H = dF_{q'}$$

Therefore we have the smooth map and local isometry

$$\mathcal{F} = F \cup G : S^n \to M$$

and it remains only to check that it is a diffeomorphism. It is surjective because $\mathcal{F}(S^n)$ is compact (and therefore closed) and $\mathcal{F}(S^n)$ is open (because local isometries are local diffeomorphisms) so we have $\mathcal{F}(S^n) = M$ (because M is connected). Why is $\mathcal{F}$ a diffeomorphism? Well, being a surjective local diffeomorphism, $\mathcal{F}$ is a covering map. (Proving this is a homework which I did on paper). And S^n is compact, so F is a diffeomorphism because M is simply connected. (What does S^n being compact have to do with that conclusion?). And we have completed the classification theorem for spaces of constant curvature!

10 Some Results that Follow

Theorem 13: Let G be an isometry subgroup for the Riemannian manifold $(\tilde{M}, \tilde{g})$. Suppose G acts on M properly discontinuously. Then $\tilde{g}$ descends to a metric on the quotient $M = \tilde{M}/G$ (i.e. such that $\pi : \tilde{M} \to M$ is a local isometry).

Proof: M is a smooth manifold in a natural way (we will not go over this) such that π is a local diffeomorphism. π is a regular (or "normal" covering). Moreover, the group of deck transformations of π *is* G. Consider any $p \in M$. There is a neighborhood U of p such that $\pi^{-1}(U)$ is the union of the disjoint open sets $\{V_\alpha \mid \alpha \in J\}$ each of which is mapped diffeomorphically onto U. Notice that because G is the deck transformation group and it acts transitively on the fibers of π, we have

$$(\pi|_{V_\alpha}^{-1})^* \tilde{g} = (\pi_{V_\beta}^{-1})^* \tilde{g}$$

for all $\alpha, \beta \in J$. So if we define a metric g on M by letting $g(x)$ be $(\pi|_{V_\alpha}^{-1})^* \tilde{g}\ (x)$ for some choice of $\alpha \in J$ for the appropriate set $\{V_\alpha \mid \alpha \in J\}$ (which is based on $\pi(x)$), then it isn't all that hard to show that g is smooth and makes π a local isometry.

∎

Theorem 14: Let M be a complete connected Riemannian manifold with constant sectional curvature $0, 1$, or -1. Then M is isometric to N/Γ where N is $\mathbb{R}^n$, S^n, or H^n respectively and Γ is some isometry subgroup of N which acts on N properly discontinously.

Proof: Consider the universal cover $\tilde{M}$ of M, $\pi : \tilde{M} \to M$. Let $\tilde{g} = \pi^* g$. Then from our classification theorem we have an isometry $\Phi : \tilde{M} \to N$ where N is the appropriate model space defined in the statement of the theorem. Let G be the group of deck transformations of π. Define $\Gamma = \{\Phi^\circ h \circ \Phi^{-1} \mid h \in G\}$. Then it is easily checked that Γ acts properly discontinuously on N Give ${}^{N}\!/_{\Gamma}$ the induced metric as in the theorem above. Then there is a well-defined way to write an isometry ${}^{N}\!/_{\Gamma} \to M$ (we use Φ the fact that Φ sends fibers to fibers to define our map, then it is not difficult to verify that it is an isometry).

$\blacksquare$

11 Some Hodge Theory

Let (M, g) be a compact oriented Riemannian manifold, and suppose $p \in M$ and $\{e_1, \cdots, e_n\}$ is a positively oriented orthonormal basis for T_pM. Let $\{\omega^1, \cdots \omega^n\}$ be the corresponding dual basis. The inner product $g(p)$ for T_pM induces an inner product on $\Lambda^k T_pM$, in which we *declare* the basis $\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \mid i_1 < \cdots < i_k\}$ to be orthonormal. This turns out to be independent of the choice of $\{e_1, \cdots, e_n\}$. (homework)

We define an inner product on $\Omega^*(M)$ as follows: If $\alpha, \beta \in \Omega^k(M)$, then

$$\langle \alpha \ , \ \beta \rangle_{L^2} = \int_M \langle \alpha \ , \ \beta \rangle \ \mathrm{d}vol$$

and if α and β have different degree then they are orthogonal.

We define the Hodge Laplatian by

$$\Delta_H = dd^* + d^*d$$

where d^* is the *formal adjoint* of d (that is, d^* is some linear map that takes k-forms to $(k-1)$-forms such that $\langle d\alpha \ , \ \beta \rangle_{L^2} = \langle \alpha \ , \ d^*\beta \rangle_{L^2}$). Let's calculate d^*. Before we begin, let us define the Hodge star, $*$.

Suppose $p \in M$. Define $* : \Lambda^k T_pM \to \Lambda^{n-k} T_pM$ by

$$\alpha \wedge *\beta = \langle \alpha \ , \ \beta \rangle \ \mathrm{d}vol$$

Let $\{e_1, \cdots, e_n\}$ be a positively oriented orthonormal basis for T_pM, with dual basis $\{\omega^1, \cdots, \omega^n\}$. Then by playing with the formula above we find that

$$\begin{aligned}
&* \, \omega^1 = \omega^2 \wedge \cdots \wedge \omega^n \\
&* \, \omega^i = (-1)^{n+1} \omega^1 \wedge \cdots \omega^{i-1} \wedge \omega^{i+1} \wedge \cdots \wedge \omega^n \\
&* \, (\omega^1 \wedge \omega^2) = \omega^3 \wedge \cdots \omega^n \\
&\quad \text{(and so on, they're easy to compute)} \\
&*^2 = (-1)^{k(n-k)} \cdot \mathrm{id}(\Lambda^k T_pM) \qquad \text{for } * : \Lambda^k T_pM \to \Lambda^{n-k} T_pM
\end{aligned}$$

And we can now compute, for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$:

$$\begin{aligned}
\langle d\alpha \ , \ \beta \rangle_{L^2} &= \int_M \langle d\alpha \ , \ \beta \rangle \ \mathrm{d}vol \\
&= \int_M d\alpha \wedge *\beta \\
&= \int_M (d(\alpha \wedge *\beta) + (-1)^{k+1} \alpha \wedge d(*\beta)) \quad \text{(btw M is compact)} \\
&= (-1)^{k+1} \int_M \alpha \wedge d(*\beta) \\
&= (-1)^{k+1} \int_M \alpha \wedge * *^{-1} \, d(*\beta) \\
&= (-1)^{k+1} \int_M \langle \alpha \ , \ *^{-1} d(*\beta) \rangle \ \mathrm{d}vol \\
&= (-1)^{k(n-k)+k+1} \int_M \langle \alpha \ , \ *d(*\beta) \rangle \ \mathrm{d}vol \\
&= (-1)^{k(n-k)+k+1} \langle \alpha \ , \ *d(*\beta) \rangle_{L^2}
\end{aligned}$$

So we have found that

$$d^* = (-1)^{k(n-k)+k+1} * d*$$

And

$$\Delta_H = (-1)^{k(n-k)+k+1} (d * d * \ + \ * d * d)$$

I will not show this, but it turns out that Δ_H agrees with the ordinary laplacian when applied to 0-forms, up to a minus sign.

A form ω is *harmonic* if $\Delta_H \omega = 0$.

Theorem 15: (Hodge decomposition) Let (M^n, g) be a compact Riemannian manifold. Let $\omega \in \Omega^k(M)$, $0 \leq k \leq n$. Then $\omega = \omega_H + d\alpha + d^*\beta$ for some $\alpha \in \Omega^{k-1}(M), \beta \in \Omega^{k+1}(M), \omega_H$ a unique harmonic k-form. We also have uniquness for $d\alpha$ (and $d\beta$) as an exact (coexact) form.

This decomposition theorem is used to prove:

Theorem 16: (The Hodge Theorem) Let $[\alpha]$ be a deRham cohomology class. Then there is a unique harmonic form α_H in $[\alpha]$.

And proving the decomposition theorem requires the following existence and uniqueness theorem:

Theorem 17: Let (M^n, g) be a compact Riemannian manifold. Let $\gamma \in \Omega^*(M)$ such that $\gamma \perp \mathcal{H}^*(M)$, where $\mathcal{H}^*(M)$ is the space of harmonic forms on M and $\perp$ is with respect to $\langle , \rangle_{L^2}$. Then there is a unique $\alpha \in \Omega^*(M)$ such that $\Delta_H \alpha = \gamma$ and $\alpha \perp \mathcal{H}^*(M)$.