Final 09/14/2017 **Name:**

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(boxes for grader use)

Problems 1-5 are each worth 8 points. Problem 6 is a bonus for up to 4 points. So a full score is 40 points and the max score is 44 points. The exam has 6 pages; make sure you have all of them.

Notes and electronic aids are not allowed. You must be seated in your assigned row for your exam to be valid.

If you need more space, use the back of the pages. Clearly indicate when you do this.

Organize your work in a reasonable way. Scattered, unclear, or unsupported work receives little or no credit.

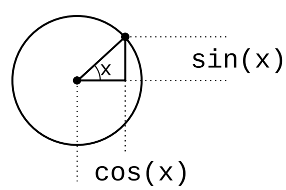
Circle your discussion section:

Wells 12:30 Wells 1:30

Peters 2:30 Peters 3:30

Things you did not need to memorize...

- The law of cooling is $\frac{dT_B}{dt} = -k(T_B - T_M)$, where T_B is the temperature of a body, T_M is the temperature of a medium in which it is immersed, t is time, and $k > 0$ is a proportionality constant.
- Unit circle:



- $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}e^x = e^x$.
- The product rule: $(fg)' = f'g + fg'$.
- The chain rule: $\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$, alternatively $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$.
- Substitution rule for integration: $\int f(u(x))\frac{du}{dx}dx = \int f(u)du$.
- Integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.
- $e^{a+ib} = e^a(\cos(b) + i\sin(b))$
- Variation of parameters: $y'' + by' + cy = Q(x)$ has a solution of the form $u_1y_1 + u_2y_2$, where u_1, u_2 are functions of x and y_1, y_2 are linearly independent solutions to the homogeneous version of the ODE. A calculation shows that $u_1' = \frac{-y_2Q}{y_1y_2' - y_1'y_2}$ and $u_2' = \frac{y_1Q}{y_1y_2' - y_1'y_2}$.
- $a\sin(\omega t) + b\cos(\omega t)$ can be written in the form $\sqrt{a^2+b^2}\sin(\omega t + \delta)$ for some δ.
- The types of fixed points we have named are: unstable/stable node, unstable/stable star, unstable/stable degenerate node, saddle point, unstable/stable spiral, and center.

1. Consider a falling object, subject to constant gravity and air resistance proportional to its velocity. Let $x(t)$ denote its height above the ground as a function of time. Then the equation of motion is given by

$$m\frac{d^2x}{dt^2} = -mg - \gamma\frac{dx}{dt}$$

where m, g, and γ are positive constants.

(a) Let $v = \frac{dx}{dt}$ be velocity and rewrite the ODE as a first order ODE in v.

$$mv' = -mg - \gamma v.$$

(b) Determine $v(t)$, velocity as a function of time, if the object was dropped from rest at $t = 0$.[1]

It's both separable and linear. So you could use the method of integrating factors, or just separate and integrate. Let me separate, and integrate with the initial conditions built into the integral:

$$\int_{v=0}^{v=v} \frac{dv}{g + \frac{\gamma}{m}v} = \int_{t=0}^{t=t} -dt.$$

Doing the integral and solving for v, we get

$$v(t) = \frac{mg}{\gamma}e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}.$$

(c) Use this to determine $x(t)$ if the object was dropped from rest from a height of x_0 at $t = 0$.

Just integrate what you got $v(t)$, since $v = \frac{dx}{dt}$. We get

$$x(t) = -\frac{m^2g}{\gamma^2}e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}t + C$$

but we still have to solve for C using the initial condition:

$$x(t) = -\frac{m^2g}{\gamma^2}e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}t + \frac{m^2g}{\gamma^2} + x_0.$$

(d) To what value does $v(t)$ limit as $t \to \infty$? (This is called the *terminal velocity.*)

The $e^{-\frac{\gamma}{m}t}$ part dies out as $t \to \infty$, and v becomes $-\frac{mg}{\gamma}$.

[1] It's easy to make small mistakes in this calculation. Think about your answer and make sure it makes sense.

2. For each given list of functions, determine whether it is linearly independent or linearly dependent. Prove your answer. Clarity of work is important here.

   (a) $\sin(x)$, x

   They are linearly independent. Suppose that $c_1 \sin(x) + c_2 x = 0$ for all x. Then, testing $x = \pi$, we deduce that c_2 must be 0. We are left with $c_1 \sin(x) = 0$, which we can test at $x = \pi/2$ to see that $c_1 = 0$.

   (b) $\sin^2(x)$, $\cos^2(x)$, 2

   They are linearly dependent; here is a dependency:

$$2\sin^2(x) + 2\cos^2(x) + (-1)(2) = 0.$$

3. The following ODE models a forced and damped oscillator:

$$4y'' + 2y' + 3y = 5\sin(t/2).$$

   Determine the steady state amplitude.

   Note: "Steady state amplitude" refers to the amplitude of oscillations in the long term, i.e. the amplitude due to the part of the solution that does not die out as $t \to \infty$.

   The solution is, as usual, $y = y_c + y_p$, where y_c is the general solution to the homogeneous version of the ODE, and y_p is a particular solution to the nonhomogeneous ODE. As we've seen with damped oscillators, the y_c part always dies out as $t \to \infty$ (though if you did not remember that you could find it and see for yourself). So the steady state motion is just y_p. To find y_p, we use the MOUC. A good guess is $y_p = A\sin(t/2) + B\cos(t/2)$. Substituing it into the ODE and solving for A and B, we obtain

$$y_p(t) = 2\sin(t/2) + \cos(t/2)$$

   This is the steady state motion. The amplitude of it is

$$\sqrt{2^2 + 1^2} = \sqrt{5}.$$

4. To save you some computation, I will tell you the exponential of the matrix $\begin{bmatrix} -5t & t \\ -t & 7t \end{bmatrix}$:

$$e^{t\left[\begin{smallmatrix} -5 & 1 \\ -1 & 7 \end{smallmatrix}\right]} = \begin{bmatrix} (1+t)e^{-6t} & te^{-6t} \\ -te^{-6t} & (1-t)e^{-6t} \end{bmatrix}.$$

   Now consider the ODE system

$$\begin{aligned} x' &= -5x + y \\ y' &= -x + 7y. \end{aligned}$$

   (a) Write down[2] the general solution of the ODE system.

   It's just:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (1+t)e^{-6t} & te^{-6t} \\ -te^{-6t} & (1-t)e^{-6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

   (b) Write down[2] the particular solution of the ODE system that satisfies $x(0) = 1$ and $y(0) = 2$.

   It's just:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (1+t)e^{-6t} & te^{-6t} \\ -te^{-6t} & (1-t)e^{-6t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

[2] Your answer should explicitly provide the functions $x(t)$ and $y(t)$. So don't leave any uncomputed matrix operations in your answer.

5. Consider the ODE system

$$R'(t) = a\,J(t)$$
$$J'(t) = b\,R(t),$$

where a and b are constants. Under each given set of assumptions below, you are asked to describe what happens to (R, J) in the long term[3] and to sketch a phase portrait[4]. The horizontal axis of your phase portraits should be R and the vertical axis should be J.

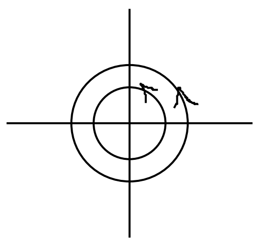
We are looking at the linear ODE system

$$\begin{bmatrix} R' \\ J' \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

The phase portrait can be determined from just the eigenvalues of the matrix, and the eigenvectors as well if they are real.

(a) Assume that $a = -1$ and $b = 1$. Sketch a phase portrait and describe what happens in the long run.

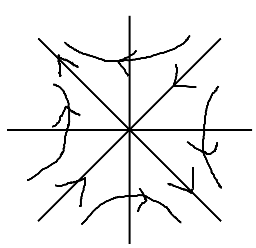
Here the eigenvalues are $\pm i$, so we get circular motion in the phase plane. We can sample a velocity vector (like shown in the forum post on drawing spirals) to know the direction the circle goes. The phase portrait looks like:



What happens in the long run? R and J just circle around forever!

(b) Assume that $a = -1$ and $b = -1$. Sketch a phase portrait and describe what happens in the long run.

Here the eigenvalues are $\lambda = \pm 1$. An eigenvector to go with $\lambda = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. An eigenvector to go with $\lambda = -1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we get a phase portrait like this:



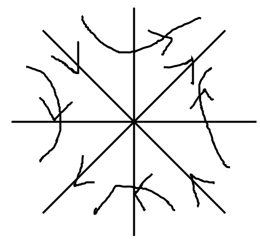
What happens in the long run? Either R becomes very negative while J becomes very positive, or vice versa. Another very unlikely possibility is that both R and J decay to zero (it's okay if you forgot about this possibility).

[3] This means "describe what happens to a solution as $t \to \infty$." If there are multiple possibilities, you should describe them all.

[4] As usual, a good phase portrait should (a) display any real eigensolutions that exist, (b) display at least one non-eigensolution of each distinct shape, and (c) display arrows on trajectories to indicate forward time. Precision is not important, but getting correct trajectory shapes is important.

(c) Assume that $a = 1$ and $b = 1$. Sketch a phase portrait and describe what happens in the long run.

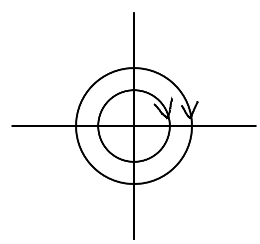
Here the eigenvalues are $\lambda = \pm 1$. An eigenvector to go with $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. An eigenvector to go with $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So we get a phase portrait like this:



What happens in the long run? Either R and J both become very positive, or they both become very negative. Another very unlikely possibility is that both R and J decay to zero (again it's okay if you forgot about this possibility).

(d) Assume that $a = 1$ and $b = -1$. Sketch a phase portrait and describe what happens in the long run.

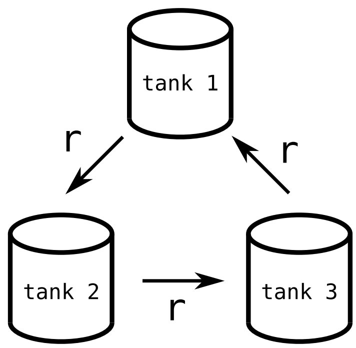
Here the eigenvalues are $\pm i$, so we get circular motion in the phase plane. We can sample a velocity vector (like shown in the forum post on drawing spirals) to know the direction the circle goes. The phase portrait looks like:



What happens in the long run? R and J just circle around forever!

6. (Bonus) In this problem, you are allowed to leave things uncomputed. So matrices don't need to be multiplied out explicitly, and matrix exponentials don't need to be computed explicitly.

   Consider three brine tanks, each of volume V and filled with brine. Pumps are attached so that brine is pumped from the first tank to the second at a rate of r units of volume per unit time, from the second to the third at the same rate, and from the third back to the first at the same rate.



   Let t denote time. Let x_1, x_2, and x_3 denote the salt concentration in the first, second, and third tanks respectively. Let a, b, and c denote the initial (at time $t = 0$) concentration in the first, second, and third tanks respectively. Assume that the brine is always well mixed inside each tank.

   Express the concentrations in the tanks as functions of time.

   The change of the concentration x_1 in tank 1 during duration dt is:

$$dx_1 = -r\left(\frac{x_1}{V}\right)dt + r\left(\frac{x_3}{V}\right)dt$$

   taking into account that brine is flowing out from tank 1 and flowing into it from tank 3. This becomes a differential equation

$$x_1' = -\frac{r}{V}x_1 + \frac{r}{V}x_3.$$

   By the same reasoning, one gets differential equations for x_2 and x_3:

$$x_2' = -\frac{r}{V}x_2 + \frac{r}{V}x_1$$
$$x_3' = -\frac{r}{V}x_3 + \frac{r}{V}x_2.$$

   Hence the three concentrations evolve according to the ODE system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \frac{r}{V}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

   The solution that satisfies the given initial conditions is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = e^{t\frac{r}{V}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$