Semisimple Lie Algebras and the Root Space Decomposition

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This document will develop just the material needed to describe a semisimple Lie algebra in terms of its root space decomposition. This takes place in section 5, which is our central focus. The major theorems that we often cite are put off until section 6, and some critical linear algebra machinery is is left for section 7, which still needs to be filled in.

1 Lie Algebra Basics

Let k be a field. A *Lie algebra* $\mathfrak{g}$ over k is a k-vector space equipped with an antisymmetric k-bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity:

$$\begin{array}{lll} (1) & \text{(antisymmetry)} & [x,x]=0 \\ (2) & \text{(Jacobi identity)} & [[x,y],z]+[[y,z],x]+[[z,x],y]=0 \end{array}$$

This bilinear map is called the *Lie bracket* of the Lie algebra. A *subalgebra* S of a Lie algebra is a vector subspace which is closed under the bracket (i.e. $[S,S] \subseteq S$). An *ideal* I of a Lie algebra is a vector subspace which is more strongly closed under the bracket in that $[\mathfrak{g}, I] \subseteq I$. A *Lie homomorphism* is a k-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the brackets:

$$\phi([x,y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}$$

The kernel of a Lie homomorphism is always an ideal. An important example is the *adjoint representation* map:

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \ .$$

That it is a Lie homomorphism is precisely the statement of the Jacobi identity. Another exact restatement of the Jacobi identity is contained in the fact that ad takes values in the subalgebra $\mathrm{Der}(\mathfrak{g})$ of derivations. The notion of a Lie algebra is meant to be an abstraction of the additive commutators of associative algebras. While lie algebras are almost never associative as algebras, they have the very property that commutators of *associative* algebras satisfy by taking place in an associative setting- the Jacobi identity.

In this spirit we often say that $x, y \in \mathfrak{g}$ "commute" when $[x,y]=0$. The ideal $Z(\mathfrak{g})$ of elements of $\mathfrak{g}$ that commute with everything in $\mathfrak{g}$ is called the *center* of $\mathfrak{g}$. It is the kernel of the adjoint representation map. We also say $\mathfrak{g}$ is "abelian" in case $Z(\mathfrak{g}) = \mathfrak{g}$.

The sum, intersection, and bracket of Lie ideals gives a Lie ideal. (Note: the bracket $[S,T]$ of two subspaces $S,T \subseteq \mathfrak{g}$ of a Lie algebra is taken to be the *subspace generated by* the elementwise brackets). If $I \subseteq \mathfrak{g}$ is an ideal, then there is a well-defined induced Lie bracket on the quotient vector space $\mathfrak{g}/I$ and a canonical surjective Lie homomorphism $\mathfrak{g} \to \mathfrak{g}/I$. We obtain standard homomorphism theorems just as in the theory of modules:

1. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie homomorphism then it induces an isomorphism ${}^{\mathfrak{g}}\!/_{\ker \phi} \cong \operatorname{im} \phi$.
2. $(S+I)/I \cong S/S \cap I$ for a subalgebra S and an ideal I of $\mathfrak{g}$.
3. For an ideal $I \subseteq \mathfrak{g}$, there is a correspondence between ideals of $\mathfrak{g}/I$ and ideals of $\mathfrak{g}$ that contain I.
4. $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$ for ideals $I \subseteq J \subseteq \mathfrak{g}$.

We define the *direct sum* of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ to be the vector space $\mathfrak{g} \oplus \mathfrak{h}$ with a bracket that kills cross-terms:

$$[a+x, b+y] = [a,b]_{\mathfrak{g}} + [x,y]_{\mathfrak{h}} \quad \text{for } a,b \in \mathfrak{g} \text{ and } x,y \in \mathfrak{h}.$$

Even though we use "$\oplus$" to denote this construction, it is only a *product* and not generally a coproduct in the category of Lie algebras.

A Lie algebra $\mathfrak{g}$ is said to be *simple* if it is nonabelian and has no proper nontrivial ideals.

2 Solvability

The *derived series* of a Lie algebra $\mathfrak{g}$ is the sequence of ideals

$$\mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}] \supseteq [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \supseteq \cdots$$

defined recursively by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$. We call $\mathfrak{g}$ *solvable* if its derived series hits rock bottom; that is, $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(n)} = 0$ for some n. For example abelian lie algebras are solvable, and simple lie algebras are not.

Theorem 1: Let $\mathfrak{g}$ be a Lie algebra.
(1) If $\mathfrak{g}$ is solvable then so are all its subalgebras and homomorphic images.
(2) If $I \subseteq \mathfrak{g}$ is a solvable ideal then: $\mathfrak{g}/I$ solvable $\Rightarrow$ $\mathfrak{g}$ solvable.
(3) The sum of solvable ideals is solvable.

Proof: If $S \subseteq \mathfrak{g}$ is a subalgebra then $S^{(i)} \subseteq \mathfrak{g}^{(i)}$. Similarly if $\phi : \mathfrak{g} \to \mathfrak{h}$ is an epimorphism, then $\mathfrak{h}^{(i)} = \phi(\mathfrak{g}^{(i)})$ (by induction). This proves (1). To prove (2) suppose that $(\mathfrak{g}/I)^{(n)} = 0$ and that I is solvable. Apply the formula used in (1) to the canonical epimorphism $\mathfrak{g} \to \mathfrak{g}/I$ to see that $\mathfrak{g}^{(n)} \subseteq I$. Then by part (1) again $\mathfrak{g}^{(n)}$ is solvable, so $\mathfrak{g}$ is obviously solvable as well.

(3): Now suppose $I, J \subseteq \mathfrak{g}$ are solvable ideals. Then by (1) $I/(I \cap J)$ is solvable, and so we may say the same of $(I+J)/J \cong I/(I \cap J)$. It follows from (2) that $I+J$ is solvable.

$\blacksquare$

Part (3) shows that a finite-dimensional Lie algebra $\mathfrak{g}$ must have a *unique* maximal solvable ideal. We call this ideal the *radical* of $\mathfrak{g}$ and denote it by $\mathrm{Rad}(\mathfrak{g})$. We say $\mathfrak{g}$ is *semisimple* if $\mathrm{Rad}(\mathfrak{g}) = 0$. Semisimplicity is then the opposite extreme condition to solvability. The word "solvable" sounds like it has particularly positive connotations, but really the two extreme cases of solvability and semisimplicity are both open to classification for Lie algebras over a sufficiently nice field. The radical of a Lie algebra, the "solvable part," ends up being the barrier to obtaining a decomposition into simple pieces.

Exercises: Show that a simple Lie algebra is semisimple. Show that a Lie algebra is semisimple iff it has no nonzero abelian ideals. Show that $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

This last fact suggests that we can try to understand all finite dimensional Lie algebras $\mathfrak{g}$ by understanding all the solvable ones (like $\mathrm{Rad}(\mathfrak{g})$) and all the semisimple ones (like $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$). Lie's theorem (section 6.2) shows that, over an algebraically closed field of characteristic 0, every solvable finite-dimensional Lie algebra is a subalgebra of an upper triangular matrix algebra. The remaining task is then to understand the structure of semisimple Lie algebras.

3 Semisimple Lie Algebras

In section 2 we defined a semisimple Lie algebra to be one with trivial radical. We will see that this is the same as saying that a particular bilinear form is nondegenerate, and that it is also the same as saying that there is a decomposition in terms of simple ideals. Throughout this section let $\mathfrak{g}$ be an arbitrary finite dimensional lie algebra over an algebraically closed field k of characteristic 0.

Define the *Killing form* $\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to k$ by

$$\kappa_{\mathfrak{g}}(x, y) := \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y))$$

This gives a symmetric bilinear form on $\mathfrak{g}$ which is *associative* in the sense that

$$\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]) \ .$$

The *radical* of a symmetric bilinear form $\beta : V \times V \to k$ on a k-vector space V is defined by

$$\mathrm{Rad}(\beta) = \{x \in V \mid 0 = \beta(x, \cdot) : V \to k\} \ .$$

Nondegenaracy of β is then equivalent to $\mathrm{Rad}(\beta) = 0$. Note that the radical of an *associative* symmetric bilinear form on a lie algebra is an ideal.

Theorem 2: Let I be an ideal of $\mathfrak{g}$. Then $\kappa_I = \kappa_{\mathfrak{g}}|_{I \times I}$.

Proof: This follows easily from the linear algebra fact that if W is a subspace of the finite dimensional vector space V, and if $\phi : V \to V$ maps V into W, then the trace of ϕ agrees with the trace of $\phi|_W : W \to W$.

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Theorem 3: $\mathfrak{g}$ is semisimple iff $\kappa_{\mathfrak{g}}$ is nondegenerate.

Proof: Suppose that $\mathrm{Rad}(\mathfrak{g}) = 0$. Let $\mathfrak{s} = \mathrm{Rad}(\kappa_{\mathfrak{g}})$. If $x \in \mathfrak{s}$ and $y \in [\mathfrak{s}, \mathfrak{s}]$ then

$$\mathrm{Tr}(\mathrm{ad}_{\mathfrak{s}}(x)\mathrm{ad}_{\mathfrak{s}} y) = \kappa_{\mathfrak{s}}(x, y) = \kappa_{\mathfrak{g}}(x, y) = 0 \ .$$

By Cartan's criterion (section 6.3) it follows that $\mathfrak{s}$ is solvable. Thus, $\mathfrak{s} \subseteq \mathrm{Rad}(\mathfrak{g}) = 0$.

Conversely, suppose that $\mathfrak{s} = 0$. We will show that $\mathfrak{g}$ has no nontrivial abelian ideals. Let I be an abelian ideal of $\mathfrak{g}$. Consider any $x \in I$ and $y \in \mathfrak{g}$. We have that $\mathrm{ad}(x)\mathrm{ad}(y)$ maps $\mathfrak{g}$ into I, and that $(\mathrm{ad}(x)\mathrm{ad}(y))^2 = 0$ since it maps $\mathfrak{g}$ into $[I, I] = 0$. Since nilpotent linear endomorphisms are traceless, we have $0 = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = \kappa(x, y)$. Since y was arbitrary, we get $x \in \mathfrak{s}$. Therefore $I \subseteq \mathfrak{s} = 0$.

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Theorem 4: Suppose that $\mathfrak{g}$ is semisimple. Then it is the direct sum of (all) its simple ideals.

Proof: Our assumption implies that $\kappa_{\mathfrak{g}}$ is nondegenerate. It can get us the required direct sum splitting.

> **Claim:** If $I \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{g} = I \oplus I^\perp$, and $I^\perp$ is an ideal as well.
> **Pf:** If $x \in I^\perp$ and $y \in \mathfrak{g}$ then for all $z \in I$ we have
>
> $$\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]) = 0 \ .$$
>
> Therefore $I^\perp$ is an ideal. If $x \in I \cap I^\perp$ and $y \in [I \cap I^\perp, I \cap I^\perp]$ then
>
> $$0 = \kappa_{\mathfrak{g}}(x, y) = \kappa_{I\cap I^\perp}(x, y) = \mathrm{Tr}(\mathrm{ad}_{I\cap I^\perp}(x)\mathrm{ad}_{I\cap I^\perp}(y)),$$
>
> so it follows from Cartan's criterion (section 6.3) that $I \cap I^\perp$ is solvable. Then since $\mathfrak{g}$ is semisimple we get $I \cap I^\perp = 0$. Since $\kappa_{\mathfrak{g}}$ is nondegenerate, this is enough to get $\mathfrak{g} = I \oplus I^\perp$ as a vector space. This is, however, automatically a direct sum in the sense of Lie algebras, for $[I, I^\perp] \subseteq I \cap I^\perp = 0$.

We continue the main proof by induction on the dimension of $\mathfrak{g}$. At low dimensions, $\mathfrak{g}$ has no proper nontrivial ideals and is therefore simple. Assume this is not the case for $\mathfrak{g}$, and assume that the theorem holds for lower dimensions. Let $\mathfrak{g}_1$ be a *minimal* proper nontrivial ideal of $\mathfrak{g}$. It is simple (it has no proper nontrivial ideals, and it is nonabelian by an exercise from section 2). By the claim we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^\perp$. Applying the induction hypothesis to $\mathfrak{g}_1^\perp$, we see that $\mathfrak{g}$ splits into a direct sum of simple ideals: $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$. Do all simple ideals of $\mathfrak{g}$ appear in this decomposition?

If I is any simple ideal of $\mathfrak{g}$ then $[I, \mathfrak{g}]$ is an ideal of I. It is nonzero because $Z(\mathfrak{g}) \subseteq \mathrm{Rad}(\mathfrak{g}) = 0$. We thus have:

$$I = [I, \mathfrak{g}] = [I, \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t] = [I, \mathfrak{g}_1] \oplus \cdots \oplus [I, \mathfrak{g}_t]$$

(it helps to notice that $[I, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$ to see why the direct sum remains in the last equality). Since I is simple all summands must be trivial except, say, the i^{th} one. Then $I = [I, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$. But $\mathfrak{g}_i$ is simple too so we get $I = \mathfrak{g}_i$.

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An easy consequence of this is that a semisimple $\mathfrak{g}$ has $[\mathfrak{g}, \mathfrak{g}]$. Another consequence is that all ideals and quotients of a semisimple $\mathfrak{g}$ are semisimple, due to:

Theorem 5: Any ideal of $\mathfrak{g}$ is a direct sum of certain of the simple ideals of $\mathfrak{g}$.

Proof: Let $I \subseteq \mathfrak{g}$ be an ideal, with $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$ the decomposition into simples. Notice that for each $\mathfrak{g}_i$, since $[I, \mathfrak{g}_i]$ is an ideal of $\mathfrak{g}_i$ and of I, it is either 0 or $\mathfrak{g}_i$. So for each i we have either $[I, \mathfrak{g}_i] = 0$ or $\mathfrak{g}_i \subseteq I$. We will show that I is spanned by the $\mathfrak{g}_i$'s that it contains; this will complete the proof.

Consider any $x \in I$. Write x as $\sum x_i$ with $x_i \in \mathfrak{g}_i$. Consider an i such that $[I, \mathfrak{g}_i] = 0$ instead of $\mathfrak{g}_i \subseteq I$. Suppose, by way of contradiction, that $x_i \neq 0$. Choose a $y \in \mathfrak{g}_i$ so that $[x_i, y] \neq 0$ (using $Z(\mathfrak{g}_i) = 0$). Then $0 = [x, y] = [x_i, y] \neq 0$.

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4 Representations of $\mathfrak{sl}_2(k)$

Let k be an algebraically closed field of characteristic 0.

Results from this section are needed for the root space decomposition to come. We begin with a very quick overview of representations, and then we classify the finite-dimensional irreducible representations of $\mathfrak{sl}_2(k)$.

4.1 Overview

A *representation* of a lie algebra $\mathfrak{g}$ over k is a Lie homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ for some vector space V over k. Since $\mathfrak{gl}(V)$ is really just $\mathrm{End}(V)$ as a set, we view a representation as an *action* of a Lie algebra on a vector space. Instead of the usual associativity rule for actions, we have

$$[x, y].v = x.(y.v) - y.(x.v)$$

With this viewpoint we often refer to a representation of $\mathfrak{g}$ as a $\mathfrak{g}$-module. There are the usual notions of submodule, $\mathfrak{g}$-homomorphism, quotient module, simple (irreducible) module, and completely reducible module. There are also notions of direct sum, tensor product, dual, and Hom for $\mathfrak{g}$-modules. We also get the usual isomorphism theorems, and a version of Schur's lemma.

An important example is the adjoint representaiton $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ which makes $\mathfrak{g}$ into a $\mathfrak{g}$-module. In this case submodules are ideals, simple modules are simple Lie algebras, etc.

According to Weyl's theorem (section 6.4), understanding the finite-dimensional representations of a semisimple Lie algebra amounts to understanding the irreducible ones.

4.2 The Classification for $\mathfrak{sl}_2(k)$

Let $\phi : \mathfrak{sl}_2(k) \to \mathfrak{gl}(V)$ be an irreducible finite-dimensional representation of $\mathfrak{sl}_2(k)$. Define $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, a basis for $\mathfrak{sl}_2(k)$ in which

$$\begin{aligned} [h, x] &= 2x \\ [h, y] &= -2y \\ [x, y] &= h\ . \end{aligned}$$

Since h is semisimple as an element of $\mathfrak{gl}_2(k)$, it's usual Jordan-Chevalley decomposition (as a matrix) is $h+0$. This must (see section 7) also be the abstract Jordan-Chevalley decomposition, so $\phi(h) + \phi(0) = \phi(h) + 0$ is the decomposition for $\phi(h) \in \mathfrak{gl}(V)$. Thus, $\phi(h)$ is semisimple. Since k is algebraically closed, it follows that h

acts *diagonally* on V; we obtain a decomposition of V as a direct sum of eigenspaces $V_\lambda = \{v \in V \mid h.v = \lambda v\}$ of the action of h. We refer to the eigenvalues of $\phi(h)$ as *weights* of h, and we refer to the eigenspaces of the action of h as *weight spaces* of h.

Suppose that $v \in V_\lambda$ is a vector of weight λ for h. What can we say about $x.v$ or $y.v$? Well,

$$\begin{aligned} h.(x.v) &= x.(h.v) + [h,x].v \\ &= \lambda(x.v) + (2x).v = (\lambda+2)x.v \end{aligned}$$

$$\begin{aligned} h.(y.v) &= y.(h.v) + [h,y].v \\ &= \lambda(y.v) + (-2y).v = (\lambda-2)y.v\ . \end{aligned}$$

We see that $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$ when $v \in V_\lambda$. So x and y shift us up and down the eigenspaces of h, with x bumping up eigenvalues by 2 and y bumping them down by 2. (At this point we only know that the eigenvalues are elements of k, so there is not really a notion of "up." But we can make sense of it later).

Starting with a particular eigenvector v for h, we can iteratively bump up its eigenvalue by applying x. Finite-dimensionality dictates that eventually this process hits 0. We obtain a vector of maximal "height", a nonzero vector $v_0 \in V_\lambda$ for some λ with $x.v_0 = 0$. Let's collect all the vectors "below", with a scale factor for convenience:

$$v_i \ := \ \frac{1}{i!}\, y^i.v_0 \qquad \text{for } i \in \mathbb{N}.$$

(Here the y^i is a notation for an i-fold application of y). Define also $v_{-1} := x.v_0 = 0$ and similarly $v_{-2} = 0$. We obtain:

$$\begin{aligned} h.v_i &= (\lambda - 2i)v_i \\ y.v_i &= (i+1)v_{i+1} \\ x.vi &= (\lambda - i + 1)v_{i-1} \end{aligned}$$

for all $i \geq -1$. The first two lines are obvious and the third can be shown by induction. The bumping down process also has to hit 0 eventually, so let m be the smallest natural number with $v_{m+1} = 0$. Of course $v_{m+1}, v_{m+2}, \cdots$ must all vanish. The formulas above show that $k\langle v_0, v_1, \cdots\rangle = k\langle v_0, \cdots, v_m\rangle$ is an $\mathfrak{sl}_2(k)$-submodule of V. It's nontrivial because it contains v_0, so by irreducibility of V we see that $v_0, \cdots, v_m$ is a *basis* for V. Also since

$$0 = x.v_{m+1} = (\lambda - m)v_m\ ,$$

we get $\lambda = m$. So weights for h are always integers. In the basis $v_0, \cdots, v_m$ of V, the endomorphsisms $\phi(h)$, $\phi(x)$, and $\phi(y)$ look like

$$\begin{bmatrix} m & & & & \\ & m-2 & & & \\ & & m-4 & & \\ & & & \ddots & \\ & & & & -m \end{bmatrix}, \quad \begin{bmatrix} 0 & m & & & \\ & 0 & m-1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & m & 0 \end{bmatrix}$$

respectively. We therefore have that there is at most *one* irreducible $\mathfrak{sl}_2(k)$-module, up to isomorphism, for each dimension $m+1$. And we see that there is *precisely* one for each dimension if we simply use the matrices above to construct them.

5 The Root Space Decompositon

Let $\mathfrak{g}$ be a nontrivial semisimple finite-dimensional Lie algebra over an algebraically closed field k of characteristic zero. In this section we put together many of the powerful tools developed throuhgout the document and we use them to study $\mathfrak{g}$.

5.1 Toral Subalgebras

Definition: A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *toral* if all its elements are ad-semisimple.

First notice that $\mathfrak{g}$ must have an element which is not ad-nilpotent, for otherwise Engel's theorem would imply that $\mathfrak{g}$ is nilpotent and therefore solvable. The semisimple part of this element would generate a one-dimensional subalgebra of $\mathfrak{g}$ consisting purely of ad-semisimple elements. In other words, nontrivial toral subalgebras exist.

Theorem 6: A toral subalgebra of a semisimple Lie algebra is abelian.

Proof: Suppose that $\mathfrak{h} \subseteq \mathfrak{g}$ is a toral subalgebra and $x \in \mathfrak{h}$. Then $\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{g}$ is a semisimple endomorphism, so its restriction $\mathfrak{h} \xrightarrow{\mathrm{ad}_{\mathfrak{h}}} \mathfrak{h}$ is too. Since k is algebraically closed, this means that $\mathrm{ad}_{\mathfrak{h}}(x)$ is diagonalizable. Therefore we can show that $\mathrm{ad}_{\mathfrak{h}}(x) = 0$ by showing that it has no nonzero eigenvalues.

Suppose, by way of contradiction, that $[x, y] = ay$ with $a \in k \setminus \{0\}$ and $y \in \mathfrak{h}$. By the same argument laid out above, y is $\mathrm{ad}_{\mathfrak{h}}$-semisimple. Thus x can be written as a linear combination of eigenvectors of $\mathrm{ad}_{\mathfrak{h}}(y) : \mathfrak{h} \to \mathfrak{h}$. So $\mathrm{ad}_{\mathfrak{h}}(y)(x)$ is a linear combination of eigenvectors of y with nonzero eigenvalue. But our assumption was that

$$\mathrm{ad}_{\mathfrak{h}}(y)(x) = [y, x] = -ay,$$

an eigenvector of $\mathrm{ad}_{\mathfrak{h}}(y)$ with eigenvalue 0.

■

Recall that the trick to classifying representations of $\mathfrak{sl}_2(k)$ revolved around the eigenspace decomposition of $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The magic that happened there was that the other basis elements of $\mathfrak{sl}_2(k)$ ended up behaving like "ladder" operators; that is, their behavior could be understood as simply shifting eigenvectors of h up and down the ladder of eigenvalues. For representations of more general Lie algebras we search for something analogous to "h." The key is to search for *toral subalgebras*!

Notation: If $L \subseteq \mathfrak{gl}(V)$ is a linear Lie algebra and $v \in V$ is a common eigenvector for the elements of L, then v has an eigenvalue $\alpha(x)$ for each $x \in L$. The resulting map $\alpha : L \to k$ turns out to be a linear functional $\alpha \in L^*$, and we call it the "eigenvalue" or, more correctly, the *weight* of v. There might be additional independent vectors with the same weight; we collect the all the vectors of weight α into a subspace which we call V_α.

5.2 The Decomposition

Now let $\mathfrak{h}$ be a *maximal toral subalgebra* of our semisimple Lie algebra $\mathfrak{g}$. We've established that $\mathfrak{h}$ is nontrivial and abelian. That it is abelian implies $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{gl}(\mathfrak{g})$ consists of *commuting* semisimple endomorphisms of $\mathfrak{g}$. It follows that they are *simultaneously diagonalizable*, and we obtain a (vector space) splitting of $\mathfrak{g}$ into weight spaces:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \ ,$$

where

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \,\rceil\, (\forall h \,\rceil\, h \in \mathfrak{h} \Rightarrow [h,x] = \alpha(h)x)\ \} \ .$$

Notice that $\mathfrak{g}_0$ is really just the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, which gives

$$\mathfrak{h} \subseteq \mathfrak{g}_0$$

since $\mathfrak{h}$ is abelian. Define

$$\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \,\rceil\, \mathfrak{g}_\alpha \neq 0\},$$

a finite set of nonzero weights which we call the set of *roots of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$. The decomposition above can be written as

$$\boxed{\mathfrak{g} = C_{\mathfrak{g}}\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha}$$

and it is called the *root space decomposition* of $\mathfrak{g}$ (relative to a maximal toral subalgebra $\mathfrak{h}$). We will soon see that in fact $C_{\mathfrak{g}}\mathfrak{h} = \mathfrak{h}$. First, let us explore the interaction of the Lie algebra structure with this vector space decomposition. I will box the important properties we collect along the way.

For convenience we shall often denote brackets using the notation of actions (the adjoint action). Suppose that $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. For $h \in \mathfrak{h}$ we have

$$h.[x,y] = [h.x,y] + [x,h.y] = (\alpha(h) + \beta(h))[x,y] = (\alpha+\beta)(h)[x,y] \ ,$$

so we have

$$\boxed{[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \qquad \text{for } \alpha, \beta \in \mathfrak{h}^*} \ .$$

Thus, if $x \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$ then $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.

Now suppose that $\alpha, \beta \in \mathfrak{h}^*$ with $\alpha + \beta \neq 0$. Choose an $h \in \mathfrak{h}$ so that $(\alpha+\beta)(h) \neq 0$. Consider any $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Let κ be the Killing form of $\mathfrak{g}$ (a symmetric associative bilinear for on $\mathfrak{g}$ which is *nondegenerate* because $\mathfrak{g}$ is semisimple). Then:

$$\begin{aligned}
(\alpha+\beta)(h)\kappa(x,y) &= \alpha(h)\kappa(x,y) + \beta(h)\kappa(x,y) \\
&= \kappa(\alpha(h)x, y) + \beta(h)\kappa(x, \beta(h)y) \\
&= \kappa(h.x, y) + \beta(h)\kappa(x, h.y) \\
&= \kappa([h,x], y) + \beta(h)\kappa(x, [h,y]) \\
&= -\kappa([x,h], y) + \beta(h)\kappa(x, [h,y]) = 0
\end{aligned}$$

so we get $\kappa(x,y) = 0$. Thus,

$$\boxed{\text{for } \alpha, \beta \in \mathfrak{h}^* \text{ with } \alpha+\beta \neq 0,\ \mathfrak{g}_\alpha \text{ is } \kappa\text{-orthogonal to } \mathfrak{g}_\beta} \ .$$

For example $\mathfrak{g}_0$ is orthogonal to $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$. An easy consequence is that κ restricts to a nondegenerate form on $\mathfrak{g}_0$.

We'll have to pause momentarily for a rather technical proof. It is reasonable to skip this proof and return to it later. For now just note that this is where the *maximality* of the toral subalgebra $\mathfrak{h}$ comes in handy.

Theorem 7: $\boxed{C_{\mathfrak{g}}\mathfrak{h} = \mathfrak{h}}$

Proof: We only need to prove $\subseteq$. Let $C = C_{\mathfrak{g}}\mathfrak{h}$.

Suppose that $x \in C$. Then $\mathfrak{h}$ is contained in the kernel of $\mathrm{ad}_{\mathfrak{g}}(x)$. If $x = s + n$ is abstract JC decomposition (section 7), then $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{g}}(s) + \mathrm{ad}_{\mathfrak{g}}(n)$ is the usual JC decomposition. Since $\mathrm{ad}(s)$ and $\mathrm{ad}(n)$ are both expressible as k-polynomials evaluated at $\mathrm{ad}(x)$ with zero constant term, they both also have $\mathfrak{h}$ in their kernel. Thus we have shown that whenever $x \in C$, its semisimple and nilpotent parts are also in C.

Suppose that $s \in C$ and s is ad-semisimple. Then $\mathfrak{h} + k\langle s\rangle$ is an abelian subalgebra of $\mathfrak{g}$ which is also toral, so by the maximality of $\mathfrak{h}$ we obtain $s \in \mathfrak{h}$. Thus we see that the semisimple things in C make it into $\mathfrak{h}$.

Suppose that $h \in \mathfrak{h}$ and that $\kappa_{\mathfrak{g}}(h, \mathfrak{h}) = 0$. We'd like to show that $h = 0$. Consider any $c \in C$, and let $c = s + n$ be its abstract JC decomposition. We saw that in this situation we have $s, n \in C$ and that we then further get $s \in \mathfrak{h}$. Then

$$\begin{aligned}\kappa(h, c) &= \mathrm{Tr}(\mathrm{ad}(h)\mathrm{ad}(s)) + \mathrm{Tr}(\mathrm{ad}(h)\mathrm{ad}(n)) \\ &= \kappa(h, s) + \mathrm{Tr}(\text{a nilpotent endomorphism}) = 0 \ .\end{aligned}$$

(The first term vanishes by the assumption $\kappa_{\mathfrak{g}}(h, \mathfrak{h}) = 0$, and the second vanishes because the composite of commuting endomorphisms is nilpotent when one of them is nilpotent). Since c was arbitrary, we get $\kappa(h, C) = 0$. But we argued above that κ was nondegenerate on $\mathfrak{g}_0 = C$, so $h = 0$. Thus we have shown that κ is nondegenerate on $\mathfrak{h}$.

Suppose that $x \in C$ with abstract JC decomposition $x = s + n$. We saw that $s \in C$ and therefore $s \in \mathfrak{h}$. In that case $\mathrm{ad}_C(s) = 0$. Therefore $\mathrm{ad}_C(x) = \mathrm{ad}_C(n)$ is nilpotent, and this holds for arbitrary $x \in C$. Thus we have shown that C is nilpotent (by Engel's theorem, section 6.1).

Suppose that $h \in \mathfrak{h}$ and $c, c' \in C$. Then

$$\begin{aligned}\kappa_{\mathfrak{g}}(h, [c, c']) &= \kappa_{\mathfrak{g}}([h, c], c') \\ &= \kappa_{\mathfrak{g}}(0, c') = 0\end{aligned}$$

so $\mathfrak{h}$ is $\kappa_{\mathfrak{g}}$-orthogonal to $[C, C]$. Since we already saw that $\kappa_{\mathfrak{g}}$ is nondegenerate on $\mathfrak{h}$, it follows that $\mathfrak{h} \cap [C, C] = 0$.

We now cite the fact that a nontrivial ideal of a nilpotent Lie algebra nontrivially intersects its center (this can be shown using the first claim in my proof of Engel's theorem; view the ideal as a module over the entire Lie algebra and find a nonzero element that is killed by the action). Suppose, by way of contradiction, that $[C, C] \neq 0$. Since C is nilpotent, there is a nonzero $z \in [C, C] \cap Z(C)$. Then $z \notin \mathfrak{h}$ since $\mathfrak{h} \cap [C, C] = 0$. So z cannot be semisimple, for that would give $z \in \mathfrak{h}$. Let n be the nilpotent part of z; now we know that $n \neq 0$. We have $n \in C$. We also have $n \in Z(C)$, because $\mathrm{ad}(n)$ is a k-polynomial evaluated at $\mathrm{ad}(z)$. It follows that $\kappa_{\mathfrak{g}}(n, C) = 0$, since its elements are traces of nilpotent maps. This contradicts the nondegeneracy of $\kappa_{\mathfrak{g}}$ on C. Thus we have proven that $[C, C] = 0$.

Finally, suppose that $C \neq \mathfrak{h}$ and choose an $x \in C \setminus \mathfrak{h}$. Let $x = s + n$ be its JC decomposition. Then $n \neq 0$, since otherwise $x = s \in \mathfrak{h}$. And $n \in C$. Since $[C, C] = 0$, $\mathrm{ad}(n)$ commutes with $\mathrm{ad}(y)$ for any $y \in C$. Then for any $y \in C$ we get $\kappa(n, y) = \mathrm{Tr}(\mathrm{ad}(n)\mathrm{ad}(y)) = 0$, contradicting the nondegeneracy of κ on C.

$\blacksquare$

In light of this fact and the comment above, we see that

$$\boxed{\kappa_{\mathfrak{g}} \text{ restricts to a nondegenerate form on } \mathfrak{h}} \ .$$

The restriction of κ to $\mathfrak{h}$ therefore establishes a linear isomorphism $\mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^*$. For $\alpha \in \mathfrak{h}^*$, let $t_\alpha \in \mathfrak{h}$ denote the image of α under the inverse of this isomorphism. In other words, for each $\alpha \in \mathfrak{h}^*$, t_α is the unique element of $\mathfrak{h}$ with

$$\alpha(h) = \kappa(h, t_\alpha) \ .$$

5.3 A First Look at the Roots

The set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ then corresponds to a certain finite set of nonzero vectors in $\mathfrak{h}$:

$$\{t_\alpha \mid \alpha \in \Phi\} \subseteq \mathfrak{h} \ .$$

Another fact that will come in handy is that

$$\boxed{\Phi \text{ spans } \mathfrak{h}^*} \ ,$$

otherwise, since span(Φ) would be a *proper* subspace of $\mathfrak{h}^*$, it would correspond to a *nontrivial* subspace of $\mathfrak{h}$. The "corresponding" subspace of $\mathfrak{h}$ is

$$\bigcap_{f \in \operatorname{span}(\Phi)} \ker(f) = \bigcap_{\alpha \in \Phi} \ker(\alpha) \ .$$

So we would have a nonzero $h \in \mathfrak{h}$ with $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then $0 = x.h = [x, h]$ for all $x \in \mathfrak{g}_\alpha$, for any $\alpha \in \Phi$. And we already have $[h, h'] = 0$ for $h' \in \mathfrak{g}_0 = \mathfrak{h}$, so this places h in $Z(\mathfrak{g})$. Impossible!

Now suppose that $\alpha \in \Phi$. Then if $-\alpha \notin \Phi$ we would get that $\mathfrak{g}_\alpha$ is orthogonal to all of $\mathfrak{g}$, which contradicts the nondegeneracy of κ. So Φ is closed under negatives:

$$\boxed{\alpha \in \Phi \ \Rightarrow \ -\alpha \in \Phi} \ .$$

Next we express brackets in terms of κ and Φ. Suppose that $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, and $y \in \mathfrak{g}_{-\alpha}$. Then for arbitrary $h \in \mathfrak{h}$ we have

$$\begin{aligned}
\kappa(h, [x, y]) &= \kappa([h, x], y) \\
&= \alpha(h)\kappa(x, y) \\
&= \kappa(h, t_\alpha)\kappa(x, y) \\
&= \kappa(h, \kappa(x, y)t_\alpha)
\end{aligned}$$

so we get

$$\boxed{[x, y] = \kappa(x, y)t_\alpha \qquad \text{for } \alpha \in \Phi,\ x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}}$$

from our earlier observation that κ is nondegenerate on $\mathfrak{h}$. So each $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$ is at most one-dimensional. In fact,

$$\boxed{\text{for } \alpha \in \Phi, \quad [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = k\langle t_\alpha\rangle \text{ is one-dimensional}} \ .$$

To see that it is nonzero: Choose a nonzero $x \in \mathfrak{g}_\alpha$ and notice that $\kappa(x, \mathfrak{g}_{-\alpha}) \neq 0$ (because we would otherwise get that x is orthogonal to all of $\mathfrak{g}$).

Next, let's show that the t_α themselves have nonzero "norm":

$$\boxed{\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \qquad \text{is nonzero for } \alpha \in \Phi} \ .$$

There are $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ with $\kappa(x, y)$ nonzero, since otherwise we would have $\mathfrak{g}_\alpha$ orthogonal to all of $\mathfrak{g}$. After rescaling we may assume that $\kappa(x, y) = 1$. Suppose, by way of contradiction, that $\alpha(t_\alpha) = 0$. Then:

$$\begin{aligned}
[t_\alpha, x] &= t_\alpha.x = \alpha(t_\alpha)x = 0 \\
[t_\alpha, y] &= t_\alpha.y = -\alpha(t_\alpha)y = 0 \\
[x, y] &= \kappa(x, y)t_\alpha = t_\alpha \ .
\end{aligned}$$

Let $\mathfrak{s} = k\langle x, y, t_\alpha\rangle$; the relations we've found show that this is a 3 dimensional solvable subalgebra. By Lie's theorem (section 6.2), $\mathfrak{s}$ stabilizes a flag in $\mathfrak{g}$ and hence acts by upper triangular matrices. Since $t_\alpha \in [\mathfrak{s}, \mathfrak{s}]$, it must act by a *strictly* upper triangular matrix. Therefore $\mathrm{ad}_\mathfrak{g}(t_\alpha)$ is a nilpotent endomorphism of $\mathfrak{g}$. But it is also semisimple since $t_\alpha \in \mathfrak{h}$ and $\mathfrak{h}$ is toral. It follows that $\mathrm{ad}_\mathfrak{g}(t_\alpha) = 0$, and then that $t_\alpha = 0$, a contradiction.

For every $\alpha \in \Phi$ and every nonzero $x \in \mathfrak{g}_\alpha$ we can find a little copy of $\mathfrak{sl}_2(k)$ containing x. First use the fact that $\kappa(x, \mathfrak{g}_{-\alpha}) \neq 0$ (by nondegeneracy of κ) to obtain a $y \in \mathfrak{g}_{-\alpha}$ with $\kappa(x, y) \neq 0$. Rescale y to ensure that

$$\kappa(x, y) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$$

and define h by

$$h = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \ .$$

We then obtain the relations

$$\begin{aligned}
[x, y] &= \kappa(x, y)t_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = h \\
[h, x] &= h.x = \alpha(h)x = \frac{2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha)x = 2x \\
[h, y] &= h.y = \alpha(h)y = \frac{-2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha)y = -2y \ .
\end{aligned}$$

Summarizing,

$$\boxed{\text{for } \alpha \in \Phi \text{ and } x \in \mathfrak{g}_\alpha \text{ nonzero, there are } y \in \mathfrak{g}_{-\alpha} \text{ and } h \in \mathfrak{h} \text{ so that } k\langle x, y, h\rangle \cong \mathfrak{sl}_2(k)} \ .$$

Throughout the next subsection fix a particular $\alpha \in \Phi$, and let $\mathfrak{s}_\alpha$ denote the copy of $\mathfrak{sl}_2(k)$ that we just plucked out of $\mathfrak{g}$.

5.4 The Action of $\mathfrak{s}_\alpha$

The next stage in understanding the root space decomposition will be to understand how $\mathfrak{s}_\alpha$ acts on $\mathfrak{g}$, which is an $\mathfrak{s}_\alpha$-module via

$$\mathfrak{s}_\alpha \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{g}) \ .$$

Define the subspace $\mathfrak{m} \subseteq \mathfrak{g}$ by

$$\mathfrak{m} = \bigoplus_{c \in k} \mathfrak{g}_{c\alpha} = \mathfrak{h} \oplus \bigoplus_{c \in k^\times} \mathfrak{g}_{c\alpha}$$

and notice that $\mathfrak{m}$ is an $\mathfrak{s}_\alpha$-submodule of $\mathfrak{g}$. What are the weights of h on $\mathfrak{m}$ (i.e. what are the eigenvalues of $\mathrm{ad}_{\mathfrak{m}}(h) : \mathfrak{m} \to \mathfrak{m}$)? There's 0, on elements of $\mathfrak{h}$. Then, if $c \in k^\times$ and $c\alpha \in \Phi$, there's the weight $2c$:

$$h.z = c\alpha(h)z = c\alpha\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right) z = 2cz \qquad \text{for nonzero } z \in \mathfrak{g}_{c\alpha}\ .$$

By Weyl's theorem (section 6.4) and the fact that $\mathfrak{s}_\alpha$ is semisimple, every $\mathfrak{s}_\alpha$-module is a direct sum of irreducible $\mathfrak{s}_\alpha$-modules. So $\mathfrak{m}$ is a direct sum of irreducble $\mathfrak{s}_\alpha$-modules, and we know exactly what those look like- there's one for each dimension, and an $(m+1)$-dimensional one has the following list of weights for h: m, $m-2$, $m-4$, $\cdots$, $-m$. In particular the weights are all integers. So the weights of h on $\mathfrak{m}$ are all integers. Therefore if $c \in k$ with $c\alpha \in \Phi$, then c must be a half-integer.

Notice that m, $m-2$, $\cdots$, $-m$ all have the same parity, so any *even* weight for h in $\mathfrak{m}$ would be for an eigenvector of $\mathrm{ad}_{\mathfrak{m}}(h)$ that lives in an irreducible $\mathfrak{s}_\alpha$-submodule of $\mathfrak{m}$ that *also* contains something with weight *zero* for h. Can we account for all the irreducible submodules with zero-weight-for-h vectors in $\mathfrak{m}$?

> **Claim:** All $z \in \mathfrak{m}$ with $h.z = 0$ lie in $\mathfrak{h}$ and therefore in the $\mathfrak{s}_\alpha$-submodule $\mathfrak{h} + \mathfrak{s}_\alpha$.
> **Pf:** By construction $\mathfrak{m} = \bigoplus_{c \in k} \mathfrak{g}_{c\alpha}$ is the eigenspace decomposition for h (i.e. for $\mathrm{ad}_{\mathfrak{m}}(h)$), with $\mathfrak{g}_{c\alpha}$ being the eigenspace corresponding to $2c$. So $\mathfrak{g}_0 = \mathfrak{h}$ indeed contains everything that is killed by h. To see that $\mathfrak{h} + \mathfrak{s}_\alpha$ is a submodule, notice that $\mathfrak{h} = \ker(\alpha) \oplus k\langle h \rangle$ (as a vector space) and that $\mathfrak{s}_\alpha$ acts trivially on $\ker(\alpha)$:
>
> $$\begin{aligned} \text{for } z \in \ker(\alpha), \qquad & h.z = 0 \\ & x.z = -[z, x] = -\alpha(z)x = 0 \\ & y.z = -[z, y] = \alpha(z)y = 0 \end{aligned}$$
>
> It follows that $\mathfrak{h} + \mathfrak{s}_\alpha = \ker(\alpha) + \mathfrak{s}_\alpha$ is a submodule.

The irreducibles that this submodule decomposes into are precisely $\mathfrak{s}_\alpha$ and a bunch of one-dimensional submodules that live in $\ker(\alpha)$, and these are apparently the only irreducible submodules in the decomposition of $\mathfrak{m}$ that house zero-weight-for-h vectors. Any *even* weight for h in $\mathfrak{m}$ then corresponds to a vector which either (1) lives in $\mathfrak{s}_\alpha$ or (2) lives in one of these one-dimensional submodules in $\ker(\alpha)$ (in which case the weight is 0). So any *nonzero even* weight for h in $\mathfrak{m}$ corresponds to a vector which lives in $\mathfrak{s}_\alpha$, where we know the weights for h are $2, 0, -2$. The *only* even weights for h in $\mathfrak{m}$ are $2, 0, -2$, and the vectors corresponding to ± 2 live in $\mathfrak{s}_\alpha$ itself.

One consequence of this is that $2\alpha \notin \Phi$, for otherwise 4 would arise as a weight for h in $\mathfrak{m}$. Of course α was arbitrary, so this is a general principle: twice a root is never a root! Since α is by assumption a root, $\frac{1}{2}\alpha$ cannot be. Therefore 1 never occurs as a weight for h in $\mathfrak{m}$.

Knowing this, we have accounted for *all* the $\mathfrak{s}_\alpha$-submodules of $\mathfrak{m}$ in its decomposition into a direct sum of irreducibles:

- Any irreducible $\mathfrak{s}_\alpha$-submodule of even dimension must have 1 as a weight of h on it, so no even-dimensional irreducibles appear.
- Any irredicble $\mathfrak{s}_\alpha$-submodule of odd dimension greater than 3 would have 4 as a weight of h on it, so none of those appear either.

- Only one 3-dimensional irreducible appears: $\mathfrak{s}_\alpha$. This is because the only vectors in $\mathfrak{m}$ corresponding to a weight of ± 2 for h are the ones in $\mathfrak{s}_\alpha$.
- A 1-dimensional $\mathfrak{s}_\alpha$-submodule of $\mathfrak{m}$ is really just spanned by a vector that h kills. Any such vector lies in $\mathfrak{g}_0 = \mathfrak{h}$.

All this implies that

$$\boxed{\mathfrak{m} = \mathfrak{h} + \mathfrak{s}_\alpha} \ .$$

From this we also get that $\mathfrak{g}_\alpha = k\langle x\rangle$, that $\mathfrak{g}_{-\alpha} = k\langle y\rangle$, and that $\mathfrak{g}_{c\alpha} = 0$ for $c \in k \setminus \{1, 0, -1\}$. This really holds for arbitrary α:

$$\boxed{\text{if } \alpha \in \Phi \text{ then } -\alpha \in \Phi \text{ and } \dim(\mathfrak{g}_{\pm\alpha}) = 1, \text{ but no other multiples of } \alpha \text{ are in } \Phi} \ .$$

At this point we understand pretty well how $\mathfrak{s}_\alpha$ acts on $\mathfrak{m}$. What about the rest of $\mathfrak{g}$? Fix a $\beta \in \Phi \setminus \{\pm\alpha\}$, so that β is a root which is not a multiple of α. Define the following $\mathfrak{s}_\alpha$-submodule of $\mathfrak{g}$:

$$\mathfrak{m}_\beta = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha} \ .$$

By the assumption on β we are guaranteed that $\beta + i\alpha$ is never 0. So each summand above has dimension 1 or 0. If $z \in \mathfrak{g}_{\beta+i\alpha}$ with $i \in \mathbb{Z}$, then

$$h.z = (\beta + i\alpha)(h)z = (\beta(h) + 2i)z$$

so the weights of h on $\mathfrak{m}_\beta$ are all of the form $\beta(h) + 2i$ with $i \in \mathbb{Z}$. These are all distinct weights and each occurs at most "once" (in the sense that each has a weight space of dimension 1 or 0).

> **Claim:** $\mathfrak{m}_\beta$ is an *irreducible* $\mathfrak{s}_\alpha$-module
> **Pf:** By Weyl's theorem $\mathfrak{m}_\beta$ is a direct sum of irreducibles. If there were more than one irreducible in the direct sum, then one of two impossibilities would obtain: (a) 0 or 1 occurs more than once as a weight for h or (b) 0 and 1 both occur as weights for h (this could not be if the weights are all in $\{\beta(h) + 2i \,|\, i \in \mathbb{Z}\}$). Case (a) occurs when $\mathfrak{m}_\beta$ has two odd-dimensional or two even-dimensional irreducible summands in its decomposition, and case (b) occurs when $\mathfrak{m}_\beta$ has an irreducible summand of odd dimension and one of even dimension.

Now the structure of $\mathfrak{m}_\beta$ as an $\mathfrak{s}_\alpha$-module is completely pinned down by one number- its dimension. Let $q \in \mathbb{Z}$ be the largest integer for which $\beta + q\alpha \in \Phi$, and let $r \in \mathbb{Z}$ be the largest integer for which $\beta - r\alpha \in \Phi$ (note that $r, q \geq 0$). Then $\beta(h) + 2q$, $\beta(h) - 2r$ are respectively the highest and lowest weights for h on $\mathfrak{m}_\beta$. We have a couple of ways of listing all the weights of h on $\mathfrak{m}_\beta$:

$$\begin{array}{l} \beta(h)+2q,\ \beta(h)+2q-2,\ \cdots,\ -(\beta(h)+2q) \qquad \text{or} \\ \beta(h)+2q,\ \beta(h)+2q-2,\ \cdots,\ \beta(h)-2r\ . \end{array}$$

Equating the two ways of writing the lowest weight yields $r = \beta(h) + q$. We also get a nice list of all the roots of the form $\beta + i\alpha$:

$$\beta + q\alpha,\ \beta + (q-1)\alpha,\ \cdots,\ \beta,\ \cdots,\ \beta - r\alpha \ \ \in \ \Phi$$

Interestingly, we have shown that $\beta(h)$ is an integer (namely $r - q$). This is true for arbitrary $\beta \in \Phi \setminus \{\pm\alpha\}$, but of course we also have that $\pm\alpha(h) = \pm 2$ is an integer. So in general for arbitrary roots $\alpha, \beta \in \Phi$ we have that

$$\beta\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right)$$

is an integer. These are called *Cartan integers*. In our case since $\beta(h) = r - q$, we see that $-r \leq -\beta(h) \leq q$ and consequently $\beta - \beta(h)\alpha$ makes it into the list above (i.e. into Φ). This is true for arbitrary $\beta \in \Phi \setminus \{\pm\alpha\}$, but we also have

$$\pm\alpha \mp \alpha(h)\alpha = \pm\alpha \mp 2\alpha = \mp\alpha \in \Phi \ .$$

So in general for arbitrary $\alpha, \beta \in \Phi$ we have that

$$\beta - \beta\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right)\alpha \in \Phi \ .$$

5.5 The Structure of $(\mathfrak{h}^*, \kappa, \Phi)$

We can turn κ into a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ via

$$\langle \gamma, \delta \rangle = \kappa(t_\gamma, t_\delta) \ .$$

Since Φ spans $\mathfrak{h}^*$, we can get a k-basis of $\mathfrak{h}^*$ consisting of *roots* $\alpha_1, \cdots, \alpha_l \in \Phi$. If $\beta \in \Phi$, then there are unique $c^1, \cdots, c^l \in k$ so that $\beta = \sum c^i \alpha_i$.

Claim: $c^1, \cdots, c^l$ lie in the prime field $\mathbb{Q}$ of k.
Pf: For any $1 \leq j \leq l$, $\langle \beta, \alpha_j \rangle = \sum c^i \langle \alpha_i, \alpha_j \rangle$. Each $\langle \alpha_j, \alpha_j \rangle = \kappa(t_{\alpha_j}, t_{\alpha_j})$ is nonzero. We have

$$\frac{2\langle \beta, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \sum_{i=1}^{l} \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} c^i$$

Note that the left hand side and the coefficients of the c^i on the right hand side are all Cartan integers. Define an $l \times l$ matrix $A \in M_l(k)$ by

$$A_{ji} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle},$$

and note that A in fact lies in $M_l(\mathbb{Q})$. Define $\vec{\beta}, \vec{c} \in k^l$ by

$$\vec{\beta}_j = \frac{2\langle \beta, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

and $\vec{c}_i = c^i$. Above we see that $\vec{\beta} = A\vec{c}$, and $\vec{\beta}$ in fact lies in $\mathbb{Q}^l$. A is invertible in $M_l(k)$ because $\alpha_1, \cdots, \alpha_l$ are a k-linearly independent. Therefore A is invertible in $M_l(\mathbb{Q})$ and $\vec{c} = A^{-1}\vec{\beta}$ lies in $\mathbb{Q}^l$.

This establishes that

$$E_{\mathbb{Q}} := \mathbb{Q}\langle \Phi \rangle$$

is $\mathbb{Q}$-spanned by $\alpha_1, \cdots, \alpha_l$. Therefore

$$\dim_{\mathbb{Q}}(E_{\mathbb{Q}}) = \dim_k(\mathfrak{h}^*) \ .$$

Claim: $\langle \cdot, \cdot \rangle$ is positive-definite and rational-valued on $E_{\mathbb{Q}}$.

Pf: For $\lambda, \mu \in \mathfrak{h}^*$ we have

$$\begin{aligned}\langle \lambda, \mu \rangle &= \kappa(t_\lambda, t_\mu) \\ &= \mathrm{Tr}(\mathrm{ad}_\mathfrak{g}(t_\lambda)\mathrm{ad}_\mathfrak{g}(t_\mu)) \\ &= \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu) \\ &= \sum_{\alpha \in \Phi} \langle \alpha, \lambda \rangle \langle \alpha, \mu \rangle\end{aligned}$$

so for $\beta \in \Phi$ we have

$$\langle \beta, \beta \rangle = \sum_{\alpha \in \Phi} \langle \alpha, \beta \rangle^2 \ .$$

We saw earlier that $\langle \beta, \beta \rangle = \kappa(t_\beta, t_\beta)$ is nonzero. Multiplying by $1/\langle \beta,\beta \rangle^2$,

$$\frac{1}{\langle \beta, \beta \rangle} = \sum_{\alpha \in \Phi} \frac{\langle \alpha, \beta \rangle^2}{\langle \beta, \beta \rangle^2}$$

Comparing the right hand side to the form of the Cartan integers, we see that $1/\langle \beta,\beta \rangle$ is a sum of nonnegative rational numbers. Therefore $\langle \beta, \beta \rangle$ is a positive rational. This proves positive-definiteness (we already had nondegeneracy). For $\alpha, \beta \in \Phi$ we have $2\langle \alpha,\beta \rangle/\langle \beta,\beta \rangle \in \mathbb{Z}$, so since $\langle \beta, \beta \rangle \in \mathbb{Q}$ we get $\langle \alpha, \beta \rangle \in \mathbb{Q}$ as well.

Now $E_\mathbb{Q}$ is an "inner product" space over $\mathbb{Q}$. We turn it into an actual real inner product space by defining E to be the real vector space

$$E := \mathbb{R} \otimes_\mathbb{Q} E_\mathbb{Q}$$

with an inner product extending $\langle \cdot, \cdot \rangle$ from $E_\mathbb{Q}$. Note that

$$\dim_\mathbb{R} E = \dim_\mathbb{Q} E_\mathbb{Q} = \dim_k \mathfrak{h}^* = \dim_k \mathfrak{h} \ .$$

Let us restate some of the things we've shown throughout this section, but this time in terms of the real inner product space E and the finite subset[1] Φ:

- Φ spans E and $0 \notin \Phi$.
- If $\alpha \in \Phi$, then so is $-\alpha$ and no other scalar multiples of α make it into Φ.
- If $\alpha, \beta \in \Phi$, then so is
$$\beta - \beta\left(\frac{t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right)\alpha = \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \ .$$
- If $\alpha, \beta \in \Phi$, then $\frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z} \subseteq \mathbb{R}$.

Such an object (E, Φ) is called a *root system*. We have forged a path $\mathfrak{g} \rightsquigarrow (E, \Phi)$.

...Does it depend on the choice of maximal toral subalgebra $\mathfrak{h}$?

[1] Let us refer to the image of Φ under $E_\mathbb{Q} \hookrightarrow E$ as Φ also.

6 Major Theorems

6.1 Engel's Theorem

Theorem 8: Let $\mathfrak{g}$ be a finite-dimensional lie algebra over F. If every element of $\mathfrak{g}$ is ad-nilpotent then $\mathfrak{g}$ is nilpotent.

Proof: The case $\mathfrak{g} = 0$ is trivial, so assume $\mathfrak{g} \neq 0$.

Claim: If $L \subseteq \mathfrak{gl}(V)$ is a linear lie algebra over F with $V \neq 0$ finite dimensional and if every element of L is nilpotent as an endomorphism of V, then there is a nonzero $x \in V$ so that $\ell(x) = 0$ for all $\ell \in L$.
Pf: Use induction on $\dim(L)$. The cases $\dim(L) = 0$ (or even 1) are obvious. Assume the claim for dimensions lower than $\dim(L)$.

Claim: A proper subalgebra of L cannot be self-normalizing.
Pf: Suppose $K \subsetneq L$ is a subalgebra. Let $\mathfrak{gl}(L;K)$ denote the subalgebra of $\mathfrak{gl}(L)$ consisting of endomorphisms that stabilize K. There is a natural map $\alpha : \mathfrak{gl}(L;K) \to \mathfrak{gl}(L/K)$ and it is a lie algebra homomorphism. $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a lie algebra homomorphism, and since K is a *subalgebra* of L we have $\mathrm{ad}(K) \subseteq \mathfrak{gl}(L;K)$. So we obtain a lie algebra homomorphism $\alpha \circ (\mathrm{ad}\,|_K) : K \to \mathfrak{gl}(L/K)$; let $\bar{K}$ denote its range. Recall that a nilpotent endomorphism is always ad-nilpotent, so $\overline{K}$ satisfies the hypotheses of the claim and $\dim(\bar{K}) \leq \dim(K) < \dim(L)$. By induction then we obtain a nonzero vector in L/K which is killed by $\bar{K}$. In other words, we get an $x \in L$ with $x \notin K$ so that $[k, x] \in K$ for all $k \in K$. Thus, $N_L(K)$ properly contains K.

L has a *maximal* proper subalgebra, say K. From the claim we have $N_L(K) = L$, so K is an *ideal*. K must have codimension 1, for otherwise we could a take a proper nontrivial subalgebra of L/K and pull it back to L so as to contradict the maximality of K! (L/K is a lie algebra, and abelian one dimensional subalgebras are available). Since $\dim(K) < \dim(L)$, the induction shows that

$$W := \{x \in V \mid k(x) = 0 \quad \forall k \in K\}$$

is nonempty.

Claim: W is stabilized by every $\ell \in L$.
Pf: Suppose $x \in W$ and $\ell \in L$. For any $k \in K$ we have

$$k(l(x)) = l(k(x)) + [k, l](x) = 0$$

because K is an ideal. So $\ell(x) \in W$.

Now choose some $z \in L \setminus K$. Since z stabilizes W, we have a nilpotent endomorphism $z\,|_W$ of W. Get an eigenvector $x \in W \setminus \{0\}$ of $z\,|_W$ with eigenvalue 0. Since K has codimension 1, $L = K + F\langle z \rangle$. It follows that everything in L kills x.

Claim: $Z(\mathfrak{g}) \neq 0$.
Pf: By hypothesis, the range of $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a linear lie algebra of nilpotent endomorphisms, and we assumed that $\mathfrak{g} \neq 0$. So from the claim we obtain a nonzero $x \in \mathfrak{g}$ which is killed by $\mathrm{ran}(\mathrm{ad})$. So $x \in Z(\mathfrak{g}) \setminus \{0\}$.

We prove Engel's theorem by induction on $\dim(\mathfrak{g})$. The base case is automatic; asssume Engel's theorem for lie algebras of dimension less than $\dim(\mathfrak{g})$. Well $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension than $\mathfrak{g}$ and all its elements are ad-nilpotent, so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. It follows that $\mathfrak{g}$ is nilpotent.

■

6.2 Lie's Theorem

Theorem 9: Suppose F is an algebraically closed field of characteristic 0, V is a nontrivial finite dimensional vector space over F, and $L \subseteq \mathfrak{gl}(V)$ is a lie subalgebra. If L is solvable, then L stabilizes some flag in V.

Proof: The case $\dim V = 1$ is clear (there is only one flag and it works). The following claim will drive an induction on $\dim V$.

Claim: Under the hypotheses of the theorem if $L \subseteq \mathfrak{gl}(V)$ is solvable, then there is in V an eigenvector common to all $\ell \in L$.
Pf: The case $\dim L = 0$ is trivial. Assume that $\dim L > 0$ and assume that the claim holds for lower dimensions. Our approach will be to find a codimension 1 ideal to which we can apply the induction hypothesis, and then to show that L stabilizes the resulting space of eigenvectors common to elements of the ideal.

Claim: L has an ideal of codimension 1.
Pf: Since L is solvable and $\dim L > 0$, $L^{(1)}$ is a proper ideal of L. Since $L/L^{(0)}$ is abelian, all its subspaces are ideals. Pull back any codimension 1 subspace to get a codimension 1 ideal of L.

Get a codimension 1 ideal $K \subseteq L$. K is solvable, so by the induction we obtain an eigenvector $v \in V$ common to all $k \in K$. Define $\lambda : K \to F$ to be the eigenvalue function; it is given by $k(v) = \lambda(k)v$ for $k \in K$. Observe that λ is linear. Define

$$W = \{w \in V \,\big|\, k(w) = \lambda(k)w \quad \forall\, k \in K\} \ .$$

We already know $W \neq 0$ because $v \in W$.

Claim: Every $\ell \in L$ stabilizes W.
Pf: If $w \in W$ and $\ell \in L$, then for $k \in K$ we have

$$k(\ell(w)) = \ell(k(w)) - [\ell, k](w) = \lambda(k)\ell(w) - \lambda([\ell, k])w$$

(since K is an ideal), so we need only prove the following claim:

Claim: For $\ell \in L$ and $k \in K$, $\lambda([\ell, k]) = 0$.
Pf: We will get a handle on the behavior of λ on K by deriving a formula that relates it to a trace. Let n be minimal so that $\{v, \ell v, \cdots, \ell^n v\}$ is linear dependent. For $0 \leq i \leq n$, let $W_i = F\langle v, \cdots, \ell^{i-1}v\rangle$. We've arranged things so that ℓ leaves W_n invariant, but it will take some work to understand how elements of k operate on W_n.

Claim: For $k' \in K$ and $0 \leq i < n$, $k'(\ell^i v) \equiv \lambda(k')\ell^i v$ modulo W_i.
Pf: The case $i = 0$ is obvious. Suppose that $0 < i < n$ and assume the claim below

i. Then:

$$\begin{aligned} k'(\ell^i v) &= \ell(k'(\ell^{i-1}v)) - [\ell, k'](\ell^{i-1}v) \\ &\equiv \ell(\lambda(k')\ell^{i-1}v) - \lambda([\ell, k'])\ell^{i-1}v \pmod{W_{i-1}} \\ &\equiv \lambda(k')\ell^i v \pmod{W_i} \end{aligned}$$

where we've applied the induction hypothesis to both terms and used the fact that K is an ideal.

It follows that each $k' \in K$ leaves W_n invariant and, when restricted to an endomorphism $W_n \to W_n$, has a matrix representation with $\lambda(k')$ along the diagonal, with respect to the basis $\{v, \ell v, \cdots, \ell^{n-1}v\}$. So for $k' \in K$, we get $\mathrm{tr}(k' \mid_{W_n}) = n\lambda(k')$. Applying this to $[\ell, k] \in K$ we get

$$\begin{aligned} n\lambda([\ell, k]) &= \mathrm{tr}([\ell, k] \mid_{W_n}) \\ &= \mathrm{tr}((\ell k - k\ell) \mid_{W_n}) \\ &= \mathrm{tr}(\ell \mid_{W_n} k \mid_{W_n} - k \mid_{W_n} \ell \mid_{W_n}) = 0 \end{aligned}$$

where we've used the fact that ℓ also restricts to an endomorphism of W_n. Since $\mathrm{char} F = 0$, we conclude that $\lambda([\ell, k]) = 0$.

Choose an $\ell \in L \setminus K$. Now ℓ stabilizes W and F is algebraically closed, so the endomorphism $\ell \mid_W : W \to W$ has an eigenvector $w \in W$. We have that $L = K + F\langle \ell \rangle$, that w (being in W) is a common eigenvector for the elements of K, and that w is an eigenvector of ℓ. Therefore w is a common eigenvector for the elements of L.

Assume that $n = \dim V > 1$ and that Lie's theorem holds for lower dimensions. Get a common eigenvector $w \in V$ for the elements of L. Since the elements of L leave $F\langle w \rangle$ invariant, $L \subseteq \mathfrak{gl}(V; F\langle w \rangle)$ (the lie algebra of endomorphisms of V that stabilize $F\langle w \rangle$). Let $\alpha : \mathfrak{gl}(V; F\langle w \rangle) \to \mathfrak{gl}(V/F\langle w \rangle)$ be the natural map (a lie algebra homomorphism). Apply the induction hypothesis to $\alpha(L) \subseteq \mathfrak{gl}(V/F\langle w \rangle)$ to obtain an $\alpha(L)$-invariant flag

$$0 = \bar{V}_0 \subseteq \bar{V}_1 \subseteq \cdots \subseteq \bar{V}_{n-1} = V/F\langle w \rangle \ .$$

Pull this back to V to obtain an L-invariant flag

$$0 = V_0, F\langle w \rangle = V_1, V_2, \cdots, V_n = V \ .$$

■

6.3 Cartan's Solvability Criterion

Theorem 10: Let $L \subseteq \mathfrak{gl}(V)$ be a lie subalgebra, with V a finite dimensional vector space over F, an algebraically closed field of characteristic 0. Suppose that

$$(\forall k, \ell \mid (k \in [L, L] \,\blacksquare\, \ell \in L) \Rightarrow \mathrm{tr}(k\ell) = 0).$$

Then L is solvable.

Proof: Let $M = \{x \in \mathfrak{gl}(V) \mid [x, L] \subseteq [L, L]\}$.

Claim: For any $k \in [L, L],\ x \in M$ we have $\mathrm{tr}(kx) = 0$.
Pf: It suffices to check a generator $[\ell, \ell'] \in [L, L]$ (where $\ell, \ell' \in L$). We have for $x \in M$:

$$\mathrm{tr}([\ell, \ell']x) = \mathrm{tr}(\ell[\ell', x]) = \mathrm{tr}([\ell', x]\ell).$$

Since $x \in M$, $[\ell', x] \in [L, L]$ (this may look obvious, but keep in mind that the endomorphism x need not lie in L). So by the main hypothesis $\mathrm{tr}([\ell', x]\ell) = 0$.

The general situation of the claim above will allow us to conclude that every $k \in [L, L]$ is nilpotent, and hence ad-nilpotent. It will follow from Engel's theorem that $[L, L]$ is nilpotent, and thus solvable, and thus it will follow that L is solvable. It remains only to show that the elements of $[L, L]$ are nilpotent. The following claim should accomplish this and complete the proof.

Claim: Suppose that $A \subseteq B \subseteq V$ are subspaces of V and define

$$M = \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\} = \{x \in \mathfrak{gl}(V) \mid \mathrm{ad}_{\mathfrak{gl}(V)}(x)(B) \subseteq A\}.$$

If $k \in M$ satisfies $\mathrm{tr}(kx) = 0$ for all $x \in M$, then k is nilpotent.
Pf: Suppose that $k \in M$ and $\mathrm{tr}(kx) = 0$ for $x \in M$. Since F is algebraically closed we obtain a Jordan-Chevally decomposition $k = s + n$, where s is semisimple and n is nilpotent. Further, we know that s is a polynomial over F evaluated at k. We also know that $\mathrm{ad}(s)$ is the semisimple part of $\mathrm{ad}(k)$ and is therefore also a polynomial over F evaluated at $\mathrm{ad}(k)$. Let $\{v_1, \cdots, v_n\}$ be a basis for V in which s is diagonal, and let $a_1, \cdots, a_n \in F$ be those diagonal entries. The goal is to show that $s = 0$. Since char $F = 0$, the prime field of F is $\mathbb{Q}$ and F is a vector space over $\mathbb{Q}$. Let E be the subspace of F $\mathbb{Q}$-spanned by $a_1, \cdots, a_n$. The following claim then shows that $s = 0$.

Claim: $E^* = 0$.
Pf: Suppose that $f \in E^*$ (so $f : E \to \mathbb{Q}$ is a $\mathbb{Q}$-linear map). Let $s' \in \mathfrak{gl}(V)$ be the endomorphism with matrix $\mathrm{diag}(f(a_1), \cdots, f(a_n))$ in the $\{v_1, \cdots, v_n\}$ basis. Let $\{e_{ij}\}_{i,j}$ be the basis of $\mathfrak{gl}(V)$ corresponding to $\{v_1, \cdots, v_n\}$. In this basis $\mathrm{ad}(s)$ has the matrix $\mathrm{diag}(\{a_i - a_j\}_{i,j})$ and $\mathrm{ad}(s')$ has the matrix $\mathrm{diag}(\{f(a_i) - f(a_j)\}_{i,j})$. By Lagrange interpolation there is some polynomial over F whose value at $(a_i - a_j)$ is $(f(a_i) - f(a_j))$ for all i, j (the linearity of f allows us to avoid conflict). Evaluating this polynomial at $\mathrm{ad}(s)$ yields $\mathrm{ad}(s')$. Since $\mathrm{ad}(s')$ is a polynomial evaluated at $\mathrm{ad}(s)$, which in turn is a polynomial evaluated at $\mathrm{ad}(k)$, we have that $\mathrm{ad}(s')$ is a polynomial evaluated at $\mathrm{ad}(k)$. We assumed that $k \in M$, so $\mathrm{ad}(k)(B) \subseteq A$. It follows that $\mathrm{ad}(s')(B) \subseteq A$ and so $s' \in M$. Then by our hypothesis concerning k we have

$$0 = \mathrm{tr}(ks') = \sum_{i=1}^{n} f(a_i)a_i \ .$$

Applying f we get $\sum_{i=1}^{n} f(a_i)^2 = 0$. It follows from this that each $f(a_i)$ vanishes, and so f vanishes, and so $E^* = 0$.

■

6.4 Weyl's Theorem

Theorem 11: Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra over an algebraically closed field k of characteristic zero. If $\mathfrak{g}$ is semisimple, then every finite dimensional representation V of $\mathfrak{g}$ is completely reducible.

Proof: Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. We will prove complete reducibility by showing that any proper nontrivial $\mathfrak{g}$-submodule of V is a direct summand (i.e. has a complementary submodule in V). First we tackle the case of irreducible submodules of codimension one, then arbitrary codimension one submodules, then arbitrary submodules. Assume $\mathfrak{g} \neq 0$ to avoid the trivial case.

Claim: Suppose that $W \subseteq V$ is an irreducible submodule of codimension one. It is a direct summand.
Pf: We may safely assume that ϕ is a *faithful* representation: If ϕ were not faithful then $\ker \phi$, being an ideal of a semisimple lie algebra, is a direct sum of some of the simple ideals of $\mathfrak{g}$. So it has a complement in $\mathfrak{g}$: $\mathfrak{g} = \ker \phi \oplus \mathfrak{g}'$. Then $\phi|_{\mathfrak{g}'} : \mathfrak{g}' :\to \mathfrak{gl}(V)$ is a *faithful* representation of $\mathfrak{g}'$, and it is easy to check that $\mathfrak{g}$-submodules of V correspond *exactly* to $\mathfrak{g}'$-submodules of V. So just assume that ϕ is faithful.

Let $c_\phi : V \to V$ be the Casimir element of ϕ. Recall that c_ϕ is a $\mathfrak{g}$-module homomorphism, so $\ker c_\phi$ is a submodule. Will it do for a complement to W?

Notice that $\mathfrak{g}$ acts trivially on the quotient $\mathfrak{g}$-module ${}^{V}\!/_{W}$. This follows from the facts that $\mathfrak{g}$ is semisimple and that $\dim({}^{V}\!/_{W}) = 1$: If $\bar{\phi} : \mathfrak{g} \to \mathfrak{gl}({}^{V}\!/_{W})$ is the induced representation then

$$\bar{\phi}(\mathfrak{g}) = \bar{\phi}([\mathfrak{g}, \mathfrak{g}]) = [\bar{\phi}(\mathfrak{g}), \bar{\phi}(\mathfrak{g})] \subseteq [\mathfrak{gl}({}^{V}\!/_{W}), \mathfrak{gl}({}^{V}\!/_{W})] = \mathfrak{sl}({}^{V}\!/_{W}) = 0 \ .$$

In other words, any $x \in \mathfrak{g}$ sends V into W. So c_ϕ, being a sum of products of things in im ϕ, also sends V into W. By Schur's lemma, $c_\phi|_W : W \to W$ is a scalar multiple of 1_W. If that scalar were 0, then we would have $c_\phi^2 = 0$ (because $c_\phi(V) \subseteq W$ and $c_\phi(W) = 0$). This would give $\mathrm{tr}(c_\phi) = 0$, which contradicts the calculation that $\mathrm{tr}(c_\phi) = \dim \mathfrak{g}$. So $c_\phi|_W : W \to W$ is an isomorphism.

It is now apparent that $V = W \oplus \ker c_\phi$: We have $W \cap \ker c_\phi = \ker(c_\phi|_W) = 0$. And $\ker c_\phi$ is one dimensional because im $c_\phi = W$ has codimension one.

Claim: Suppose that $W \subseteq V$ is a submodule of codimension one. It is a direct summand.
Pf: Use induction on $\dim V$; the base case $\dim V = 1$ is trivial. Assume that codimension 1 submodules of $\mathfrak{g}$-modules of lower dimension are direct summands. The previous claim covers the case in which W is irreducible, so suppose that W contains a proper nontrivial submodule W'.

Now ${}^{W}\!/_{W'}$ is a codimension one $\mathfrak{g}$-submodule of the quotient $\mathfrak{g}$-module ${}^{V}\!/_{W'}$, which has stricly lower dimension than $\dim V$. We obtain a complementary submodule ${}^{\widetilde{W}}\!/_{W'}$ so that

$${}^{V}\!/_{W'} = {}^{W}\!/_{W'} \oplus {}^{\widetilde{W}}\!/_{W'} \ .$$

Since $\dim({}^{\widetilde{W}}\!/_{W'}) = 1$, W' is a codimension one submodule of $\widetilde{W}$, which has stricly lower dimension than $\dim V$. We obtain a complementary submodule $X \subseteq \widetilde{W}$ to W'.

X should do the trick. We get $W \cap X = 0$: if $x \in W \cap X$ then $x + W' \in {}^{W}\!/_{W'} \cap {}^{\widetilde{W}}\!/_{W'} = 0$, so $x \in W' \cap X = 0$. And X is one dimensional so $\dim X + \dim W = \dim V$. Thus we get $V = W \oplus X$.

Claim: Suppose that $W \subsetneq V$ is a proper submodule. It is a direct summand.
Pf: Since V and W are $\mathfrak{g}$-modules, so is $\mathrm{Hom}_k(V, W)$. Consider the subspace

$$\mathcal{V} := \{f \in \mathrm{Hom}_k(V, W) \mid f|_W = \lambda 1_W \text{ for some } \lambda \in k\}$$

and the smaller subspace

$$\mathcal{W} := \{f \in \mathcal{V} \mid f|_W = 0\}.$$

Observe that $\mathfrak{g}$ sends $\mathcal{V}$ into $\mathcal{W}$: For $f \in \mathcal{V}$ (say $f|_W = \lambda 1_W$), $x \in \mathfrak{g}$, and $w \in W$ we have

$$(x.f)(w) = x.(f(w)) - f(x.w) = \lambda\, x.w - \lambda\, x.w = 0,$$

so $(x.f)|_W = 0$. This shows that $\mathcal{V}, \mathcal{W}$ are submodules of $\mathrm{Hom}_k(V, W)$.

Now $\mathcal{W}$ has codimension one in $\mathcal{V}$. One way to see this is to define a linear surjection $F : \mathcal{V} \to k$ that takes $f \in \mathcal{V}$ with $f|_W = \lambda 1_V$ to λ, and note that its kernel is $\mathcal{W}$. Thus the previous claim applies and we obtain a one dimensional complementary submodule to $\mathcal{W}$ in $\mathcal{V}$; let $f \in \mathcal{V}$ span it. By rescaling f we may ensure that $f|_W = 1_W$.

To say that $\mathfrak{g}$ kills an element of $\mathrm{Hom}_k(V, W)$ is precisely to say that that element is a $\mathfrak{g}$-module homomorphism. Well $\mathfrak{g}$ does kill f, since one dimensional representations of semisimple lie algebras are always trivial (see argument in first claim). So f is a $\mathfrak{g}$-module homomorphism, and its kernel is then the submodule we need! We have $W \cap \ker f = \ker(f|_W) = 0$. And we have $W + \ker f = V$ because any $v \in V$ can be written as $f(v) + (v - f(v))$. So $V = W \oplus \ker f$.

■

7 The Jordan-Chevalley Decomposition

[still need to fill in proofs in this section]

Let V be a finite-dimensional vector space over an algebraically closed field k of characteristic 0, and consider an $x \in \mathrm{End}(V)$. Then:

1. There are unique $x_s, x_n \in \mathrm{End}(V)$ such that
   - x_s is semisimple
   - x_n is nilpotent
   - x_s, x_n commute
   - $x = x_s + x_n$
2. There are $p, q \in k[T]$ with a constant term of 0 such that
   - $x_s = p(x)$
   - $x_n = q(x)$

An abstract version of this decomposition can take place inside a semisimple Lie algebra. If $\mathfrak{g}$ is a semisimple Lie algebra over k, then every $x \in \mathfrak{g}$ can be written uniquely as $x = s + n$ with s being ad-semisimple, with n ad-nilpotent, and with $[s, n] = 0$.

In the case that $L \subseteq \mathfrak{gl}(V)$ is a *linear* semisimple Lie algebra, we have two versions of the decomposition for $x \in L$ (the abstract decomposition and the decomposition of x as an endomorphism). The two notions do end up agreeing, but the proof of this fact is not simple.

A very useful consequence of this is that representations of a semisimple Lie algebra are compatible with the abstract decomposition: If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over k and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional representation, then for $x \in \mathfrak{g}$

$$\begin{array}{rll} \text{the abstract Jordan-Chevalley decomposition} & x = s + n & \text{in } \mathfrak{g} \\ \text{gives the usual Jordan-Chevalley decomposition} & \phi(x) = \phi(s) + \phi(n) & \text{in } \mathfrak{gl}(V). \end{array}$$

In other words what is true by definition for the representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ turns out to be true for more general representations.

References

[1] Humphreys, J.E. (1972) Introduction to Lie Algebras and Representation Theory. Springer Verlag, New York.