The Navier-Stokes Equations

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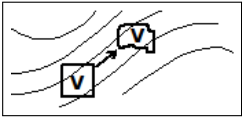
Mass and Momentum

The Navier-Stokes equations describe the non-relativistic time evolution of **mass** and **momentum** in fluid substances.

- mass density field: $\rho = \rho(t, x, y, z)$
- velocity field: $v^i = v^i(t, x, y, z), \; i = 1, 2, 3$

We will derive them by using **conservation of mass** and **force laws** on a control volume V. The control volume propagates in time, $V = V(t)$.



(We will assume an isothermal continuum, so we don't need to consider energy conservation.)

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Conservation of Mass

Conservation of mass gives a **continuity equation**.

(Demand that the mass flux through any closed surface be the change in total mass, and use divergence theorem)

$$\dot{\rho} + \partial_i(\rho v^i) = 0$$

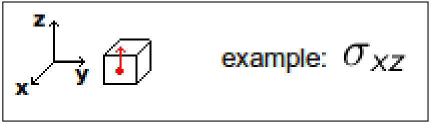
This provides us with nice simplifications if we later assume incompressibility

$$\text{incompressibility} \implies \dot{\rho} = 0 \implies \partial_i v^i = 0$$

The Forces

We start by considering the forces on a control volume V.

- There are forces on the **body**, like gravity. Call this force field f^i.
- And there are forces on the **surface** of the control volume, pressures along normals and viscous forces tangentially. These are described by a stress tensor σ_{ij}.
  - σ_{ij} can be physically defined by the way it operates on a normal n^j to a surface element of area dA. It gives the force on that surface element: $\sigma_{ij} n^j dA$
  - σ_{ij} is the stress (the force per area) in the j direction acting on the cube face with normal in the i direction. Thus the diagonal components are pressures and the off-diagonal components are shear stresses.



The Force Law

Now we write the force law on a control volume V:

$$\frac{d}{dt}\int_{V(t)} (\rho v^i)\, d^3x = \int_{V(t)} (f^i)\, d^3x + \int_{\partial V(t)} (\sigma^{ij} n_j)\, dA$$

$$\frac{d}{dt}\vec{p} \quad = \quad \vec{F}_{\text{body}} \quad + \quad \vec{F}_{\text{surface}}$$

Divergence theorem on the surface force term gives:

$$\frac{d}{dt}\int_{V(t)} (\rho v^i)\, d^3x = \int_{V(t)} (f^i)\, d^3x + \int_{V(t)} (\partial_j \sigma^{ij})\, d^3x$$

It now remains to turn the first term into a volume integral. The time derivative cannot simply be pushed in because the volume of integration is time-dependent.

Reynold's Transport Theorem

We need to deal with the time derivative on $\frac{d}{dt}\int_{V(t)} \alpha(t)\ d^3x$ for a function $\alpha = \alpha(t,x,y,z)$ and a time-dependent volume of integration $V(t)$.

$$\frac{d}{dt}\int_{V(t)} \alpha(t) =$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\int_{V(t+\Delta t)} \alpha(t+\Delta t)\ -\ \int_{V(t)} \alpha(t)\right) =$$

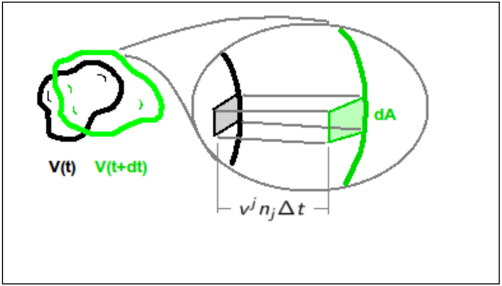
$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\int_{V(t+\Delta t)} \alpha(t+\Delta t)\ -\ \int_{V(t)} \alpha(t+\Delta t)\ +\ \int_{V(t)} \alpha(t+\Delta t)\ -\ \int_{V(t)} \alpha(t)\right) =$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\int_{V(t+\Delta t)} \alpha(t+\Delta t)\ -\ \int_{V(t)} \alpha(t+\Delta t)\right)\ +\ \int_{V(t)} \frac{\partial \alpha}{\partial t} =$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\int_{\Delta V} \alpha(t+\Delta t)\right)\ +\ \int_{V(t)} \frac{\partial \alpha}{\partial t}$$

Reynold's Transport Theorem

The integral over the difference in volume, $\int_{\Delta V} \alpha(t + \Delta t)$, can be expressed in terms of an integral of the changes in volume over the surface.



The differential volume change at dA is $v^j n_j \Delta t \ dA$.

Reynold's Transport Theorem

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\int_{\Delta V} \alpha(t) \right) =$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\sum_{\text{cube at each dA}} \alpha\big(\text{center of cube}\big) \left(\Delta V_{\text{at patch dA}}\right) \right) =$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\sum_{\text{cube at each dA}} \alpha\big(\text{center of cube}\big) v^j n_j \Delta t \; dA \right) =$$

$$\sum_{\text{cube at each dA}} \alpha\big(\text{center of cube}\big) v^j n_j \; dA =$$

$$\int_{\partial V(t)} \alpha(t) v^j n_j \; dA$$

Momentum Equation

And so we have Reynold's transport theorem:

$$\frac{d}{dt}\int_{V(t)} \alpha(t) = \int_{\partial V(t)} \alpha(t) v^j n_j \, dA + \int_{V(t)} \frac{\partial \alpha}{\partial t}$$

We can use it on the force equation:

$$\int_{\partial V(t)} \left(\rho v^i v^j n_j\right) dA + \int_{V(t)} \frac{\partial \left(\rho v^i\right)}{\partial t} = \int_{V(t)} f^i + \int_{V(t)} \partial_j \sigma^{ij}$$

Then use divergence theorem again, and combine the volume integrals:

$$\int_{V(t)} \left(\partial_j \left(\rho v^i v^j\right) + \frac{\partial \left(\rho v^i\right)}{\partial t} - f^i - \partial_j \sigma^{ij} \right) = 0$$

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Momentum Equation

Since this holds for any control volume, we get the differential form of the equation. We can also use mass conservation $\dot{\rho} = -\partial_j(\rho v^j) = -\partial_j \rho v^j - \rho \partial_j v^j$

$$\partial_j \left(\rho v^i v^j \right) + \frac{\partial \left(\rho v^i \right)}{\partial t} - f^i - \partial_j \sigma^{ij} = 0$$

$$\partial_j \rho v^i v^j + \rho \partial_j v^i v^j + \rho v^i \partial_j v^j + \dot{\rho} v^i + \rho \dot{v}^i - f^i - \partial_j \sigma^{ij} = 0$$

$$\partial_j \rho v^i v^j + \rho \partial_j v^i v^j + \rho v^i \partial_j v^j - \partial_j \rho v^i v^j - \rho v^i \partial_j v^j + \rho \dot{v}^i - f^i - \partial_j \sigma^{ij} = 0$$

And we are finally left with the **Cauchy momentum equation**:

$$\rho \partial_j v^i v^j + \rho \dot{v}^i - f^i - \partial_j \sigma^{ij} = 0$$

Momentum Equation

To get an actual equation we must choose a form for the stress tensor σ^{ij}.

$$\rho\partial_j v^i v^j + \rho\dot{v}^i - f^i - \partial_j\sigma^{ij} = 0$$