Worksheet 1, Problem 5: Imagine a clock where the second hand on is one unit long, and the center of the clock is at the origin, so that the tip of the second hand traces out the unit circle. What is the position of its tip t seconds after the minute?

Solution: We'd like to parameterize a unit circular arc that begins at the 12 o'clock position (which is $(0,1)$) and travels clockwise at a rate of $2\pi/60$ radians per second. Let's start with a well-known parameterization for a unit circle and then modify it until it meets our demands. We begin with

$$\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad t \in [0, 2\pi],$$

which starts at $(1,0)$ and travels counterclockwise making one revolution at one radian per second. In order to travel clockwise we replace t by $-t$,

$$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad t \in [0, 2\pi],$$

and in order to start the curve at $(0,1)$ we shift the parameter by $\pi/2$ in the appropriate direction. We should also allow time to run indefinitely into the future:

$$\begin{pmatrix} \cos(t - \pi/2) \\ -\sin(t - \pi/2) \end{pmatrix}, \quad t \in [0, \infty).$$

This can be rewritten more cleanly as

$$\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad t \in [0, \infty),$$

which could have been an appropriate curve to start with if we had the foresight! Now the curve starts at the right place and travels in the right direction, but the speed is not quite right. This can be remedied by rescaling the parameter so that a full revolution has taken place at $t = 60$. Replacing t by $({}^{2\pi}/_{60})t$ should achieve this:

$$\begin{pmatrix} \sin(\frac{2\pi}{60}t) \\ \cos(\frac{2\pi}{60}t) \end{pmatrix}, \quad t \in [0, \infty).$$

And this is the desired curve.

Reparameterizations can be confusing and it is easy to make errors. Parameter shifts are particularly notorious for this! For this reason you should check at each step that the reparameterized curve is doing the right thing.