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Research Statement

1 Overview

The question of stability for any physical phenomenon described by a differential equation is essentially this: If we perturb our initial conditions by a small amount, will the new solution remain close to the original one? This question is important because if a mathematically predicted motion is not stable we will not see it physically. My research pertains to stability for fluid motion, but a calculus textbook rotating in space (with zero gravity) serves as a good finite dimensional analog. The equations of motion predict that the book should sustain a steady rotation with constant angular velocity about any of its three axes of symmetry. However, rotation about the axis perpendicular to the book's spine is unstable, so the slightest perturbation will dramatically change this motion and we could never actually observe this rotation[1]. Moreover, this instability is inherent in the equations of motion and we can understand it mathematically in the framework of the group structure of rotation matrices as a consequence of conservation of energy and momentum, see [A, AK]. In my research I examine Euler's equation for incompressible fluids with no viscosity and try to determine whether or not certain solutions are mathematically stable.

One method for analyzing the stability of a flow is to examine its so-called linear stability. This involves linearizing Euler's equation, equation (1) below, at a particular solution to get a linear system of equations for approximating the evolution of a perturbation. To keep the analysis manageable, we typically analyze the linear stability of steady-state solutions, i.e. solutions where the velocity of the flow does not depend on time. For example, a river where the paths of flow do not change, but the water is moving, corresponds to a steady-state of the fluid motion. Linear stability or instability does not imply nonlinear stability or instability, but it can provide valuable information. Section 2.2 details work I began as a graduate student on linear instability criteria for steady-state solutions to Euler's equation.

Examining the stability of a fluid flow amounts to determining whether or not infinitely small perturbations have a finite affect on the flow in question. All stability questions take the same basic form, but there are three (perhaps subtle-seeming) factors that really define a given stability problem: What is the initial flow that we are perturbing? What is the fluid domain? What norm are we using to measure the growth of perturbations? Section 2.3 describes a collaborative project in which we are trying to strengthen existing results for a certain type of flows (so-called vortex patch solutions) by generalizing the fluid domain and reducing certain restrictions on the initial perturbations.

In addition to detailing my research projects in Sections 2.2 and 2.3 document, I also include some basic information on Euler's equation in Section 2.1. My future research plans include supervising undergraduate research, so I discuss some ideas for projects in the last section of this document.

[1] to see this phenomenon watch International Space Station officer Don Petit's youtube video: *Rotating Solid Bodies in Microgravity*

2 Current Research Projects

2.1 Background: Euler's Equation

The flow of a fluid is described by a vector field v, where the vector $v(x,t)$ is the velocity of a fluid particle traveling through point x at time t. I study incompressible fluids; this assumption imposes the mathematical requirement that the divergence of the velocity field is 0, $\partial_k v^k = 0$, at all times. The divergence-free requirement leads to the introduction of a scalar pressure $p = p(x,t)$ into the description of our fluid flow. Whereas, the Navier-Stokes equation describes viscous flows, the flows I consider have no viscosity. Thus, the evolution of the velocity v and pressure p is governed by Euler's equation:

$$\begin{cases} \partial_t v^j = -v^k \cdot \partial_k v^j - \partial_j p \\ \partial_k v^k = 0 \\ v(x,0) = v_0(x) \end{cases} \tag{1}$$

Here v_0 is the initial (divergence-free) velocity field. The vorticity of a fluid flow is the curl of the velocity field, so by Stoke's theorem we have that the circulation of the flow along any closed path is the integral of the vorticity inside that path. Vorticity plays an important role in the study of fluids for two reasons: 1) circulation is conserved by Eulerian fluid motion and 2) the velocity field can be recovered from the vorticity ω using the Biot-Savart Law:

$$v(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(t,y)\, dy.$$

2.2 Linear Instability for Steady State Flows

The goal of this project is to narrow down which perturbations are causing a given steady-state solution to be linearly unstable. The focus is on fast-oscillating perturbations, of which, two types are considered: those that preserve the circulation of the steady-state velocity field, and those that do not. The conservation of circulation motivates the interest in comparing perturbations that preserve the topological character of the vorticity with those that dramatically change it. In [Th] I find lower bounds for the growth of fast-oscillating perturbations of both types.

Our fluid domain is $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ for $n = 2,3$; this corresponds to periodic boundary conditions. To define reasonable criteria for linear instability of a given steady-state solution $u \in C^\infty(\mathbb{T}^n)$, we linearize Euler's equation (1) at u and get an equation for the linear evolution of a divergence-free perturbation $w_0 \in L^2(\mathbb{T}^2)$. This leads to a semigroup of bounded linear evolution operators indexed by time, which we denote $G(t)$, where each operator maps the initial perturbation to the first order perturbation at the corresponding time t. In other words if our perturbed initial condition is $v_0(x) = u(x) + w_0(x)$, then the velocity field for our flow at time t can be approximated by $v(x,t) \approx u(x) + G(t)w_0(x)$. The concept of linear stability requires that the perturbation $G(t)w_0$ does not grow too much as time increases.

The general approach here is based on WKB methods for analyzing fast-oscillating perturbations first applied to Euler's equation by Friedlander and Vishik [FV1, FV2, FV3] and also, independently, by Lifshitz and Hameiri [LH1, LH2]. The key observation is that the growth of fast-oscillating perturbations is governed by the growth of solutions to a system of ODEs called the bicharacteristic amplitude system:

$$\text{(BAS)} \begin{cases} \dot{x} = u(x), \\ \dot{\xi} = -\left(\frac{\partial u}{\partial x}\right)^T \xi, \\ \dot{b} = -\left(\frac{\partial u}{\partial x}\right) b + 2\left(\frac{\partial u}{\partial x} b, \xi\right) \frac{\xi}{|\xi|^2}, \\ \big(x(0), \xi(0), b(0)\big) = (x_0, \xi_0, b_0) \in \mathcal{A}, \end{cases} \tag{2}$$

where the set of initial conditions to (BAS) is restricted to $\mathcal{A}$, defined by

$$\mathcal{A} := \{(x_0, \xi_0, b_0) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n | \ \xi_0 \perp b_0, \ |\xi_0| = |b_0| = 1\}.$$

The main result of [Th] is a series of lower bounds for the growth of each type of fast-oscillating perturbation for both 2- and 3-dimensional flows. The lower bounds are defined in terms of solutions to the bicharacteristic amplitude system (BAS). Growth of perturbations that preserve circulation corresponds to growth of solutions to (BAS) with initial conditions in the support of the vorticity of our steady-state solution, curl(u). To find a lower bound for the growth of perturbations in the factor space we consider initial conditions for (BAS) outside the support of curl(u).

The problem of tracking how the initial perturbation stretches with time can be reduced to determining the spectrum of $G(t)$. A scalar $\lambda \in \mathbb{C}$ is in the spectrum of some linear operator T if the operator $\lambda I - T$ does not have a bounded inverse. For example if the operator T is a matrix, then its spectrum is its set of eigenvalues. Just as in the finite dimensional case, the spectrum of an infinite dimensional operator gives us some indication of how the operator stretches vectors in its domain. The fast-oscillating perturbations under consideration here, correspond to a subset of the spectrum, called the essential spectrum. In [V] Vishik proved that, for a given time $t > 0$, the essential spectral radius of the linear evolution operator $G(t)$ is expressed in terms of the maximal Lyapunov exponent associated with (BAS). More recently, analysis involving the theory of cocycles has led to a greater understanding of the essential spectrums of the evolution operators $G(t)$. In a series of results by Shvydkoy, Latushkin and Vishik, the authors use cocycle methods to determine that the essential spectrum for $G(t)$ in 2-dimensions is a solid annulus and in 3-dimensions is a solid torus, see [SL1, SL2, SV, S2]. Shvydkoy also applied methods from the theory of cocycles to a more general class of advective equations from hydrodynamics in [S1], which resulted in a new, shorter proof of Vishik's result for the essential spectral radius. The lower bounds in my result are lower bounds for the radius of the essential spectrum of the operator $G(t)$ restricted to each type of perturbation.

I plan to apply the theory of cocycles to the problem of singling out the effects of perturbations that preserve circulation. In particular, it may be possible to completely determine the essential spectrum of $G(t)$ restricted to circulation-preserving perturbations. Flows with a hyperbolic stagnation point provide an important class of examples with instability in the essential spectrum. Preliminary calculations indicate that flows with hyperbolic stagnation point always have instability coming from the factor space (this is the class of perturbations that dramatically change the topological character of the vorticity of the steady flow). I would like to determine if every flow with instability in the essential spectrum necessarily has instability coming from the factor space.

2.3 Nonlinear Stability of Vortex Patch Solutions

Vortex patch solutions to 2-dimensional Euler's equation (1) are flows where the initial vorticity, $\mathrm{curl}(v_0) = \partial_1 v_2(0) - \partial_2 v_1(0)$, is a characteristic function of some bounded region Ω_0. These solutions are of interest, in part, because they are the most stripped-down mathematical descriptions of hurricanes. The goal of this project is to utilize geometric consequences of certain conserved quantities for Eulerian fluids to strengthen existing nonlinear stability results in the L^1-norm for some families of rigidly rotating vortex patch solutions.

In 2-dimensions vorticity is preserved along flow lines, so the study of such a solution reduces to examining the evolution of the vortex patch under the flow. If we let g_t be the flow map defined by $\frac{d}{dt} g_t(x) = v(g_t(x), t)$, then the vorticity at time t is the characteristic function of the set $\Omega_t := g_t(\Omega_0)$. As a result, several conserved quantities of fluids have geometric consequences for the shape of a vortex patch. For example, conservation of circulation implies that the area of a vortex patch will not change as it flows. And for moment of fluid impulse, defined by

$$J(\Omega) := \int_\Omega |x|^2 \, dx,$$

the circular patch is the unique minimizer of J among all patches of a given area centered at the origin. Thus, conservation of moment of fluid impulse implies the well-known fact that circular patches are steady-state solutions.

Wan and Pulvirenti showed L^1 stability for circular patches where the fluid domain is a disk, [WP]. In [T] the nonlinear stability of circular patches in the plane was established using a similar method. The nonlinear orbital stability of certain Kirchhoff ellipse solutions in the plane was also shown in [T]. In [W] Wan establishes a more general result for the so-called Kelvin m-waves. For the nonlinear stability of the rotating solutions, both [T] and [W] require the initial perturbation to be small and its time evolution to remain in a fixed disk, a property which has not been established.

Dreitschel examined the stability of circular vortex patches in a new norm defined in terms of area and moment of fluid impulse [D]. Recently Sideris and Vega proved L^1 stability of circular patches in the plane without the smallness assumption on the initial perturbation, [SV]. Their method involved demonstrating that Dreitschel's norm on perturbations was equivalent to the L^1-norm.

Currently I am collaborating with Thomas Sideris on further investigations into the L^1 nonlinear stability of rotating vortex patch solutions with the goal of reducing the requirements on the initial perturbations and removing the assumption that perturbations remain in a fixed disk. In our examination of elliptic vortex patches we have defined a function analogous to Dreitschel's norm which maximizes the L^1-norm. In the spirit of Sederis and Vega's result for circular patches in [SV], we hope to show that for some alignment of the ellipse, this maximizing function is bounded by the L^1-norm of the initial perturbation.

3 Student Research

To my mind, there are two motivations for undergraduate research: (1) to expose students to valuable or interesting subject matter not covered in traditional mathematics courses in a way that facilitates greater appreciation for and comfort with mathematical concepts, and (2) to give especially talented students the opportunity to work to the limits of their abilities and perhaps create original research. The ultimate goal in both cases is to foster a love for mathematics and build critical thinking skills at a level beyond that of the classroom. I look forward to supervising research projects of both types.

If the goal of the research project is more in keeping with exposure to new and interesting mathematics, the choice of project would likely depend on the particular student's taste. If a student is more inclined towards algebraic concepts, I might suggest looking at quaternions and their connection to 3D rotations or an extension of topics covered in linear algebra. If the student has interests in physics or appreciates more 'visualizable' questions I would suggest some topics connecting calculus with physics–perhaps an in-depth look at the variations of Stokes' Theorem. If the student feels more comfortable with experimentation or programming I would suggest a project utilizing computer programs (or developing new programs) to explore ideas beyond the reach of analytic tools available to the student.

I also have project ideas more closely related to my own research to challenge the especially talented student. I would love to introduce the functional calculus of bounded self-adjoint operators to an undergraduate student. The subject is so algebraic in nature that many of the key ideas do not require heavy background in analysis. Analyzing the stability of a particular fluid flow can lead to interesting questions that do not require graduate level background. For example, some aspects of stability for plane parallel shear flow (i.e. flow straight down a pipe) only require vector calculus and basic ODEs. Certain steady-state solutions to Euler's equation correspond to very manageable (BAS) equations. For example, any flow with a hyperbolic stagnation point corresponds to a system of ODEs with constant coefficients.

As an undergraduate student I was steered towards mathematics when my Vector Calculus professor, Dr. Kunin, proposed supervising my work on an independent research project. I did not know that the subject was so beautiful and interesting, and I certainly did not think I was capable of earning a PhD in mathematics. We continued that work for several semesters and he encouraged me to participate in an REU, the MASS program and Budapest Semesters. Dr. Kunin's mentoring had a profound positive effect on my career and I would be honored to do the same for one of my own students someday.

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