Handout 9

September 3, 2015

Problem 1

$$y'' + 9y = g(t)$$

$$y(0) = 2, y'(0) = -3$$

1. Find the transfer function $H(s)$ for the system.
2. Find the impulse response function $h(t)$.
3. Give a formula for the solution to the initial value problem.

Problem 2 Determine $\mathcal{L}(f)$, where the periodic function f is described by the following graph:

Unit Step Function The function $u(t)$ defined by $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$.

Laplace Transform of Step Function

$$\mathcal{L}(u(t-a))(s) = \frac{e^{-as}}{s}$$

Translation in t Let $F(s) = \mathcal{L}(f)(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

$$\mathcal{L}(f(t-a)u(t-a))(s) = e^{-as}F(s)$$

Convolution Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The convolution of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv$$

Properties of Convolution Let $f(t), g(t)$ and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

- $f * g = g * f$
- $f * (g + h) = (f * g) + (f * h)$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0$

Convolution Theorem Let $f(t)$ and $g(t)$ be piecewise continous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathcal{L}(f)(s)$ and $G(s) = \mathcal{L}(g)(s)$. Then

$$\mathcal{L}(f * g)(s) = F(s)G(s)$$

Transfer Function The transfer function $H(s)$ of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, under the assumption that all initial conditions are zero. That is, $H(s) = Y(s)/G(s)$. For example, if the linear system is governed by the differential equation

$$ay'' + by' + cy = g(t), \qquad t > 0$$

where a, b, c are constants, we can compute the transfer function as follows. Take the Laplace transform of both sides to get

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s)$$

Since the initial conditions are assumed to be zero, the equation reduces to

$$(as^2 + bs + c)Y(s) = G(s)$$

Thus the transfer function is

$$H(s) = \frac{Y(s)}{G(s)} = \frac{1}{as^2 + bs + c}$$

Note similarity to finding the roots of the auxiliary equation of the homogeneous equation. Indeed, to invert $Y(s) = G(s)/(as^2 + bs + c)$ the first step is to find the roots of the denominator and use partial fractions.

Impulse Response Function The function $h(t) = \mathcal{L}^{-1}(H)(t)$ is called the impulse response function for the system.

Solution Using Impulse Response Function Consider the initial value problem

$$ay'' + by' + cy = g, \qquad y(0) = y_0, \qquad y'(0) = y_1$$

Let y_k be the solution to the corresponding homogeneous system (when $g = 0$). Let h be the impulse response function. Then the unique solution is given by

$$y(t) = (h * g)(t) + y_k(t) = \int_0^t h(t - v)g(v)dv + y_k(t)$$