NOETHERIAN HOPF ALGEBRA DOMAINS OF GELFAND-KIRILLOV DIMENSION TWO

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This paper is dedicated to Susan Montgomery on the occasion of her 65th birthday.

Abstract. We classify all noetherian Hopf algebras $H$ over an algebraically closed field $k$ of characteristic zero which are integral domains of Gelfand-Kirillov dimension two and satisfy the condition $\text{Ext}_H^1(k, k) \neq 0$. The latter condition is conjecturally redundant, as no examples are known (among noetherian Hopf algebra domains of GK-dimension two) where it fails.

0. Introduction

A study of general infinite dimensional noetherian Hopf algebras was initiated by Brown and the first-named author in 1997-98 [9, 7], where they summarized many nice properties of noetherian Hopf algebras, most of which are either PI (namely, satisfying a polynomial identity) or related to quantum groups, and posted a list of interesting open questions. Some progress has been made, and a few of the open questions have been solved since then. For example, the homological integral was introduced for Artin-Schelter Gorenstein noetherian Hopf algebras [19], and it became a powerful tool in the classification of prime regular Hopf algebras of Gelfand-Kirillov dimension one [11]. Some interesting examples were also discovered in [11]. In another direction, the pointed Hopf algebra domains of finite Gelfand-Kirillov dimension with generic infinitesimal braiding have been classified by Andruskiewitsch and Schneider [3] and Andruskiewitsch and Angiono [2]. To further understand general infinite dimensional noetherian Hopf algebras, and to more easily reveal their structure, a large number of new examples would be extremely helpful. After the work [11], it is natural to consider Hopf algebras of Gelfand-Kirillov dimension two.

In the present work, we aim to understand noetherian and/or affine Hopf algebras $H$ under the basic assumption

$(H)$ $H$ is an integral domain with Gelfand-Kirillov dimension two, over an algebraically closed field $k$ of characteristic zero.

Our analysis requires an additional homological assumption, namely

$(\natural)$ $\text{Ext}_H^1(Hk, Hk) \neq 0$, where $Hk$ denotes the trivial left $H$-module.

The condition $(\natural)$ is equivalent to the condition that the corresponding quantum group contains a classical algebraic group of dimension 1 (see Theorem 3.9 for the

2000 Mathematics Subject Classification. 16A39, 16K40, 16E10, 16W50.

Key words and phrases. Hopf algebra, noetherian, Gelfand-Kirillov dimension.

This research was partially supported by grants from the NSF (USA) and by Leverhulme Research Interchange Grant F/00158/X (UK).
algebraic version of this statement). The latter condition is analogous to the fact that every quantum projective 2-space in the sense of Artin-Schelter [4] contains a classical commutative curve of dimension 1.

The combination of (2) with (H) will be denoted (H♯). Our main result can be stated as follows.

Theorem 0.1. Let $H$ be a Hopf algebra satisfying (H♯). Then $H$ is noetherian if and only if $H$ is affine, if and only if $H$ is isomorphic to one of the following:

(I) The group algebra $k\Gamma$, where $\Gamma$ is either
   (Ia) the free abelian group $\mathbb{Z}^2$, or
   (Ib) the nontrivial semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$.

(II) The enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ is either
   (IIa) the 2-dimensional abelian Lie algebra over $k$, or
   (IIb) the Lie algebra over $k$ with basis $\{x, y\}$ and $[x, y] = y$.

(III) The Hopf algebras $A(n, q)$ from Construction 1.1, for $n \geq 0$.

(IV) The Hopf algebras $B(n, p_0, \ldots, p_s, q)$ from Construction 1.2.

(V) The Hopf algebras $C(n)$ from Construction 1.4, for $n \geq 2$.

Aside from the cases $A(0, q) \cong A(0, q^{-1})$, the Hopf algebras listed above are pairwise non-isomorphic.

The Hopf algebras $B(n, p_0, \ldots, p_s, q)$ in Theorem 0.1(IV) provide a negative answer to [26, Question 0.4] and [8, Questions K and J]; see Remark 1.7. Geometrically, Theorem 0.1 provides a list of 2-dimensional quantum groups that are connected and satisfy the condition (♯).

Following the above classification one can establish the following common properties for these Hopf algebras.

Proposition 0.2. Let $H$ be a Noetherian Hopf algebra satisfying (H♯).

(a) $\text{Kdim} H = 2$, and $\text{gldim} H = 2$ or $\infty$.

(b) $H$ is Auslander-Gorenstein and GK-Cohen-Macaulay, with injective dimension 2.

(c) $\text{Spec } H$ has normal separation.

(d) $H$ satisfies the strong second layer condition.

(e) $\text{Spec } H$ is catenary, and $\text{height}(P) + \text{GKdim}(H/P) = 2$ for all prime ideals $P$ of $H$.

(f) $H$ satisfies the Dixmier-Moeglin equivalence.

(g) $H$ is a pointed Hopf algebra.

(h) $\dim_k \text{Ext}^1_H(k, k) = 1$ if and only if $H$ is not commutative.

We conjecture that Proposition 0.2(a,b,c,d,e,f) hold for affine and/or noetherian prime Hopf algebras of GK-dimension two. Let $A$ be the group algebra given in [19, Example 8.5]; then the quotient Hopf algebra $A/[A, A]$ is isomorphic to the group algebra $k(\mathbb{Z}/2\mathbb{Z})^d$ which is finite dimensional. By Theorem 3.9, $\text{Ext}^1_A(k, k) = 0$, and hence (♯) fails for this prime affine noetherian Hopf algebra of GK-dimension two.

The condition (♯) may follow from (H) as we have no counterexamples. It is only natural to conjecture (♯) when $\text{GKdim } H \leq 2$, since examples abound in higher GK-dimensions, starting with $U(\mathfrak{sl}_2(k))$. Completely different ideas are needed to handle Hopf algebras satisfying (H) but not (♯). We emphasize this key question:

Question 0.3. Does (♯) follow from (H)?
Observation 0.4. As we show in Proposition 3.8, an affine or noetherian Hopf algebra $H$ satisfies (7) if and only if $H$ has an infinite dimensional quotient algebra which is commutative. It is helpful to note part of this result in advance:

If the Hopf algebra $H$ contains an algebra ideal $I$ such that $I \subseteq \text{ker} \epsilon$ and $H/I$ is an infinite dimensional commutative affine domain, then $H$ satisfies (7).

To see this, observe that $Hk$ is an ($H/I$)-module, and that all simple ($H/I$)-modules have non-split self-extensions. Thus $\text{Ext}^1_{H/I}(k, k) \neq 0$, whence $\text{Ext}^1_H(k, k) \neq 0$.

0.5. Let $k$ be a commutative base field which is algebraically closed of characteristic 0. All algebras, tensor products, linear maps, and algebraic groups in the paper are taken over $k$. Throughout, $H$ will denote a Hopf algebra over $k$, but we do not impose any of the hypotheses such as $(H)$ at first. Our reference for basic material and notation about Hopf algebras is [23]. In particular, we denote all counits, comultiplications, and antipodes by the symbols $\epsilon$, $\Delta$, and $S$, respectively, sometimes decorated by subscripts indicating the name of the Hopf algebra under consideration. We use the notations $U(g)$, $k\Gamma$, and $\mathcal{O}(G)$ for enveloping algebras, group algebras, and coordinate rings of algebraic groups, respectively, taken with their standard Hopf algebra structures.

1. Constructions of Hopf algebras satisfying $(H_2)$

In this section, we construct three families of Hopf algebras which, together with two group algebras and two enveloping algebras, cover the Hopf algebras classified in Theorem 0.1. All of them are generated (as algebras) by grouplike and skew primitive elements, and only one family requires more than two such generators (counting a grouplike element and its inverse as one generator).

Recall that a grouplike element in a Hopf algebra $H$ is an element $g$ such that $\epsilon(g) = 1$ and $\Delta(g) = g \otimes g$. It is not necessary to specify $S(g)$, since that is forced by the antipode axiom: the equations $\epsilon(g)1 = S(g)g = gS(g)$ require that $S(g) = g^{-1}$. The set of grouplike elements in $H$ forms a subgroup of the group of units of $H$, denoted $G(H)$. A skew primitive element in $H$ is an element $p$ such that $\Delta(p) = g \otimes p + p \otimes h$ for some grouplike elements $g$ and $h$. Here the value of $\epsilon(p)$ is already forced by the counit axiom: since $p = 1p + \epsilon(p)h$ and $h \neq 0$, we must have $\epsilon(p) = 0$. The antipode axiom yields $0 = g^{-1}p + S(p)h$, which requires $S(p) = -g^{-1}ph^{-1}$. If desired, we can change to a situation in which either $g = 1$ or $h = 1$, since $\Delta(g^{-1}p) = 1 \otimes g^{-1}p + g^{-1}p \otimes g^{-1}h$ and $\Delta(ph^{-1}) = gh^{-1} \otimes ph^{-1} + ph^{-1} \otimes 1$. The term primitive element is used when both $g = h = 1$, that is, $\Delta(p) = 1 \otimes p + p \otimes 1$.

When defining a Hopf algebra structure on an algebra $A$ given by generators and relations, it suffices to check the Hopf algebra axioms on a set of algebra generators for $A$. This is obvious for the counit and coassociativity axioms, which require that various algebra homomorphisms coincide. As for the antipode axiom, it suffices to check it on monomials in the algebra generators, and that follows from the case of a single generator by induction on the length of a monomial.

Construction 1.1. Let $n \in \mathbb{Z}$ and $q \in k^\times$, and set $A = k \langle x^{\pm 1}, y \mid xy = qyx \rangle$. 


(a) There is a unique Hopf algebra structure on $A$ under which $x$ is grouplike and $y$ is skew primitive, with $\Delta(y) = y \otimes 1 + x^n \otimes y$. This Hopf algebra satisfies $(H)$. Let us denote it $A(n,q)$.

(b) Let $m \in \mathbb{Z}$ and $r \in k^\times$. Then $A(m,r) \cong A(n,q)$ if and only if either $(m,r) = (n,q)$ or $(m,r) = (-n,q^{-1})$.

**Remark.** The Hopf algebra $A(1,q^2)$ is well known – it is the quantized enveloping algebra of the positive Borel subalgebra of $\mathfrak{sl}_2(k)$ (e.g., [10, I.3.1, I.3.4]). Moreover, $A(2,q)$ appears as the quantized Borel subalgebra of the variant $U_{q^{-1}}(\mathfrak{sl}_2(k))$ [17, §3.1.2, p. 57]. All of the Hopf algebras $A(n,q)$, for $q$ not a root of unity, appear among the Hopf algebras with generic infinitesimal braiding constructed by Andruskiewitsch and Schneider [3, Section 4]. Namely, $A(n,q) \cong U(D)$ for suitable generic data $D$ of Cartan type $A_1$.

Because of (b) above, we may always assume that $n \geq 0$ when discussing $A(n,q)$.

**Proof.** (a) It is clear that there exist unique $k$-algebra homomorphisms $\epsilon : A \to k$ and $\Delta : A \to A \otimes A$ and a unique $k$-algebra anti-automorphism $S$ on $A$ such that

$$\begin{align*}
\epsilon(x) &= 1 & \Delta(x) &= x \otimes x & S(x) &= x^{-1} \\
\epsilon(y) &= 0 & \Delta(y) &= y \otimes 1 + x^n \otimes y & S(y) &= -x^{-n}y.
\end{align*}$$

The Hopf algebra axioms are easily checked on $x$ and $y$; we leave that to the reader. That $A$ satisfies $(H)$ is clear. Condition (ii) follows from Observation 0.4, because $A/(y) \cong k[x^\pm 1]$. (b) Label the canonical generators of $A(m,r)$ as $u$, $u^{-1}$, $v$. If $(m,r) = (-n,q^{-1})$, there is a Hopf algebra isomorphism $A(m,r) \to A(n,q)$ sending $x \mapsto u^{-1}$ and $y \mapsto v$.

Now assume we are given a Hopf algebra isomorphism $\phi : A(m,r) \to A(n,q)$. Since $u$ generates the group of grouplikes in $A(m,r)$, its image $\phi(u)$ must generate the group of grouplikes in $A(n,q)$. As that group consists of the powers of $x$, the only possibilities are $\phi(u) = x$ and $\phi(u) = x^{-1}$. By the first paragraph, we may assume that $\phi(u) = x$. Let $f = \phi(v)$. Since $A(m,r)$ is generated by $u^\pm 1$ and $v$ with relation $uv = rvu$, the algebra $A(n,q)$ is generated by $x^\pm 1$ and $f$ with relation $xf = rf$. This implies that $y = \sum_{i \geq 0} c_i(x)f^i$ where $c_i(x) \in k[x^{\pm 1}]$. By counting the $y$-degree of $\sum_{i \geq 0} c_i(x)f^i$, one sees that $f$ has $y$-degree 1 and $c_i(x) = 0$ for all $i > 1$. Writing $f = a(x) + b(x)y$ for $a(x), b(x) \in k[x^\pm 1]$ with $b(x) \neq 0$, we have

$$ra(x)x + rb(x)yx = rf = x = af(x) + b(x)yx.$$

Since $b(x) \neq 0$, this equation forces $r = q$. We also have

$$y = c_0(x) + c_1(x)(a(x) + b(x)y).$$

This implies that $b(x) = \alpha x^s$ and $c_1(x) = \alpha^{-1}x^{-s}$ for some $\alpha \in k^\times$, $s \in \mathbb{Z}$. Applying $\phi$ to $\Delta(v) = v \otimes 1 + u^m \otimes v$, we obtain $\Delta(f) = f \otimes 1 + x^m \otimes f$, or $a(x \otimes x) + \alpha(x \otimes x)^s(y \otimes 1 + x^n \otimes y) = (a(x) + \alpha x^s y) \otimes 1 + x^m \otimes (a(x) + \alpha x^s y)$. This equation forces $n = m$ and therefore $(n,q) = (m,r)$.

Given an automorphism $\sigma$ of a $k$-algebra $A$, we write $A[x^\pm 1; \sigma]$ for the corresponding skew Laurent polynomial ring, with multiplication following the rule $xa = \sigma(a)x$ for $a \in A$.

**Construction 1.2.** Let $n, p_0, p_1, \ldots, p_s$ be positive integers and $q \in k^\times$ with the following properties:
To this end, use multi-index notation \( x^\alpha \) space over \( k \), \( R/I \) is an isomorphism. It suffices to exhibit elements of \( \phi \) and observe that the elements \( \mu_i \) of \( \sigma \) for the corresponding cosets in \( A \). The Hopf algebra \( B \) satisfies \( (H_2) \). We shall denote it \( B(n, p_0, \ldots, p_s, q) \).

**Remark.** The simplest case of Construction 1.2 is when \( n = p_0 = 1, p_1 = 2 \) and \( p_2 = 3 \). The resulting Hopf algebra \( B(1, 1, 2, 3, q) \) is new, as far as we are aware.

**Proof.** Note that \( m_i p_i = m = m_j p_j \) for all \( i, j > 0 \). We claim that \( A \) can be presented as the commutative \( k \)-algebra with generators \( y_1, \ldots, y_s \) and relations \( y_i^p = y_i^q \) for \( 1 \leq i < j \leq s \). Consequently, \( B \) can be presented as the \( k \)-algebra with generators \( x, x^{-1}, y_1, \ldots, y_s \) and the following relations:

\[
\begin{align*}
xx^{-1} &= x^{-1}x = 1 \\
yi &= q^{m_i}yi \\
yj &= yj (1 \leq i < j \leq s) \\
yi^p &= yj^p (1 \leq i < j \leq s)
\end{align*}
\]

Let \( R = k[x_1, \ldots, x_s] \) be a polynomial ring in \( s \) indeterminates, and \( I \) the ideal of \( R \) generated by \( x_i^p - x_j^p \) for \( 1 \leq i < j \leq s \). There is a surjective \( k \)-algebra homomorphism \( \phi : R/I \rightarrow A \) sending each \( x_i + I \rightarrow y_i \), and we need to show that \( \phi \) is an isomorphism. It suffices to exhibit elements of \( R/I \) which span \( k \) (as a vector space over \( k \)) and which are mapped by \( \phi \) to linearly independent elements of \( A \). To this end, use multi-index notation \( x^d \) for monomials in \( R \), and write \( \overline{x^d} = x^d + I \) for the corresponding cosets in \( R/I \). Set

\[
D = \{ d = (d_1, \ldots, d_s) \in \mathbb{Z}_{\geq 0}^s \mid d_i < p_i \text{ for } i = 2, \ldots, s \},
\]

and observe that the elements \( x^d \) for \( d \in D \) span \( R/I \). The map \( \phi \) sends

\[
x^d = x_1^{d_1}x_2^{d_2} \cdots x_s^{d_s} \longmapsto y_1^{d_1}y_2^{d_2} \cdots y_s^{d_s} = y^{\mu \cdot d} \in k[y],
\]

where \( \mu = (m_1, \ldots, m_s) \) and \( \mu \cdot d \) denotes the inner product \( m_1 d_1 + \cdots + m_s d_s \). Thus, it suffices to show that the map \( d \mapsto \mu \cdot d \) from \( D \rightarrow \mathbb{Z}_{\geq 0}^s \) is injective.

If \( d, d' \in D \) and \( \mu \cdot d = \mu \cdot d' \), then

\[
m_1(d_1 - d'_1) + \cdots + m_s(d_s - d'_s) = 0.
\]

Since \( p_i \mid m_j \) for \( i \neq j \) and \( p_i, m_i \) are relatively prime, \( p_i \mid d_i - d'_i \) for all \( i \). For \( i \geq 2 \), it follows that \( d_i = d'_i \), given that \( 0 \leq d_i, d'_i < p_i \). This leaves \( m_1(d_1 - d'_1) = 0 \), whence \( d_1 = d'_1 \) and consequently \( d = d' \), as desired. The claim is now established.

It is clear that there is a \( k \)-algebra homomorphism \( \epsilon : B \rightarrow k \) such that \( \epsilon(x) = 1 \) and \( \epsilon(y_i) = 0 \) for all \( i \).

To construct a comultiplication map, we show that \( x \otimes x \), \( x^{-1} \otimes x^{-1} \), and the elements \( \delta_i = y_i \otimes 1 + x^{m_i} \otimes y_i \) in \( B \otimes B \) satisfy the relations (E1.2.1). First, note...
that \((x \otimes x) \delta_i = q^{m_i} \delta_i (x \otimes x)\) for all \(i\). Next, since \(m \mid m_i m_j\) and hence \(\ell \mid m_i m_j n\) for \(i \neq j\), we have
\[
x^{m_i, n} y_j = q^{m_i, m_j, n} y_j x^{m_i, n} = y_j x^{m_i, n}
\]
and similarly \(x^{m_i, n} y_i = y_i x^{m_i, n}\) for \(1 \leq i < j \leq s\). It follows that \(\delta_1, \ldots, \delta_s\) commute with each other. Finally,

\[
\delta^p_i = \sum_{r=0}^{p_i} \binom{p_i}{r} y_i^{p_i - r} (x^{m_i, n})^r \otimes y_i^r
\]

making use of the fact that the \(q^{m_i, n}\)-binomial coefficients \(\binom{p_i}{r} q^{m_i, n}\) vanish for \(0 < r < p_i\), since \(q^{m_i, n}\) is a primitive \(p_i\)-th root of unity. To see the latter, observe that on one hand, \(q^{m_i, n p_i} = 1\) because \(\ell\) divides \(m n = m_i n p_i\). On the other hand, if \(q^{m_i, n t} = 1\) for some positive integer \(t\), then \(\ell \mid m_i^2 n t\) and \(m n = p_0 \ell\) divides \(p_0 m_i^2 n t\), whence \(p_1 p_2 \cdots p_s = m\) divides \(p_0 n t \prod_{j \neq i} p_j^2\). Since \(p_0, \ldots, p_s\) are pairwise relatively prime, it follows that \(p_i \mid t\), as desired. Equations (E1.2.2) imply that \(\delta_i^p = \delta_i^j\) for all \(i, j\).

We have now shown that \((x \otimes x)^{\pm 1}, \delta_1, \ldots, \delta_s\) satisfy (E1.2.1). Therefore, there is a \(k\)-algebra homomorphism \(\Delta : B \to B \otimes B\) such that \(\Delta(x) = x \otimes x\) and \(\Delta(y_i) = \delta_i\) for all \(i\). Coassociativity and the counit axiom are easily seen to hold for the generators \(x, x^{-1}, y_1, \ldots, y_s\).

Finally, we construct a \(k\)-algebra anti-automorphism \(S\) on \(B\) such that \(S(x) = x^{-1}\) and \(S(y_i) = w_i := -x^{-m_i} y_i\) for all \(i\). We need to show that \(x^{-1}, x, w_1, \ldots, w_s\) satisfy the reverse of (E1.2.1), i.e., the corresponding relations with products reversed. The first relation is trivially satisfied, and it is clear that \(w_i x^{-1} = q^{m_i} x^{-1} w_i\) for all \(i\). Next,
\[
w_i w_j = x^{-m_i, n} y_i x^{-m_j, n} y_j = q^{m_i, m_j, n} x^{-(m_i + m_j)} y_i y_j = x^{-(m_i + m_j)} y_i y_j
\]
for \(i \neq j\), because \(\ell \mid m_i m_j n\) in that case. Hence, \(w_i w_j = w_j w_i\) for all \(i, j\). For each \(i\), we have
\[
w_i^p = (-1)^{p_i} q^{a_i} x^{-m_i, n p_i} y_i^p,
\]
where \(a_i = m_i^2 n p_i (p_i - 1)/2\).

If \(p_i\) is odd, then \(\ell \mid a_i\), whence \((-1)^{p_i} q^{a_i} = -1\). If \(p_i\) is even, then \((-1)^{p_i} q^{a_i} = q^{-m_i^2 n (p_i/2)} = -1\) because \(q^{m_i, n}\) is a primitive \(p_i\)-th root of unity (as shown above). Thus, \(w_i^p = -x^{-m_i} y_i^p\) for all \(i, j\), and so \(w_i^p = w_j^p\) for all \(i, j\). The relations needed for the existence of \(S\) are now established.

The antipode axiom is easily checked on the generators of \(B\). Therefore the algebra \(B\), equipped with the given \(\epsilon, \Delta, S\), is a Hopf algebra. It is clear that \(B\) satisfies (H), and (z) follows from Observation 0.4 because \(H/(y_1, \ldots, y_s) \cong k[x^{\pm 1}]\). \(\square\)

**Lemma 1.3.** Let \(n, p_0, \ldots, p_s \in \mathbb{Z}_{>0}, q \in k^x\) and \(n', p_0', \ldots, p'_t \in \mathbb{Z}_{>0}, r \in k^x\) satisfy conditions (a)-(c) of Construction 1.2. Then

\[
B(n, p_0, \ldots, p_s, q) \cong B(n', p_0', \ldots, p'_t, r) \iff (n, p_0, \ldots, p_s, q) = (n', p_0', \ldots, p'_t, r).
\]

**Proof.** Set \(B = B(n, p_0, \ldots, p_s, q)\) and \(B' = B(n', p_0', \ldots, p'_t, r)\), and assume there is a Hopf algebra isomorphism \(\phi : B' \to B\). Label the generators of \(B\) as \(x^{\pm 1}, y_1, \ldots, y_s\) as in Construction 1.2, and correspondingly use \(u^{\pm 1}, v_1, \ldots, v_t\) for the generators of
B'. Since the groups of grouplike elements in B and B' are infinite cyclic, generated by x and u respectively, we must have φ(u) = x or φ(u) = x⁻¹.

Since ℓ = (n/p₀)m_ipᵢ with (n/p₀)pᵢ > 1, we have qᵢ⁻¹ ≠ 1 for i = 1, ..., s. But [x, yᵢ] = (qᵢ⁻¹ − 1)yᵢx, so we obtain yᵢ ∈ [B, B]. Therefore [B, B] equals the ideal of B generated by y₁, ..., yₙ. We then compute that the algebra B₀ of coinvariants of B for the right coaction ρ : B → B ⊗ (B/[B, B]) induced from Δ is precisely k[y₁, ..., yₙ]. Likewise, the corresponding algebra B₀' of coinvariants in B' must equal k[v₁, ..., vₙ]. On the other hand, B₀ and B₀' are defined purely in terms of the Hopf algebra structures of B and B', so φ must map B₀' isomorphically onto B₀. This isomorphism extends to the integral closures of these domains, i.e., to an isomorphism ˆφ : k[v] → k[y]. Observe that since φ maps B₀' ∩ ker ε to B₀ ∩ ker ε, we must have ˆφ(v) = λy for some λ ∈ k⁎. In particular, ˆφ is a graded isomorphism.

Now m₁, ..., mₙ are the minimal generators of {j ∈ ℤ≥₀ | yᵢj ∈ B₀}, and similarly for m′₁, ..., m′ₙ. Hence, these two sets of integers must coincide, and since they are listed in descending order, we conclude that s = t and mᵢ′ = mᵢ for i = 1, ..., s. Recall that m = p₁p₂ · · · pₙ is the least common multiple of the mᵢ, and similarly for m′ = p′₁p′₂ · · · p′ₙ. Consequently, m = m′, and so pᵢ′ = m/mᵢ = pᵢ for i = 1, ..., s.

Now φ sends each vᵢ → λᵢyᵢ where λᵢ = λmᵢ. In particular, since (φ ⊗ φ)Δ(vᵢ) = Δ(φ(vᵢ)), we obtain

\[ λ₁y₁ ⊗ 1 + x^{±m₁′}ν ⊗ λ₁y₁ = λ₁y₁ ⊗ 1 + λ₁x^{m₁n} ⊗ y₁, \]

whence φ(u) = x and n' = n. The relations uvᵢ = rᵢmᵢvᵢu now imply that xyᵢ = rᵢmᵢyᵢx, whence rᵢmᵢ = rᵢmᵢ = qᵢ⁻¹ for all i. In particular, since rᵢmᵢ is a primitive ℓᵢ/mᵢ-th root of unity and qᵢ⁻¹ is a primitive ℓ/m₁-th root of unity, we obtain (n'p₀)p₁′ = ℓ′/m₁′ = ℓ/m₁ = (n/p₀)p₁, and consequently p₀′ = p₀. Finally, since gcd(m₁, ..., mₙ) = 1, there are integers a₁, ..., aₙ such that a₁m₁ + · · · + aₙmₙ = 1, whence

\[ q = (q₁)m₁a₁(q₂)m₂a₂ ... (qₙ)mₙaₙ = (r₁)m₁(r₂)m₂ ... (rₙ)mₙ = r. \]

Therefore (n, p₀, ..., pₙ, q) = (n', p₀', ..., pₙ', r). □

Given a derivation δ on a k-algebra A, we write A[x; δ] for the corresponding skew polynomial ring, with multiplication following the rule xa = ax + δ(a) for a ∈ A.

**Construction 1.4.** Let n be a positive integer, and set C = k[y⁺¹][x; (y^n − y)^d/dy].

(a) There is a unique Hopf algebra structure on C such that y is grouplike and x is skew primitive, with Δ(x) = x ⊗ y⁻¹ + 1 ⊗ x. The Hopf algebra C satisfies (H²), and we shall denote it C(n).

(b) For m, n ∈ ℤ≥₀, the Hopf algebras C(m) and C(n) are isomorphic if and only if m = n. In fact, they are isomorphic as rings only if m = n.

**Remark.** Since C(1) ≅ A(0, 1), we will usually assume that n ≥ 2 when discussing C(n). These Hopf algebras appear to be new, as far as we are aware.

**Proof.** (a) There is clearly a unique k-algebra homomorphism ε : C → k such that ε(x) = 0 and ε(y) = 1. Since

\[ [x ⊗ y_{n−1} + 1 ⊗ x, y ⊗ y] = (y^n − y) ⊗ y + y ⊗ (y^n − y) = (y ⊗ y)^n − y ⊗ y, \]

there is a unique k-algebra homomorphism Δ : C → C ⊗ C such that Δ(y) = y ⊗ y and Δ(x) = x ⊗ y⁻¹ + 1 ⊗ x. The relation xy − yx = y^n − y implies xy⁻¹ − y⁻¹x =
\[ y^{-1} - y^{n-2}, \text{ whence } \\
\quad y^{-1}(-xy^{1-n}) - (-xy^{1-n})y^{-1} = (y^{-1} - y^{n-2})y^{1-n} = y^{-n} - y^{-1}. \]

Consequently, there is a unique \( k \)-algebra anti-automorphism \( S \) of \( C \) such that \( S(x) = -xy^{1-n} \) and \( S(y) = y^{-1}. \)

It is routine to check the Hopf algebra axioms for the generators \( x, y, y^{-1} \) of the algebra \( C \), and it is clear that \( C \) satisfies \( (H) \). For \( (2) \), observe that \( C/C(y - 1) \cong k[x] \) and invoke Observation 0.4.

(b) First note that \( C(m) \) is commutative if and only if \( m = 1 \), and similarly for \( C(n) \). Hence, we need only consider the cases when \( m, n \geq 2 \). Under that assumption, we claim that the quotients of \( C(m) \) and \( C(n) \) modulo their commutator ideals have Goldie ranks \( m - 1 \) and \( n - 1 \), respectively. Statement (b) then follows.

Fix an integer \( n \geq 2 \) and \( H = C(n) \), and note that \( [H, H] = (y^n - y)H \). Set \( \partial = (y^n - y)\frac{d}{dy} \), and let \( \xi_1, \ldots, \xi_{n-1} \) be the distinct \((n-1)\text{st}\) roots of unity in \( k \).

For \( i = 1, \ldots, n-1 \), the ideal \((y - \xi_i)k[y^\pm 1]\) is a \( \partial \)-ideal of \( k[y^\pm 1] \), whence \((y - \xi_i)H\) is an ideal of \( H \), and \( H/(y - \xi_i)H \cong k[x] \). Since the \( y - \xi_i \) generate pairwise relatively prime ideals of \( k[y^\pm 1] \) whose intersection equals the ideal generated by \( y^n - y \), the corresponding statements hold for ideals of \( H \). Consequently, \( H/(y^n - y)H \cong \prod_{i=1}^{n-1} H/(y - \xi_i)H \), a direct product of \( n-1 \) copies of \( k[x] \). Therefore \( H/(y^n - y)H \) has Goldie rank \( n - 1 \), as claimed.

\[ \square \]

\textbf{Remark 1.5.} K.A. Brown has pointed out that some of the Hopf algebras constructed here are lifts of others in the sense of Andruskiewitsch and Schneider (e.g., [3]). First, the Hopf algebra \( C(n) \) of Construction 1.4 is a lift of the Hopf algebra \( A(n - 1, 1) \) of Construction 1.1 — the associated graded ring of \( C(n) \) with respect to its coradical filtration is isomorphic (as a graded Hopf algebra) to \( A(n - 1, 1) \). This raises the question of lifting the \( A(-, -) \) in general, which is easily done. For instance, given \( n \in \mathbb{Z} \) and \( q \in k^\times \), let \( C(n, q) \) be the \( k \)-algebra given by generators \( y^\pm 1 \) and \( x \) and the relation \( xy = qyx + y^n - y \). It is easy to check that there is a unique Hopf algebra structure on \( C(n, q) \) such that \( y \) is grouplike and \( x \) is skew primitive, with \( \Delta(x) = x \otimes y^{n-1} + 1 \otimes x \). The associated graded ring of this Hopf algebra, with respect to its coradical filtration, is isomorphic to \( A(n - 1, q^{-1}) \). However, \( C(n, q) \) is not new — it turns out that \( C(n, q) \cong A(n - 1, q^{-1}) \) when \( q \neq 1 \).

\textbf{Proposition 1.6.} The Hopf algebras listed in Theorem 0.1(I)–(V) are pairwise non-isomorphic, aside from the cases \( A(0, q) \cong A(0, q^{-1}) \) for \( q \in k^\times \).

\textbf{Proof.} The commutative Hopf algebras in this list are the group algebra \( k[\mathbb{Z}] = k[x^\pm 1, y^\pm 1] \), the enveloping algebra \( U(\mathfrak{g}) = k[x, y] \), and the \( A(n, 1) = k[x^\pm 1, y] \) for \( n \geq 0 \). These three types are pairwise non-isomorphic because their groups of grouplike elements are different — they are free abelian of ranks 2, 0, and 1, respectively. Further, \( A(m, 1) \not\cong A(n, 1) \) for distinct \( m, n \in \mathbb{Z}_{\geq 0} \) by Construction 1.1(b).

The noncommutative, cocommutative Hopf algebras in the list are the group algebra \( k[\mathbb{Z} \times \mathbb{Z}] \), the enveloping algebra \( U(\mathfrak{g}) \) with \( \mathfrak{g} \) non-abelian, and the \( A(0, q) \) with \( q \neq 1 \). The three types are again pairwise non-isomorphic because their groups of grouplike elements are different — namely \( \mathbb{Z} \times \mathbb{Z} \), the trivial group, and the infinite cyclic group. Moreover, \( A(0, q) \cong A(0, r) \) (for \( q, r \in k^\times \)) if and only if \( q = r^{\pm 1} \) by Construction 1.1(b).
Each \( B(-) = B(n, p_0, \ldots, p_s, q) = k[y_1, \ldots, y_s][x^{\pm 1}; \sigma] \) as algebras. Since \( s \geq 2 \), the subalgebra \( k[y_1, \ldots, y_s] \) has infinite global dimension, from which it follows that \( B(-) \) has infinite global dimension (e.g., [20, Theorem 7.5.3(ii)]). However, all the other Hopf algebras in the list have global dimension 2. Therefore no \( B(-) \) is isomorphic to any of the others. Moreover, the \( B(-) \) themselves are pairwise non-isomorphic by Lemma 1.3.

The abelianization \( A(n, q)/[A(n, q), A(n, q)] \) is isomorphic to either \( k[x^{\pm 1}] \) (when \( q \neq 1 \)) or \( k[x^{\pm 1}, y] \) (when \( q = 1 \)). As shown in the proof of Construction 1.4, the abelianization \( C(m)/[C(m), C(m)] \), when \( m \geq 2 \), is isomorphic to a direct product of \( m - 1 \) copies of \( k[x] \). Therefore \( C(m) \), when \( m \geq 2 \), is not isomorphic to any \( A(n, q) \). Taking account of Constructions 1.1(b) and 1.4(b), the proposition is proved. \( \Box \)

Remark 1.7. In [8, Question K], Brown asked whether a noetherian Hopf algebra which is a domain over a field of characteristic zero must have finite global dimension. A similar question was asked by Wu and the second-named author in [26, Question 0.4]. The Hopf algebras \( B(n, p_0, \ldots, p_s, q) \) provide a negative answer to this question, since they have infinite global dimension as shown in the proof of Proposition 1.6. The Hopf algebras \( B(n, p_0, \ldots, p_s, q) \) also provide a negative answer to [8, Question J].

2. Commutative Hopf domains of GK-dimensions 1 and 2

While our main interest is in Hopf algebras of GK-dimension 2, these may have subalgebras and/or quotients of GK-dimension 1, and it is helpful to have a good picture of that situation. Thus, we begin by classifying commutative Hopf algebras which are domains of GK-dimension 1 (= transcendence degree 1) over \( k \).

Recall that the antipode of any commutative (or cocommutative) Hopf algebra \( H \) satisfies \( S^2 = 1 \) [23, Corollary 1.5.12]. From this, we can see that \( H \) is the directed union of its affine Hopf subalgebras, as follows. Any finite subset \( X \subseteq H \) must be contained in a finite dimensional subcoalgebra, say \( C \), of \( H \) [23, Theorem 5.1.1]. Since \( S \) is an anti-coalgebra morphism, \( S(C) \) is a subcoalgebra of \( H \), and hence so is \( D := C+S(C) \). Moreover, \( S(D) \subseteq D \). The subalgebra \( A \) of \( H \) generated by \( D \) is thus a Hopf subalgebra containing \( X \). Choose a basis \( Y \) for \( C \). Then \( A \) is generated by \( Y \cup S(Y) \), proving that \( A \) is affine.

Proposition 2.1. Assume that the Hopf algebra \( H \) is a commutative domain with \( \text{GKdim} \, H = 1 \). Then \( H \) is isomorphic to one of the following:

(a) An enveloping algebra \( U(g) \), where \( \text{dim}_k g = 1 \).
(b) A group algebra \( k\Gamma \), where \( \Gamma \) is infinite cyclic.
(c) A group algebra \( k\Gamma \), where \( \Gamma \) is a non-cyclic torsionfree abelian group of rank 1, i.e., a non-cyclic subgroup of \( \mathbb{Q} \).

Proof. If \( H \) is affine, then \( H \cong \mathcal{O}(G) \) for some connected 1-dimensional algebraic group \( G \) over \( k \). There are only two possibilities: either \( G \cong k^* \) or \( G \cong k^+ \) [6, Theorem III.10.9]. These correspond to cases (b) and (a), respectively.

Now suppose that \( H \) is not affine, and view \( H \) as the directed union of its affine Hopf subalgebras.

Since \( H \) is a domain, its only finite dimensional subalgebra is \( k1 \). Hence, any nontrivial affine Hopf subalgebra of \( H \) has GK-dimension 1, and so must have one of the forms (a) or (b). Among these Hopf algebras, there are no proper embeddings
of the type \( U(g) \rightarrow U(g') \), because the only primitive elements of \( U(g') \) are the elements of \( g' \), and there are no embeddings of the type \( k\Gamma \rightarrow U(g) \), because the only units of \( U(g) \) are scalars. Thus, \( H \) must be a directed union \( \bigcup_{i \in I} H_i \) of Hopf subalgebras \( H_i \cong k\Gamma_i \), where the \( \Gamma_i \) are infinite cyclic groups.

For each index \( i \), the group \( G(H_i) \) is isomorphic to \( \Gamma_i \), and is a \( k \)-basis for \( H_i \). The group \( \Gamma = G(H) \) is the directed union of the \( G(H_i) \), so it is torsionfree abelian of rank 1, and it is a \( k \)-basis for \( H \). Therefore \( H = k\Gamma \). We conclude by noting that \( \Gamma \) is not cyclic, since \( H \) is not affine.

Turning to the case of GK-dimension 2, we need to describe the 2-dimensional connected algebraic groups over \( k \). While the result is undoubtedly known, we have not located it in the literature, and so we sketch a proof.

**Lemma 2.2.** Let \( G \) be a 2-dimensional connected algebraic group over \( k \). Then \( G \) is isomorphic (as an algebraic group) to either

(a) One of the abelian groups \((k^\times)^2, (k^+)^2, k^+ \times k^\times\), or

(b) One of the semidirect products \( k^+ \rtimes k^\times \) where \( k^\times \) acts on \( k^+ \) by \( b \cdot a = b^n a \), for some positive integer \( n \).

**Remark.** Note that the three groups listed in (a) are pairwise non-isomorphic. The group described in (b) can be written as the set \( k^+ \times k^\times \) equipped with the product \((a, b) \cdot (a', b') = (a + b^n a', b b')\). The center consists of the elements \((0, b)\) with \( b^n = 1 \), so the center is cyclic of order \( n \). Thus, none of these groups is abelian, and no two (for different \( n \)) are isomorphic. It is also possible to form a semidirect product with the multiplication rule \((a, b) \cdot (a', b') = (a + b^{-n} a, b b')\), but this group is isomorphic to the previous one, via \((a, b) \mapsto (a, b^{-1})\).

**Proof of Lemma 2.2.** According to [6, Corollary IV.11.6], 2-dimensional groups are solvable. Then, by [6, Theorem III.10.6], \( G = T \times G_u \) where \( T \) is a maximal torus and \( G_u \) is the unipotent radical of \( G \).

(i) If \( \dim T = 2 \), then \( G = T \) is a torus, isomorphic to \((k^\times)^2\).

(ii) If \( \dim T = 0 \), then \( G = G_u \) is unipotent. Another part of [6, Theorem III.10.6] and its proof shows that \( G \) has a closed connected normal subgroup \( N \), contained in \( Z(G) \), such that \( N \) and \( G/N \) are 1-dimensional. Being unipotent, \( N \cong G/N \cong k^+ \). Choose \( x \in G \setminus N \) and let \( H \) be the subgroup of \( G \) generated by \( N \cup \{x\} \). Then \( H \) is abelian and \( H/N \) is infinite. The latter implies \( H/N \) is dense in \( G/N \), so \( H \) is dense in \( G \). Thus, \( G \) is abelian. Unipotence then implies \( G \cong (k^+)^2 \).

(iii) If \( \dim T = 1 \), then also \( \dim G_u = 1 \), so \( T \cong k^\times \) and \( G_u \cong k^+ \) (since \( G_u \) is connected [6, Theorem III.10.6]). Hence, \( G \cong k^+ \rtimes k^\times \), for some homomorphism \( \phi \) from \( k^\times \) to the group \( \text{Aut}(k^+) \) of algebraic group automorphisms of \( k^+ \). But \( \text{Aut}(k^+) \cong k^\times \), where \( b \in k^\times \) corresponds to the automorphism \( a \mapsto ba \). The homomorphisms \( \phi \) thus amount to the (rational) characters of \( k^\times \), namely the maps \( k^\times \rightarrow k^\times \) given by \( b \mapsto b^n \) for \( n \in \mathbb{Z} \). The case \( n = 0 \) corresponds to the trivial action of \( k^\times \) on \( k^+ \), in which case \( G \cong k^+ \times k^\times \). The cases \( n > 0 \) give the groups listed in (b), and the cases \( n < 0 \) give these same groups again.

**Proposition 2.3.** Assume that the Hopf algebra \( H \) is a commutative domain with \( \text{GKdim} \ H = 2 \). Then \( H \) is noetherian if and only if \( H \) is affine, if and only if \( H \) is isomorphic to one of the following:

(a) An enveloping algebra \( U(g) \), where \( g \) is 2-dimensional abelian.
(b) A group algebra $k\Gamma$, where $\Gamma$ is free abelian of rank 2.
(c) A Hopf algebra $A(n, 1)$, for some nonnegative integer $n$.

Proof. If $H$ is affine, then because it is commutative, it is noetherian. The converse is Molnar’s theorem [22].

The Hopf algebras listed in (a)–(c) are clearly affine. Conversely, if $H$ is affine, then $H \cong \mathcal{O}(G)$ for some 2-dimensional connected algebraic group $G$. The structure of $H$ follows from the descriptions of $G$ given in Lemma 2.2. Namely, the cases $G \cong (k^\times)^2$, $(k^+)^2$, or $k^+ \times k^\times$ give $H \cong k\Gamma$, $U\langle g \rangle$, or $A(0, 1)$, respectively, while case (b) of Lemma 2.2 corresponds to $H \cong A(n, 1)$ with $n > 0$. □

3. Ext$^1_H(k, k)$ and Condition (2)

We discuss the relevance of Ext$^1_H(k, k)$ (actually, just its dimension) to the structure of general Hopf algebras $H$ over $k$. To simplify notation, we have written $k$ rather than $H$ for the trivial left $H$-module, and we define

$$e(H) := \dim_k \text{Ext}_H^1(k, k).$$

It follows from Lemma 3.1(a) below that $e(H)$ also equals $\dim_k \text{Ext}_H^1(k_H, k_H)$.

If $H$ is an affine commutative Hopf algebra $\mathcal{O}(G)$, then $e(H)$ equals the dimension of $G$.

For purposes of the present paper, the key result of this section is that $(H_2)$ together with affineness or noetherianness implies that $H$ has a Hopf quotient of the form $k[t^{\pm 1}]$ or $k[t]$ (Theorem 3.9).

Lemma 3.1. Let $m = \ker \epsilon$ where $\epsilon$ is the counit of $H$.

(a) $e(H) = \dim_k (m/m^2)^*$.
(b) If $H$ is either noetherian or affine, then $e(H) < \infty$.

Proof. (a) Since $mk = 0$, the restriction map $\text{Hom}_H(H, k) \to \text{Hom}_H(m, k)$ is zero. Inspecting the long exact sequence for $\text{Ext}_H^1(\cdot, k)$ relative to the short exact sequence $0 \to m \to H \to k \to 0$, we see that the connecting homomorphism $\text{Hom}_H(m, k) \to \text{Ext}_H^1(k, k)$ is an isomorphism. On the other hand, all maps in $\text{Hom}_H(m, k)$ factor through $m/m^2$, so that $\text{Hom}_H(m, k) \cong \text{Hom}_H(m/m^2, k)$. Thus, $\text{Ext}_H^1(k, k) \cong (m/m^2)^*$, and part (a) follows.

(b) If $H$ is noetherian, then $m/m^2$ is a finitely generated left module over $H$, and hence also over $H/m = k$, whence $\dim_k m/m^2 < \infty$, and so $e(H) < \infty$ by (a).

If $H$ is generated as a $k$-algebra by elements $h_1, \ldots, h_n$, write each $h_i = \alpha_i + m_i$ for some $\alpha_i \in k$ and $m_i \in m$, and note that $H$ is also generated as a $k$-algebra by $m_1, \ldots, m_n$. It follows that $m/m^2$ is spanned by the cosets $m_i + m^2$. Again, $\dim_k m/m^2 < \infty$ and $e(H) < \infty$. □

Further information will be obtained by working with the associated graded ring of $H$ relative to the descending filtration given by powers of $m$. Since $m$ is a Hopf ideal of $H$ (because $\epsilon$ is a Hopf algebra homomorphism), this associated graded ring will be a Hopf algebra, as follows. This result is presumably well known, but since we could not locate it in the literature, we sketch a proof.

Recall that a positively graded Hopf algebra is a Hopf algebra $B$ which is also a positively graded $k$-algebra of the form $B = \bigoplus_{i \geq 0} B_i$, such that $\Delta_{|i}, \epsilon_{|i}$, and $S_B$ are graded maps of degree zero. (We assume that $B \otimes B$ is positively graded by total degree, that is, $(B \otimes B)_i = \bigoplus_{i+m=i} B_i \otimes B_m$, for all $i \geq 0$.)
Lemma 3.2. Let $K$ be a Hopf ideal of $H$, and $B = \bigoplus_{i \geq 0} K^i/K^{i+1}$ the corresponding associated graded ring (where $K^0$ means $H$). Set $L = H \otimes K + K \otimes H$, and let $C = \bigoplus_{n \geq 0} L^n/L^{n+1}$ be the corresponding associated graded ring of $H \otimes H$.

(a) There is a graded algebra isomorphism $\theta : B \otimes B \to C$ which sends
\[
(a + K^{i+1}) \otimes (b + K^{j+1}) \mapsto (a \otimes b) + L^{i+j+1}
\]
for all $a \in K^i$ and $b \in K^j$, $i, j \geq 0$.

(b) The algebra $B$ becomes a positively graded Hopf algebra with
- $\Delta_B = \theta^{-1}\Delta$, where $\Delta : B \to C$ is the graded algebra homomorphism induced by $\Delta$;
- $\epsilon_B = \epsilon_B \pi$, where $\pi : B \to B_0$ is the canonical projection;
- $S_B$ is the graded algebra anti-endomorphism of $B$ induced by $S_H$.

Proof. (a) Observe that the equalities $\bigoplus_{i=0}^n K^i \otimes K^{n-i} = L^n$ induce linear isomorphisms $(B \otimes B)_n \to L^n/L^{n+1}$ for all $n \geq 0$. The direct sum of these isomorphisms gives a graded linear isomorphism $\theta : B \otimes B \to C$ which acts as described, and one checks that $\theta$ is an algebra homomorphism.

(b) Since $K$ is a Hopf ideal of $H$, the coproduct $\Delta_H$ maps $K$ into $L$, and hence induces a graded algebra homomorphism $\Delta : B \to C$. Composing this map with $\theta^{-1}$, we obtain a graded algebra homomorphism $\Delta_B : B \to B \otimes B$. Coassociativity of $\Delta_B$ follows easily from coassociativity of $\Delta_H$.

It is clear that $\epsilon_B$ is a graded algebra homomorphism from $B$ to $k$, where $k$ is concentrated in degree 0. The counit axiom for $B$ follows directly from that for $H$. Finally, since $S_H$ is an algebra anti-endomorphism of $H$ mapping $K \to K$, it induces a graded algebra anti-endomorphism $S_B$ of $B$. All that remains is to check the antipode axiom.

Lemma 3.3. Let $m = \ker \epsilon$, and write $H$ for the graded algebra $\bigoplus_{i=0}^\infty m^i/m^{i+1}$.

(a) With the operations described in Lemma 3.2, $H$ is a positively graded Hopf algebra.

(b) As a $k$-algebra, $H$ is connected graded and generated in degree 1.

(c) $H$ is also connected as a coalgebra.

(d) All homogeneous elements of degree 1 in $H$ are primitive elements.

(e) $H$ is cocommutative.

Proof. Let $B := H$ be the graded algebra defined in Lemma 3.2 with $K = m$.

(a) See Lemma 3.2.

(b) It is clear that the algebra $B$ is generated in degree 1, and it is connected graded because $B_0 = H/m = k$.

(d) Let $x \in B_1$. Since $B_0 = k$, Lemma 3.2 shows that we can write $\Delta_B(x) = 1 \otimes y + z \otimes 1$ for some $y, z \in B_1$. Applying the counit axiom together with the fact that $\epsilon_B(y) = \epsilon_B(z) = 0$, we find that $y = z = x$. This shows that $x$ is a primitive element.

(e) Since $B$ is generated by $B_1$, cocommutativity follows from (d).

(c) Since $B$ is cocommutative and $k$ is algebraically closed, $B$ is pointed [23, p. 76]. Thus, any simple subcoalgebra of $B$ has the form $kg$ for a grouplike element $g$. To prove that $B$ is a connected coalgebra, it remains to show that $g = 1$. Write
Since each $\Delta(x) = \sum_{i=0}^{n} \Delta_B(x_i) = \Delta_B(g) = g \otimes g = \sum_{i,m=0}^{n} x_i \otimes x_m,$

with $x_n \otimes x_n \neq 0$. Since the homogeneous components of $\Delta_B(g)$ have degree at most $n$, we can have $x_n \otimes x_n \neq 0$ only when $n = 0$. Hence, $g \in k1$, and so $g = 1$ because $\epsilon(g) = 1$. This proves that $B$ is indeed connected. $\square$

In the following result, the dimension of an infinite dimensional vector space $V$ is considered to be the symbol $\infty$, rather than an infinite cardinal, so that $\dim_k V^* = \dim_k V$.

**Proposition 3.4.** Let $m = \ker \epsilon$, and write $\text{gr} H$ for the positively graded Hopf algebra $\bigoplus_{i=0}^{\infty} m^i/m^{i+1}$.

(a) $\text{gr} H \cong U(\mathfrak{g})$ for some positively graded Lie algebra $\mathfrak{g}$ which is generated in degree 1.

(b) $e(H) = \dim_k (\text{gr} H)_1 = \dim_k \mathfrak{g}_1 \leq \dim_k \mathfrak{g} \leq \text{GKdim} H$.

**Proof.** (a) By Lemma 3.3, $B := \text{gr} H$ is a connected cocommutative Hopf algebra, generated in degree 1 as an algebra. The Cartier-Kostant Theorem (e.g., [23, Theorem 5.6.5]) thus implies that $B \cong U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of primitive elements of $B$.

We next claim that $\mathfrak{g}$ is a positively graded Lie algebra, $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathfrak{g} \cap B_i$ for $i \geq 1$. To see this, consider $x \in \mathfrak{g}$, and write $x = x_0 + \cdots + x_n$ with $x_i \in B_i$ for $i = 0, \ldots, n$. Then

$$\sum_{i=0}^{n} \Delta(x_i) = \Delta(x) = x \otimes 1 + 1 \otimes x = \sum_{i=0}^{n} (x_i \otimes 1 + 1 \otimes x_i).$$

Since each $\Delta(x_i) \in (B \otimes B)_1$, we see that $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$. Thus, $x_i \in \mathfrak{g}$ for all $i$. Moreover, $B_0 = k$ contains no nonzero primitive elements, so $x_0 = 0$. Thus, $x$ lies in $\sum_{i \geq 1} \mathfrak{g}_i$, establishing the claim.

If $\mathfrak{h}$ is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_1$, then $B_1 = \mathfrak{g}_1 \subseteq \mathfrak{h} \subseteq U(\mathfrak{h}) \subseteq U(\mathfrak{g}) = B$.

Since $B$ is generated by $B_1$, we have $U(\mathfrak{h}) = U(\mathfrak{g})$, whence $\mathfrak{h} = \mathfrak{g}$. Therefore $\mathfrak{g}$ is generated in degree 1.

(b) In view of Lemmas 3.1(a) and 3.3(d), we have

$$e(H) = \dim_k m/m^2 = \dim_k B_1 = \dim_k \mathfrak{g}_1 \leq \dim_k \mathfrak{g}.$$

To prove that $\dim_k \mathfrak{g} \leq \text{GKdim} H$, we may assume that $\text{GKdim} H = d < \infty$. Suppose on the contrary that $\dim_k \mathfrak{g} > d$. Then we pick a finite dimensional subspace $W \subseteq \mathfrak{g}_1 = m/m^2$ such that $\text{GKdim} U(L(W)) = \dim_k L(W) > d$, where $L(W)$ is the Lie subalgebra of $\mathfrak{g}$ generated by $W$. Lift $W$ to a finite dimensional vector subspace $V \subseteq m$ such that $V \cap m^2 = 0$ and $(V + m^2)/m^2 = W$. Then, for arbitrarily small $b > 0$, there is an $a > 0$ such that $\dim_k (k + V)^a \leq an^{d+b}$ for all
n > 0. Note that
\[
\dim_k(k + V)^n = \dim_k \sum_{i=0}^{n} V^i \geq \sum_{i=0}^{n} \dim_k(V^i + m^{i+1})/m^{i+1}
\]
\[
= \sum_{i=0}^{n} \dim_k W^i = \dim_k(k + W)^n \geq a_1 n \dim_k L(W),
\]
for some \(a_1 > 0\) and all \(n \geq 0\), since \(\dim_k L(W) = \text{GKdim} U(L(W))\). Thus we have
\[
a_n d + b \geq a_1 n \dim_k L(W)
\]
for all \(n > 0\). Then \(d + b \geq \dim_k L(W)\) and hence \(d \geq \dim_k L(W)\) since \(b\) can be arbitrarily small.

We now have \(\dim_k W \leq \dim_k L(W) \leq d\) for any finite dimensional subspace \(W \subseteq \mathfrak{g}_1\) such that \(\dim_k L(W) > d\). Since the result holds for all finite dimensional subspaces of \(\mathfrak{g}_1\) containing such a \(W\), we must have \(\dim_k \mathfrak{g}_1 \leq d\). Taking the case \(W = \mathfrak{g}_1\), we obtain \(\dim_k \mathfrak{g} \leq d\), contradicting our assumption. Therefore \(\dim_k \mathfrak{g} \leq \text{GKdim} H\).

Lemma 3.5. Let \(m = \ker \epsilon\) and \(J_m = \bigcap_{i \geq 0} m^i\), and write \(\text{gr} H\) for the graded Hopf algebra \(\bigoplus_{i=0}^{\infty} m^i/m^{i+1}\). If \(\text{gr} H\) is commutative, then \(H/J_m\) is commutative.

Proof. Proposition 3.4 shows that \(B := \text{gr} H \cong U(\mathfrak{g})\) for a positively graded Lie algebra \(\mathfrak{g}\) generated in degree 1. By hypothesis, \(U(\mathfrak{g})\) is commutative, whence \(\mathfrak{g}\) is abelian and \(\mathfrak{g} = \mathfrak{g}_1\).

To prove that \(H/J_m\) is commutative, it suffices to show that \(H/m^n\) is commutative for all \(n \geq 0\). Note that \(H/m^2 = k1 \oplus m/m^2\) is automatically commutative.

Now assume that \(H/m^n\) is commutative for some \(n \geq 2\). Adopt the notation of Lemma 3.2, with \(K = m\). Since
\[
(H \otimes H)/(H \otimes m^n + m^n \otimes H) \cong (H/m^n) \otimes (H/m^n)
\]
is commutative, so is \((H \otimes H)/L^n\). In particular, it follows that \([L, L] \subseteq L^n\). An easy induction then yields \([L^i, L^j] \subseteq L^{i+j+n-2}\) for all \(i, j \geq 1\).

Consider any elements \(y_1, y_2 \in m\), and set \(x_i = y_i + m^2 \in B_1 = \mathfrak{g}\) for \(i = 1, 2\). Since the \(x_i\) are primitive elements, we see that each \(\Delta(y_i) = y_i \otimes 1 + 1 \otimes y_i + u_i\) for some \(u_i \in L^2\). From the previous paragraph, \([u_i, L] \subseteq L^{n+1}\) for \(i = 1, 2\), and \([u_1, u_2] \in L^{n+2}\). It follows that
\[
\Delta([y_1, y_2]) = [\Delta(y_1), \Delta(y_2)] \equiv [y_1, y_2] \otimes 1 + 1 \otimes [y_1, y_2] \pmod{L^{n+1}},
\]
and so \(z := [y_1, y_2] + m^{n+1}\) is a primitive element in \(B_n\). As \(B = \text{gr} H \cong U(\mathfrak{g})\) with \(\mathfrak{g}\) in degree 1, all primitive elements of \(B\) are in degree 1. Consequently, \(z = 0\) and \([y_1, y_2] \in m^{n+1}\). Since \(m\) generates \(H\), this proves that \(H/m^{n+1}\) is commutative, establishing the induction step. This finishes the proof.

Proposition 3.6. Let \(H\) be either noetherian or affine. Set \(m = \ker \epsilon\), and write \(\text{gr} H\) for the graded Hopf algebra \(\bigoplus_{i=0}^{\infty} m^i/m^{i+1}\). Suppose that \(\bigcap_{i \geq 0} m^i = 0\). Then the following are equivalent:

(a) \(H\) is commutative.
(b) \(\epsilon(H) = \text{GKdim} H\).
(c) \(\text{gr} H\) is commutative.
Lemma 3.7. seen as follows.

Proof. By Lemma 3.1, \(e(H) = \dim_k m/m^2 < \infty\).

(a) \(\Rightarrow\) (b): Assuming \(H\) is commutative, it is affine either by assumption or by Mohar’s theorem [22]. Hence, \(H \cong O(G)\) for some algebraic group \(G\). Since algebraic groups are smooth and homogeneous as varieties [6, Proposition I.1.2], \(H\) is regular and equidimensional. Consequently,

\[\text{GKdim } H = \dim G = \text{gldim } H = \dim_k m/m^2 = e(H)\]

(b) \(\Rightarrow\) (c): With \(g\) as in Proposition 3.4, condition (b) implies that \(\dim_k g = \dim_k g_1\). This implies that \(g\) is abelian. Therefore \(\text{gr } H \cong U(g)\) is commutative.

(c) \(\Rightarrow\) (a): Lemma 3.5.

The next lemma, which was stated in [19, Lemma 4.3] without proof, is easily seen as follows.

**Lemma 3.7.** The commutator ideal \([H, H]\) is a Hopf ideal of \(H\), and so \(H/[H, H]\) is a commutative Hopf algebra.

**Proof.** Since \(\epsilon\) is a homomorphism from \(H\) to a commutative ring, its kernel must contain \([H, H]\). It is clear from the anti-multiplicativity of the antipode that \(S([H, H]) \subseteq [H, H]\). If \(p : H \to H/[H, H]\) is the quotient map, then \((p \otimes p)\Delta\) is a homomorphism from \(H\) to a commutative ring, so

\([H, H] \subseteq \ker(p \otimes p)\Delta = \Delta^{-1}(\ker(p \otimes p)) = \Delta^{-1}([H, H] \otimes H + H \otimes [H, H])\).

Thus, \(\Delta([H, H]) \subseteq [H, H] \otimes H + H \otimes [H, H]\).

**Proposition 3.8.** Let \(H\) be a Hopf algebra.

(a) If \(e(H) \neq 0\), then \(H/[H, H]\) is infinite dimensional.

(b) Suppose \(H\) has an affine (or noetherian) commutative Hopf quotient that is infinite dimensional. Then \(e(H) \neq 0\).

(c) Suppose \(H\) is either affine or noetherian. Then \(H\) satisfies \(\zeta\) if and only if \(H/[H, H]\) is infinite dimensional.

**Proof.** (a) Let \(A = H/[H, H]\). Since \(H/m^2\) is commutative, there is a surjective algebra map \(A \to H/m^2\). This shows that \(\ker(\epsilon_A) \neq \ker(\epsilon_A)^2\), or equivalently \(e(A) \neq 0\) by Lemma 3.1(a). By Proposition 3.4 (applied to \(A\) in place of \(H\)) and the fact that \(\dim g_1 = e(A) \neq 0\), the algebra \(\text{gr } A \cong U(g)\) is infinite dimensional. Therefore \(A\) is infinite dimensional.

(b) Suppose that \(B\) is an affine commutative Hopf quotient of \(H\) and that \(B\) is infinite dimensional. Then the proof of (a) \(\Rightarrow\) (b) in Proposition 3.6 shows that \(e(B) = \text{GKdim } B > 0\). Since \(B\) is a Hopf quotient of \(H\), \(e(H) \geq e(B) > 0\).

(c) Note that the condition \(\zeta\) means \(e(H) > 0\). The assertion follows from parts (a) and (b).

Now we consider the case when \(H\) has GK-dimension 2 (or satisfies \(\zeta\)).

**Theorem 3.9.** Let \(H\) be a Hopf algebra of GK-dimension at most 2, and assume that \(H\) is either affine or noetherian. Then \(H\) satisfies \(\zeta\) if and only if \(H\) has a Hopf quotient isomorphic to either \(k[t^{\pm 1}]\) with \(t\) grouplike or \(k[t]\) with \(t\) primitive.

**Proof.** One implication is Proposition 3.8(b). For converse we assume that \(\zeta\) holds. By Proposition 3.8(a), \(H/[H, H]\) is infinite dimensional. By Lemma 3.7, \(H/[H, H]\) is a commutative Hopf algebra. Since \(H\) is affine or noetherian, \(H/[H, H]\) has one of these properties. By Mohar’s theorem [22], \(H/[H, H]\) is necessarily affine. Thus,
\[ H/[H, H] \cong \mathcal{O}(G) \] for some infinite algebraic group \( G \) of dimension at most 2. Now \( \mathcal{O}(G) \) is a Hopf quotient of \( \mathcal{O}(G) \), where \( G^\circ \) is the connected component of the identity in \( G \), and so \( \mathcal{O}(G) \) is a Hopf quotient of \( H \). If \( \dim G^\circ = 1 \), then \( G^\circ \) is isomorphic to either \( k^\times \) or \( k^+ \), and the assertion follows. Since \( G^\circ \) is infinite, it remains to consider the case when \( \dim G^\circ = 2 \). In this case, Lemma 2.2 implies that \( G^\circ \) has a one-dimensional closed connected subgroup, say \( G_1 \). Then \( \mathcal{O}(G_1) \) is a Hopf quotient of \( \mathcal{O}(G^\circ) \), isomorphic to \( k[t^\pm 1] \) or \( k[t] \). \[ \square \]

4. Hopf quotients and other preliminaries

Our analysis of \( H \) under \( (H^\natural) \) is first divided up according to the GK-dimension of the commutator quotient \( H/[H, H] \). This quotient can have GK-dimension 2 only when \([H, H] = 0\), that is, when \( H \) is commutative. That case is covered in Section 2.

In case \( H \) is affine or noetherian, the only other possibility is \( \text{GKdim } H/[H, H] = 1 \), by Theorem 3.9.

We now set up some general machinery for Hopf quotients of \( H \), which we will apply later to the quotients arising from Theorem 3.9.

4.1. Let \( K \subset H \) be a Hopf ideal, which will be fixed in later sections, and let \( \overline{H} \) be the Hopf quotient algebra \( H/K \), with quotient map \( \pi : H \to \overline{H} \). It is obvious that \( H \) is a right and left comodule algebra over \( \overline{H} \) via
\[ \rho = (\text{id} \otimes \pi)\Delta : H \to H \otimes \overline{H} \]
and
\[ \lambda = (\pi \otimes \text{id})\Delta : H \to \overline{H} \otimes H \]
respectively. Using these two maps, we can define two subalgebras of coinvariants as follows:
\[ H_0 = H^{co \rho} := \{ h \in H \mid \rho(h) = h \otimes 1 \} \]
and
\[ 0H = ^{co \lambda}H := \{ h \in H \mid \lambda(h) = 1 \otimes h \}. \]

There are various useful relations among \( \pi, \rho, \lambda \) on the one hand, and the comultiplication and antipode maps on the other, as follows.

**Lemma 4.2.** Let \( \tau : H \otimes H \to H \otimes H \) be the flip, and let \( S \) denote the antipodes in both \( H \) and \( \overline{H} \).

(a) \( (\lambda \otimes \text{id})\Delta = (\pi \otimes \Delta)\Delta = (\text{id} \otimes \Delta)\lambda \).
(b) \( (\text{id} \otimes \rho)\Delta = (\Delta \otimes \pi)\Delta = (\Delta \otimes \text{id})\rho \).
(c) \( (\text{id} \otimes \rho)\lambda = (\pi \otimes \rho)\Delta = (\lambda \otimes \pi)\Delta = (\lambda \otimes \text{id})\rho \).
(d) \( \rho S = \tau(S \otimes S)\lambda \).
(e) \( \lambda S = \tau(S \otimes S)\rho \).

**Proof.** First, compose the map \( \pi \otimes \text{id} \otimes \text{id} \) with \( (\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta \), to get
\[ (\pi \otimes \Delta)\Delta = (\lambda \otimes \text{id})\Delta. \]

On the other hand,
\[ (\pi \otimes \Delta)\Delta = (\text{id} \otimes \Delta)(\pi \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\lambda. \]
This gives (a), and (b) holds symmetrically.

Next, composing the map \( \pi \otimes \text{id} \otimes \pi \) with \( (\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta \), we have
\[ (\pi \otimes \rho)\Delta = (\lambda \otimes \pi)\Delta. \]
Since $\pi \otimes \rho = (\text{id} \otimes \rho)(\pi \otimes \text{id})$, we have $(\pi \otimes \rho)\Delta = (\text{id} \otimes \rho)\lambda$, and similarly $(\lambda \otimes \pi)\Delta = (\lambda \otimes \text{id})\rho$. This establishes (c).

Finally, if we compose the map $\text{id} \otimes \pi$ with $\Delta S = \tau(S \otimes S)\Delta$, we obtain

$$\rho S = \tau(\pi \otimes \text{id})(S \otimes S)\Delta = \tau(S \otimes S)(\pi \otimes \text{id})\Delta = \tau(S \otimes S)\lambda.$$ 

This yields (d), and (e) holds symmetrically. □

**Lemma 4.3.** Retain the notation as above.

(a) $\Delta(H_0) \subseteq H \otimes H_0$ and $\Delta(\rho H) \subseteq \rho H \otimes H$.
(b) $S(H_0) \subseteq \rho H$ and $S(\rho H) \subseteq H_0$.
(c) If $H_0 = \rho H$, then $H_0$ is a Hopf subalgebra of $H$.
(d) $H_0 = k1 + (H_0 \cap K)$ and $\rho H = k1 + (\rho H \cap K)$.

**Proof.** Parts (a)–(c) are generalizations of [11, Proposition 2.1(c,e,h)].

(a) If $h \in H_0$, then Lemma 4.2(b) implies

$$(\text{id} \otimes \rho)\Delta(h) = \Delta(h) \otimes 1.$$ 

Now write $\Delta(h) = \sum_j a_j \otimes b_j$ with the $a_j$ linearly independent over $k$. Then $\sum_j a_j \otimes \rho(b_j) = \sum_j a_j \otimes b_j \otimes 1$, and consequently $\rho(b_j) = b_j \otimes 1$ for all $j$. Thus all $b_j \in H_0$, and so $\Delta(h) \in H \otimes H_0$, proving the first inclusion. The second is symmetric.

(b) If $h \in H_0$, then in view of Lemma 4.2(e), $\lambda S(h) = \tau(S \otimes S)(h \otimes 1) = 1 \otimes S(h)$, so $S(h) \in \rho H$. This establishes the first inclusion. The second is symmetric.

(c) From part (a), we get

$$\Delta(H_0) \subseteq (H_0 \otimes H) \cap (H \otimes H_0) = H_0 \otimes H_0,$$

and from part (b) we get $S(H_0) \subseteq H_0$, which implies the assertion.

(d) Note that $\rho(H_0) = H_0 \otimes 1$ in $H \otimes H$ implies $\Delta(H_0) \subseteq H_0 \otimes 1 + H \otimes K$ in $H \otimes H$. Hence,

$$H_0 = (\epsilon \otimes \text{id})\Delta(H_0) \subseteq k1 + kK.$$ 

Since $K \subseteq \ker \epsilon$, it follows that

$$H_0 \cap \ker \epsilon \subseteq H_0 \cap K \subseteq H_0 \cap \ker \epsilon.$$ 

Therefore $H_0 = k1 + (H_0 \cap \ker \epsilon) = k1 + (H_0 \cap K)$, and similarly for $\rho H$. □

We end this section by recording two known results that will be needed later.

**Lemma 4.4.** Let $G$ be a totally ordered group (e.g., $G = \mathbb{Z}^n$) and $A$ a $G$-graded domain.

(a) Every invertible element in $A$ is homogeneous.
(b) Suppose $A$ is a Hopf algebra such that $A \otimes A$ is a domain. If $x \in A$ such that $\Delta(x)$ is invertible in $A \otimes A$ (in particular, if $x$ is invertible in $A$), then $x \in A_g$ and $\Delta(x) \in A_g \otimes A_g$ for some $g \in G$.
(c) Continue the assumptions of part (b). If, further, $A_g$ is 1-dimensional, then $x$ is a scalar multiple of a grouplike element.

**Proof.** (a) This follows from the hypothesis that $G$ is totally ordered.

(b) The $G$-grading on $A$ induces a $(G \times G)$-grading on $A \otimes A$. Since $G \times G$ is totally ordered and $\Delta(x)$ is invertible, part (a) implies that $\Delta(x)$ is homogeneous, so $\Delta(x) \in A_g \otimes A_h$ for some $g, h \in G$. By the counit property, $x \in A_g \cap A_h$, and so $g = h$.

(c) This is clear. □
**Lemma 4.5.** All domains of GK-dimension \( \leq 1 \) over \( k \) are commutative.

**Proof.** Since \( k \) is algebraically closed, the only domain of GK-dimension zero over \( k \) is \( k \) itself. The GK-dimension one case is well known; e.g., it is embedded in the proof of [5, Theorem 1.1]. It appears explicitly in [19, Corollary 7.8 (a)⇒(b)]. \( \square \)

5. **Hopf quotient** \( \overline{H} = k[t^{\pm 1}] \) **in general**

From now on through Section 7, we assume that \( H \) has a Hopf quotient \( \overline{H} = H/K = k[t^{\pm 1}] \) for some Hopf ideal \( K \subset H \), where \( t \) is a grouplike element in \( \overline{H} \). We do not impose the hypotheses \((H\#)\) until later.

Because \( \overline{H} = k[t^{\pm 1}] \), the right and left \( \overline{H} \)-comodule algebra structures given by \( \rho \) and \( \lambda \) correspond to \( \mathbb{Z} \)-gradings on \( H \), namely

\[
H = \bigoplus_{n \in \mathbb{Z}} H_n = \bigoplus_{n \in \mathbb{Z}} nH
\]

where

\[
H_n = \{ h \in H \mid \rho(h) = h \otimes t^n \}
\]
\[
nH = \{ h \in H \mid \lambda(h) = t^n \otimes h \},
\]

(cf. [23, Example 4.1.7]). Write \( \pi^r_n \) and \( \pi^l_n \) for the respective projections from \( H \) onto \( H_n \) and \( nH \) in the above decompositions. Thus,

\[
\rho(h) = \sum_{n \in \mathbb{Z}} \pi^r_n(h) \otimes t^n \quad \text{and} \quad \lambda(h) = \sum_{n \in \mathbb{Z}} t^n \otimes \pi^l_n(h)
\]

for \( h \in H \).

Note that for \( h \in H \) we have

\[
\sum_{m,n \in \mathbb{Z}} t^m \otimes \pi^r_m \pi^l_n(h) \otimes t^n = (\text{id} \otimes \rho)(\lambda(h)) = (\lambda \otimes \text{id})\rho(h) = \sum_{m,n \in \mathbb{Z}} t^m \otimes \pi^l_m \pi^r_n(h) \otimes t^n
\]

in view of Lemma 4.2(c), which implies that

(E5.0.1) \[
\pi^r_m \pi^l_n = \pi^l_m \pi^r_n
\]

for all \( m, n \in \mathbb{Z} \). Consequently, \( \pi^l_m(H_n) \subseteq H_n \) and \( \pi^r_n(mH) \subseteq mH \) for all \( m, n \), which shows that the \( \lambda \) - and \( \rho \) - gradings on \( H \) are compatible in the sense that

(E5.0.2) \[
H_n = \bigoplus_{m \in \mathbb{Z}} (H_n \cap mH) \quad \text{and} \quad mH = \bigoplus_{n \in \mathbb{Z}} (mH \cap H_n)
\]

for all \( m, n \).

**Lemma 5.1.** Retain the notation as above.

(a) \( \Delta \pi^r_n = (\text{id} \otimes \pi^r_n)\Delta \) and \( \Delta \pi^l_n = (\pi^l_n \otimes \text{id})\Delta \) for all \( n \in \mathbb{Z} \).
(b) \( \Delta(H_n) \subseteq H \otimes H_n \) and \( \Delta(nH) \subseteq nH \otimes H \) for all \( n \in \mathbb{Z} \).
(c) \( S\pi^r_n = \pi^l_{-n}S \) and \( S\pi^l_n = \pi^r_{-n}S \) for all \( n \in \mathbb{Z} \).
(d) \( S(H_n) \subseteq -nH \) and \( S(nH) \subseteq H_{-n} \) for all \( n \in \mathbb{Z} \).
Proof. (a) For $h \in H$, apply Lemma 4.2(b) to obtain
$$\sum_{n \in \mathbb{Z}} \Delta \pi^n_r(h) \otimes t^n = (\Delta \otimes \text{id}) \rho(h)$$
$$= (\text{id} \otimes \rho) \Delta(h) = \sum_{(h)} h_1 \otimes \rho(h_2)$$
$$= \sum_{(h)} \sum_{n \in \mathbb{Z}} h_1 \otimes \pi^n_r(h_2) \otimes t^n,$$
and consequently $\Delta \pi^n_r(h) = \sum_{(h)} h_1 \otimes \pi^n_r(h_2) = (\text{id} \otimes \pi^n_r) \Delta(h)$ for $n \in \mathbb{Z}$. This establishes the first formula, and the second is symmetric.

(b) If $h \in H_n$, then $\Delta(h) = \Delta \pi^n_r(h) = (\text{id} \otimes \pi^n_r) \Delta(h) \in H \otimes H_n$. This gives the first inclusion, and the second is symmetric.

(c) For $h \in H$, apply Lemma 4.2(e) to obtain
$$\sum_{m \in \mathbb{Z}} t^m \otimes \pi^m_l S(h) = \lambda S(h) = \tau(S \otimes S) \rho(h) = \sum_{n \in \mathbb{Z}} t^{-n} \otimes S \pi^n_r(h).$$
This yields $S \pi^n_r(h) = \pi^{-n}_r S(h)$. Similarly, $S \pi^n_l(h) = \pi^{-n}_l S(h)$.

(d) These inclusions are clear from (c).

Lemma 5.2. For each $n \in \mathbb{Z}$, there is an element $t_n \in H_n \cap nH$ such that $\pi(t_n) = t^n$. In particular, $H_n$ and $nH$ are nonzero for all $n$.

Proof. Fix $n$, and choose some $x \in H$ for which $\pi(x) = t^n$. Then
$$\sum_{i \in \mathbb{Z}} \pi \pi^n_r(x) \otimes t^i = (\pi \otimes \text{id}) \rho(x) = (\pi \otimes \pi) \Delta(x) = \Delta(t^n) = t^n \otimes t^n,$$
whence $\pi \pi^n_r(x) = t^n$. This allows us to replace $x$ by $\pi^n_r(x)$, that is, we may now assume that $x \in H_n$.

Since the $\lambda$- and $\rho$-gradings are compatible, all $\pi^j_l(x) \in H_n$ for all $j \in \mathbb{Z}$. As in the previous paragraph, $\pi \pi^j_l(x) = t^n$, and we may replace $x$ by $\pi^j_l(x)$. Now $x \in H_n \cap nH$, as desired.

Lemma 5.3. The Hopf algebra $H$ is strongly graded with respect to both the $\rho$-grading and the $\lambda$-grading.

Proof. Let $n \in \mathbb{Z}$ and $t_n \in H_n \cap nH$ as in Lemma 5.2. Note that $\epsilon(t_n) = \epsilon(t^n) = 1$. By Lemma 5.1(b)(c), $\Delta(t_n) = \sum a_i \otimes b_i$ for some $a_i \in nH$ and $b_i \in H_n$, and then $S(a_i) \in H_{-n}$ and $S(b_i) \in -nH$ for all $i$. Now
$$1_H = \sum a_i b_i \in H_{-n} H_n \text{ and } 1_H = \sum a_i S(b_i) \in n H_{-n} H ,$$
and thus $H_{-n} H_n = H_0$ and $n H_{-n} H = 0 H$. Since $n$ was arbitrary, it follows that the $\rho$- and $\lambda$-gradings are strong.

The following results will be helpful in working with gradings on $H$ and its subalgebras.

Lemma 5.4. Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded $k$-algebra which is a domain.

(a) If $\dim_k A_0 < \infty$, and if $A_l \neq 0$ and $A_{-m} \neq 0$ for some $l, m > 0$, then $\dim_k A_w \leq 1$ for all $w \in \mathbb{Z}$, and $\text{GKdim} A = 1$. Further, $A = k[x^{\pm 1}]$ for some nonzero homogeneous element $x$ of positive degree.
(b) If $A_n \neq 0$ for some $n \neq 0$, then $\text{GKdim } A \geq \text{GKdim } A_0 + 1$.

(c) Suppose that $\text{GKdim } A = 1$, and that $A_n \neq 0$ for some $n \neq 0$. Then \( \dim_k A_i \leq 1 \) for all $i \in \mathbb{Z}$. Further, if $\sigma$ is any graded $k$-algebra automorphism of $A$, there is a nonzero scalar $q \in k$ such that $\sigma(a) = qa$ for all $a \in A_i$, $i \in \mathbb{Z}$.

**Proof.** (a) Since $k$ is algebraically closed and $A_0$ is a finite dimensional domain over $k$, we have $A_0 = k$.

Since $A$ is a domain, $\dim_k A_w \leq \dim_k A_{w+v}$ for $v, w \in \mathbb{Z}$ with $A_v \neq 0$, because multiplication by any nonzero element of $A_w$ embeds $A_w$ into $A_{w+v}$. Suppose $A_w \neq 0$ for some $w \neq 0$. Then $A_{lw(m)} \supseteq A_{w}^{lm-1}$ for $l, m > 0$. Further, $A_{lw(m)} \supseteq A_{w}^{lm} \neq 0$ if $w > 0$, while $A_{lw(m)} \supseteq A_{l}^{w(-m)} \neq 0$ if $w < 0$. Hence, 

$$\dim_k A_w \leq \dim_k A_{lw} \leq \dim_k A_{lw(m)} = \dim_k A_0 = 1.$$ 

Now consider an arbitrary finite dimensional subspace $V \subseteq A$, and note that $V \subseteq \bigoplus_{s=0}^{\infty} A_i$ for some $s > 0$. Consequently, $V^n \subseteq \bigoplus_{i=-ns}^{ns} A_i$, and so $\dim_k V^n \leq 2ns + 1$ for all $n > 0$. Therefore $\dim_k V^n$ grows at most linearly with $n$. On the other hand, if $V = kx$ for some nonzero element $x \in A_l$, then $\dim_k V^n = 1$ for all $n > 0$. Therefore $\text{GKdim } A = 1$.

For the last statement, let $x \in A$ be a nonzero homogeneous element of minimal positive degree and $y \in A$ a nonzero homogeneous element of largest negative degree. Then $xy$ has degree 0, and we may assume that $y = x^{-1}$. If $z$ is a nonzero homogeneous element, then $deg z$ must be a multiple of $deg x$ by the minimality of $deg x$. Therefore $A = k[x^{\pm 1}]$.

(b) Pick a nonzero element $x \in A_n$. Then $x^i \in A_{in}$ for $i \geq 0$, from which we see that the nonnegative powers $x^i$ are left linearly independent over $A_0$.

Let $B$ be an affine subalgebra of $A_0$, choose a finite dimensional generating subspace $V$ for $B$ with $1 \in V$, and set $W = V + kx$. For any $n > 0$, the power $W^n$ contains $\sum_{i=0}^{n} V^{n-i}x^i$, and so $\dim_k W^n \geq \sum_{i=0}^{n} \dim_k V^{n-i}$ because the $x^i$ are left linearly independent over $A_0$. Consequently, $\dim_k W^n \geq \dim_k Z^n$ for all $n > 0$, where $Z = V + kx$ is a finite dimensional generating subspace for a polynomial ring $B[z]$. Since $\text{GKdim } B[z] = 1 + \text{GKdim } B$, it follows that $\text{GKdim } A \geq 1 + \text{GKdim } B$.

(c) Let $B$ be the graded ring of fractions of $A$ (which exists because $A$ is commutative). Then $B$ is a domain over $k$ with $\text{GKdim } B = 1$, and $B_l$ and $B_{-m}$ are nonzero for some $l, m > 0$. Part (b) implies that $\text{GKdim } B_0 = 0$, whence $B_0 = k$. Now part (a) implies that $B = k[x^{\pm 1}]$ for some nonzero homogeneous element $x$ of positive degree. In particular, $\dim_k A_i \leq \dim_k B_i \leq 1$ for all $i \in \mathbb{Z}$.

Any graded $k$-algebra automorphism $\sigma$ of $A$ extends uniquely to a graded $k$-algebra automorphism of $B$, which we also denote $\sigma$. Clearly there is a nonzero scalar $r \in k$ such that $\sigma(x) = rx$. Let $d = \text{deg } x$, and let $q \in k$ be a $d$-th root of $r$. Then $\sigma(x^i) = r^i x^i = q^{di} x^i$ for all $i \in \mathbb{Z}$. Since $\text{deg } x^i = dl$, it follows that $\sigma(b) = qb$ for all nonzero $b \in B_i$, $i \in \mathbb{Z}$. This equation holds trivially when $b = 0$, yielding the desired description of $\sigma$. 

\[\begin{array}{ll}
6. \quad \overline{H} = k[t^{\pm 1}] \text{ AND } H_0 = 0H
\end{array}\]

In this section, we classify all affine or noetherian Hopf algebras under the hypothesis ($H_2$) in the case that $H$ has a Hopf quotient $\overline{H} = k[t^{\pm 1}]$ with $t$ grouplike and $H_0 = 0H$. The assumptions ($H_2$), $\overline{H} = k[t^{\pm 1}]$, and $H_0 = 0H$ are to hold throughout this section; finiteness conditions are only imposed in Theorem 6.3.
Recall from Lemma 4.3(c) that $H_0$ must be a Hopf subalgebra of $H$ in these circumstances.

**Lemma 6.1.** Either $H_0 = k[y]$ with $y$ primitive, or $H_0 \cong k\Gamma$ where $\Gamma$ is a torsion-free abelian group of rank 1.

**Proof.** By Lemma 5.3, the $\mathbb{Z}$-graded algebra $H = \bigoplus_n H_n$ is strongly graded. In particular, $H_n \neq 0$ for all $n$. Lemma 5.4(a)(b) then implies that $\dim_k H_0 = \infty$ and $\text{GKdim } H_0 \leq 1$, and so by Lemma 4.5, $H_0$ is commutative. Since $H_0$ is a domain and $k$ is algebraically closed, the only finite dimensional subalgebra of $H_0$ is $k1$. Thus, $\text{GKdim } H_0 = 1$, and the desired conclusion follows from Proposition 2.1.  

**Lemma 6.2.** As a $k$-algebra, $H = H_0[x^{\pm 1}; \phi]$ where $x \in H_1$ and $\phi$ is a $k$-algebra automorphism of $H_0$. Moreover, $x$ can be chosen to be a grouplike element of $H$ satisfying $\pi(x) = t$.

**Proof.** By Lemma 5.3, $H_1H_{-1} = H_0$, and so there are elements $x_1, \ldots, x_r \in H_1$ and $y_1, \ldots, y_r \in H_{-1}$ such that $\sum_i x_i y_i = 1$. In particular, it follows that $H_1 = \sum_i x_i H_0$.

In view of Lemma 6.1, $H_0$ is isomorphic to either a polynomial ring $k[y]$ or a group ring $k\Gamma$, where $\Gamma$ is free abelian of rank 1. In either case, $H_0$ is a directed union of commutative PIDs, and hence a Bezout domain. At least one $y_j \neq 0$, and left multiplication by $y_j$ embeds $H_1$ in $H_0$. Thus, $H_1$ must be a cyclic right $H_0$-module, say $H_1 = x H_0$.

Since $xH_{-1} = xH_0H_{-1} = H_1H_{-1} = H_0$, we see that $x$ is right invertible, and thus invertible (because $H$ is a domain). It follows that $H_n = x^n H_0 = H_0 x^n$ for all $n \in \mathbb{Z}$. In particular, from $xH_0 = H_0 x$ we see that conjugation by $x$ restricts to a $k$-algebra automorphism $\phi$ of $H_0$. Since $H = \bigoplus_n H_n$ is a free left $H_0$-module with basis $\{x^n | n \in \mathbb{Z}\}$, we conclude that $H = H_0[x^{\pm 1}; \phi]$.

By Lemma 4.3(d), $\pi(H_0) = k1$, and so $\pi(H_1) = k\pi(x)$. By Lemma 5.2, there is some $t_1 \in H_1$ such that $\pi(t_1) = t$, and hence $\pi(x)$ must be a nonzero scalar multiple of $t$. After replacing $x$ by a scalar multiple of itself, we may assume that $\pi(x) = t$. In particular, it follows that $\epsilon(x) = \epsilon(t) = 1$.

Next, observe that $H \otimes H = (H_0 \otimes H_0)[z_1^{\pm 1}, z_2^{\pm 1}; \phi_1, \phi_2]$ is an iterated skew-Laurent polynomial ring over $H_0 \otimes H_0$, with respect to the commuting automorphisms $\phi_1 = \phi \circ \text{id}$ and $\phi_2 = \text{id} \circ \phi$. Since $H_0 \otimes H_0$ is isomorphic to either a polynomial ring $k[y] \otimes k[y] \cong k[y, z]$ or a group algebra $k(\Gamma \times \Gamma)$ over a torsion-free abelian group, it is a domain, and thus $H \otimes H$ is a domain. Lemma 4.4(b) now implies that $\Delta(x) \in H_n \otimes H_n$ for some $n \in \mathbb{Z}$. On the other hand, $\Delta(x) \in H \otimes H_1$ by Lemma 5.1(b), so $n = 1$ and $\Delta(x) \in H_1 \otimes H_1 = x H_0 \otimes x H_0$. Thus, $\Delta(x) = (x \otimes x)w$ for some $w \in H_0 \otimes H_0$. Since $\Delta(x)$ and $x \otimes x$ are invertible in $H \otimes H$, so is $w$. In view of the skew-Laurent polynomial structure of $H \otimes H$, it follows that $w$ is invertible in $H_0 \otimes H_0$.

If $H_0 \cong k[y]$, then $w = \alpha 1 \otimes 1$ for some $\alpha \in k^\times$, and $\Delta(x) = \alpha x \otimes x$. Since $\epsilon(x) = 1$, the counit property implies $x = \alpha x$, forcing $\alpha = 1$. Therefore $x$ is grouplike in this case.

Finally, suppose that $H_0 \cong k\Gamma$, so that $H_0 \otimes H_0 \cong k(\Gamma \times \Gamma)$. Since the units in $k(\Gamma \times \Gamma)$ are scalar multiples of elements of $\Gamma \times \Gamma$, we conclude that $w = \alpha u \otimes v$ for some grouplike elements $u, v \in H_0$. Now $\Delta(x) = \alpha xu \otimes xv$. The counit property implies $x = \alpha xu = \alpha xv$, whence $\alpha = 1$ in $k$ and $u = v = 1$ in $H_0$. Therefore $\Delta(x) = x \otimes x$, and so $x$ is grouplike in this case also.  

$\square$
Theorem 6.3. Assume that $H$ is a Hopf algebra satisfying $(H\sharp)$ and having a Hopf quotient $\overline{H} = k[t^{\pm 1}]$ with $t$ grouplike and $H_0 = _0H$. Then $H$ is affine if and only if $H$ is noetherian, if and only if $H$ is isomorphic to one of the following Hopf algebras:

(a) $A(0, q)$, for some nonzero $q \in k$.
(b) A group algebra $k\Lambda$, where $\Lambda$ is either free abelian of rank 2 or the semidirect product $\mathbb{Z} \rtimes \mathbb{Z} = \langle x, y \mid xyx^{-1} = y^{-1}\rangle$.

Proof. Note first that the Hopf algebras listed in (a) and (b) are both affine and noetherian. Conversely, assume that $H$ is either affine or noetherian.

Continue with the notation above, and consider first the case where $H_0 = k[y]$ with $y$ primitive. The automorphism $\phi$ of $H_0$ must send $y \mapsto qy + b$ for some $q \in k^\times$ and $b \in k$, and so $xy = (qy + b)x$. Applying $\epsilon$ to the latter equation yields $0 = b$, so $xy = qyx$. In view of Lemma 6.2, we obtain $H \cong A(0, q)$ in this case.

Now assume that $H_0 = k\Gamma$ for some torsionfree abelian group $\Gamma$ of rank 1. Note that since $x$ is grouplike, conjugation by $x$ maps grouplike elements of $H_0$ to grouplike elements. Hence, $\phi$ restricts to an automorphism of $\Gamma$, and therefore $H = k\Lambda$ for some semidirect product $\Lambda = \Gamma \rtimes \mathbb{Z}$.

If $H$ is noetherian, then in view of Lemma 6.2, $H_0$ must be noetherian. Consequently, $\Gamma$ is infinite cyclic, and case (b) holds.

Finally, assume that $H$ is affine, which implies that $\Lambda$ is a finitely generated group. Note also that $\Lambda$ is solvable. Since $\text{GKdim}\ H = 2 < \infty$, the group $\Lambda$ cannot have exponential growth, and a theorem of Milnor [21, Theorem] (or see [18, Corollary 11.6]) shows that $\Lambda$ must be polycyclic. This forces $\Gamma$ to be polycyclic, whence $\Gamma \cong \mathbb{Z}$. Therefore we are in case (b) again. \qed

7. $\overline{H} = k[t^{\pm 1}]$ and $H_0 \neq _0H$

In this section, we continue to assume that $H$ satisfies $(H\sharp)$ and that $H$ has a Hopf quotient $\overline{H} = k[t^{\pm 1}]$ with $t$ grouplike, but we now assume that $H_0 \neq _0H$.

Lemma 7.1. Both $H_0$ and $_0H$ have GK-dimension 1. Consequently, these algebras are commutative.

Proof. Since $\text{GKdim}\ H > 1$, Lemmas 5.2 and 5.4(a) imply that $H_0$ and $_0H$ are infinite dimensional over $k$. Since they are domains, and $k$ is algebraically closed, the only finite dimensional subalgebra of either $H_0$ or $_0H$ is $k$. Thus, $H_0$ and $_0H$ have GK-dimension at least 1.

By Lemmas 5.2 and 5.4(b), $\text{GKdim}\ H_0 \leq 1$. Therefore $\text{GKdim}\ H_0 = 1$, and similarly for $_0H$. The consequence follows from Lemma 4.5. \qed

In the following results, we will need the notation $H_{i,j} = H_i \cap _jH$, for $i, j \in \mathbb{Z}$. In view of (E5.0.2), $H = \bigoplus_{m,n \in \mathbb{Z}} H_{m,n}$ is a $\mathbb{Z}^2$-graded algebra. Similarly, $H_0 = \bigoplus_{n \in \mathbb{Z}} H_{0,n}$ and $_0H = \bigoplus_{n \in \mathbb{Z}} H_{n,0}$ are $\mathbb{Z}$-graded algebras. We have $H_0 \nsubseteq _0H$ or $_0H \nsubseteq H_0$, say $H_0 \nsubseteq _0H$. Then $H_{0,n} \neq 0$ for some $n \neq 0$. By Lemmas 7.1 and 5.4(c),

$$\dim_k H_{0,i} \leq 1$$

for all $i \in \mathbb{Z}$. In particular, $\dim_k H_{0,0} \leq 1$, and so $_0H \nsubseteq H_0$. Proceeding as with $H_0$, we find that

$$\dim_k H_{i,0} \leq 1$$

for $i \in \mathbb{Z}$.


Lemma 7.2. The intersections $H_{n,n}$ are all one-dimensional, and
\[ \bigoplus_{n \in \mathbb{Z}} H_{n,n} = k[z^{\pm 1}] \]
where $z \in H_{1,1}$ is a grouplike element of $H$ such that $\pi(z) = t$. Further,
\[ H = H_0[z^{\pm 1}; \sigma_r] = 0 H[z^{\pm 1}; \sigma_l] \]
where $\sigma_r$ and $\sigma_l$ are graded $k$-algebra automorphisms of $H_0$ and $0 H$, respectively.

Proof. As noted above, dim$_k H_{0,i} \leq 1$ for all $i \in \mathbb{Z}$. Thus, $H_{0,0} \otimes H_0 = \bigoplus_{m,n \in \mathbb{Z}} H_{0,m} \otimes H_{0,n}$ is a $\mathbb{Z}^2$-graded algebra in which each nonzero homogeneous component is 1-dimensional, spanned by a regular element. Since $\mathbb{Z}^2$ is an ordered group, it follows that $H_0 \otimes H_0$ is a domain.

For each $n \in \mathbb{Z}$, Lemma 5.2 provides us with a nonzero element $t_n \in H_{n,n}$ such that $\pi(t_n) = t^n$. Left multiplication by $t_0^{-1}$ provides an embedding of $H_{n,n}$ into $H_{0,0}$. Hence, $H_{n,n}$ must be 1-dimensional. In particular, $H_{1,1} = k z$ where $z = t_1$. Note that $\epsilon(z) = \epsilon(t) = 1$. Now $t_{-1} z$ and $zt_{-1}$ are nonzero elements of $H_{0,0} = k$, hence units. Therefore $z$ is a unit in $H$, with $z^{-1} \in H_{-1,1}$. The one-dimensionality of the spaces $H_{n,n}$ now forces $H_{n,n} = k z^n$ for all $n$, whence $\bigoplus_{n \in \mathbb{Z}} H_{n,n} = k[z^{\pm 1}]$.

The existence of a unit $z \in H_1$ implies that $H_n = H_0 z^n = z^n H_0$ for all $n$, and thus $H = \bigoplus_{n \in \mathbb{Z}} H_0 z^n$. Further, $z H_0 z^{-1} = H_0$, and so conjugation by $z$ restricts to a $k$-algebra automorphism $\sigma_r$ of $H_0$. We conclude that $H = H_0[z^{\pm 1}; \sigma_r]$, and similarly $H = 0 H[z^{\pm 1}; \sigma_l]$ where $\sigma_l$ is the restriction of $0 H$ by $\epsilon(z) = 1$. By Lemmas 4.4(c) says that $z$ is a scalar multiple of a grouplike element. Since $\epsilon(z) = 1$, we conclude that $z$ itself is grouplike.

The situation of Lemma 7.2 does occur, even in the commutative case. For instance, let $H = A(n,1)$ with $n \geq 1$, and take $K = \langle y \rangle$ and $t = x + K$. Here $\lambda(y) = t^n \otimes y$ and $\rho(y) = y \otimes 1$, so $x \in H_0 \setminus 0 H$. The element $z$ of the lemma is just $x$.

Lemma 7.3. If $H$ is not commutative, then either $H_0 = H_{0,\geq 0}$ or $H_0 = H_{0,\leq 0}$.

Proof. Suppose to the contrary that $H_0 \neq H_{0,\geq 0}$ and $H_0 \neq H_{0,\leq 0}$. By Lemmas 7.2 and 5.4(a), $H_0 = k[x^{\pm 1}]$ for some nonzero homogeneous element $x$ of positive degree, say degree $d$. Then $x \in \partial H$, so $\lambda(x) = t^d \otimes x$, and consequently the element $\pi(x) \in \overline{H}$ satisfies $\Delta \pi(x) = t^d \otimes \pi(x)$.

Further, we have $H = H_0[z^{\pm 1}; \sigma_r]$ as in Lemma 7.2. Hence, $zz z^{-1} = qx^m$ for some $q \in k^x$ and $m \in \{\pm 1\}$. Applying $\Delta \pi$ to the equation $zx = qx^m z$, we obtain
\[ (t \otimes t)(t^d \otimes \pi(x)) = q(t^{md} \otimes \pi(x)^m)(t \otimes t), \]
from which we see that $m = 1$ and $q = 1$. But now $H$ is commutative, contradicting our assumptions.
For the rest of this section, we assume that $H$ is not commutative. By symmetry, we may further assume that $H_0 = H_{0,\geq 0}$. This is possible because interchanging $t$ and $t^{-1}$ in $H$ results in interchanging $nH$ and $-nH$ for all $n$. Recall that $\text{GKdim} H_0 = 1$ whereas $H_{0,0} = k$, whence $H_{0,j}$ must be nonzero for infinitely many $j > 0$.

We will need the following well known facts about nonzero additive submonoids $M$ of $\mathbb{Z}_{\geq 0}$. First, if $n$ is the greatest common divisor of the positive elements of $M$, then $M = nS = \{ns \mid s \in S\}$ for a submonoid $S \subseteq \mathbb{Z}_{\geq 0}$, and the greatest common divisor of the positive elements of $S$ is $1$. Second, $S$ has a unique minimal set of generators, consisting of those positive elements of $S$ which cannot be written as sums of two positive elements of $S$. We shall refer to these elements as the \textit{minimal generators} of $S$. Third, $S$ has only finitely many minimal generators, and $\mathbb{Z}_{\geq 0} \setminus S$ is finite.

**Lemma 7.4.** Set $n = \gcd\{j > 0 \mid H_{0,j} \neq 0\}$ and $S = \{i \geq 0 \mid H_{0,ni} \neq 0\}$. Let $z \in H_{1,1}$ and $\sigma_r \in \Aut H_0$ as in Lemma 7.2.

(a) There is a scalar $q \neq 0, 1$ in $k$ such that $\sigma_r(h) = q^i h$ for all $h \in H_{0,ni}$, $i \geq 0$.

(b) If $m$ is a minimal generator of $S$, then each $h \in H_{0,nm}$ is a skew primitive element satisfying $\Delta(h) = h \otimes 1 + z^{nm} \otimes h$.

(c) If $S = \mathbb{Z}_{\geq 0}$, then $H \cong A(n,q)$.

(d) If $S \neq \mathbb{Z}_{\geq 0}$, then $q$ is a root of unity.

**Proof.** (a) By Lemma 5.4(c), there is a nonzero scalar $s \in k$ such that $\sigma_r(h) = s^j h$ for all $h \in H_{0,j}$, $j \geq 0$. Setting $q = s^n$, we have $\sigma_r(h) = q^i h$ for all $h \in H_{0,ni}$, $i \geq 0$.

Now since we have assumed that $H = H_0[z^{\pm 1}; \sigma_r]$ is not commutative, $\sigma_r$ is not the identity on $H_0$. But $H_0 = \bigoplus_{i \geq 0} H_{0,ni}$ by choice of $n$, so we must have $q \neq 1$.

(b) Since the spaces $H_{0,ni}$ are at most one-dimensional, we can choose elements $h_i$ such that $H_{0,ni} = kh_i$ for all $i \geq 0$. In particular, we choose $h_0 = 1$. Since we are assuming that $H$ is noncommutative, $\sigma_r$ is nontrivial, and there is some $j > 0$ such that $h_j \neq 0$ and $q^j \neq 1$. Since $zh_j = q^j h_j z$ and $\epsilon(z) = 1$, it follows that $\epsilon(h_j) = 0$.

For all $i > 0$, we have $h_i \in H_{0,ni} = kh_j$, and so $\epsilon(h_i) = 0$.

Now $H_0 = \bigoplus_{i \geq 0} kh_i$ and $H = H_0[z^{\pm 1}; \sigma_r] = \bigoplus_{i \geq 0} k[z^{\pm 1}] h_i$. Since $z \in 1H$, it follows that $H = \bigoplus_{s \geq 0} k z^n h_s$. By Lemma 5.1(b), each $\Delta(h_i) \subseteq niH \otimes H_0$, and so

$$
\Delta(h_i) = \sum_{s,t \geq 0} \beta_{st}^i z^{n(i-s)} h_s \otimes h_t
$$

for some scalars $\beta_{st}^i \in k$. Whenever $h_s = 0$ or $h_t = 0$, we choose $\beta_{st}^i = 0$. In particular, this means that when $h_i = 0$, we must have $\beta_{st}^i = 0$ for all $s$, $t$. The counit axiom now implies that

$$
h_i = \sum_{t \geq 0} \beta_{ti}^0 h_t = \sum_{s \geq 0} \beta_{si}^t z^{n(i-s)} h_s,
$$

from which we see that if $h_i \neq 0$, then $\beta_{ti}^0 = \beta_{si}^0 = 1$, while $\beta_{ti}^t = 0$ for $t \neq i$ and $\beta_{si}^s = 0$ for $s \neq i$. Therefore

$$
\Delta(h_i) = h_i \otimes 1 + z^{ni} \otimes h_i + u_i \quad \text{where} \quad u_i = \sum_{s,t \geq 1} \beta_{st}^i z^{n(i-s)} h_s \otimes h_t.
$$

This formula also holds, trivially, when $h_i = 0$. 
Assume that \( m \) is a minimal generator of \( S \). We must show that \( u_m = 0 \). Observe that

\[
\begin{align*}
h_m \otimes 1 \otimes 1 + z^{nm} \otimes h_m \otimes 1 + z^{nm} \otimes z^{nm} \otimes h_m + z^{nm} \otimes u_m \\
+ \sum_{s,t \geq 1} \beta_{st}^{m} z^{n(m-s)} h_s \otimes (h_t \otimes 1 + z^{nt} \otimes h_t + u_t)
\end{align*}
\]

\[
= (\text{id} \otimes \Delta)(\Delta(h_m)) = (\Delta \otimes \text{id})(\Delta(h_m))
\]

\[
= h_m \otimes 1 \otimes 1 + z^{nm} \otimes h_m \otimes 1 + u_m \otimes 1 + z^{nm} \otimes z^{nm} \otimes h_m
\]

\[
+ \sum_{s,t \geq 1} \beta_{st}^{m} (z \otimes z)^{n(m-s)} (h_s \otimes 1 + z^{ns} \otimes h_s + u_s) \otimes h_t.
\]

To analyze the terms of these equations, we view \( H \otimes H \otimes H \) as the direct sum of the spaces \( k[z^\pm 1]h_r \otimes k[z^\pm 1]h_s \otimes k[z^\pm 1]h_t \), which we refer to as the \((r,s,t)\)-components of this tensor product. Fix \( s, t \geq 1 \), and note that the terms \( h_s \otimes u_t \) and \( h_s \otimes h_t \) have no \((s,0,t)\)-components. Hence, comparing \((s,0,t)\)-components in the above equation, we find that

\[
\beta_{st}^{m} z^{n(m-s)} h_s \otimes z^{nt} \otimes h_t = \beta_{st}^{m} z^{n(m-s)} h_s \otimes z^{n(m-s)} \otimes h_t.
\]

By our choice of coefficients, \( \beta_{st}^{m} \) cannot be nonzero unless \( h_s \) and \( h_t \) are both nonzero, that is, \( s, t \in S \). In this case, we obtain \( z^{nt} = z^{n(m-s)} \) from the equation above. But then \( m = s + t \), contradicting the assumption that \( m \) is a minimal generator of \( S \). Therefore \( \beta_{st}^{m} = 0 \) for all \( s, t \geq 1 \), yielding \( u_m = 0 \) as desired.

(c) If \( S = \mathbb{Z}_{\geq 0} \), then 1 is the minimal generator of \( S \), and \( H_0 = k[y] \) where \( y = h_1 \). By part (b), \( y \) is a skew primitive element satisfying \( \Delta(y) = y \otimes 1 + z^n \otimes y \), and therefore \( H \cong A(n, q) \) in this case.

(d) If \( S \neq \mathbb{Z}_{\geq 0} \), then 1 \( \not\in S \) and \( S \) has at least two minimal generators. Choose minimal generators \( i < j \). Then \( h_i^j \) and \( h_j^i \) are nonzero elements of \( H_{0, ni,j} \), and so \( h_j^i = \lambda h_i^j \) for some \( \lambda \in k^\times \). Since \( z^{ni} h_i = q^{ni} h_i z^{ni} \) and \( z^{nj} h_j = q^{nj} h_j z^{nj} \), we have, because of part (b),

\[
\Delta(h_i^j) = (h_i \otimes 1 + z^{ni} \otimes h_i)^j = \sum_{s=0}^{j} \binom{j}{s} q^{nj} h_i^{j-s} z^{nis} \otimes h_i^s \in \bigoplus_{s=0}^{j} H \otimes H_{0, ni,s}
\]

\[
\Delta(h_j^i) = (h_j \otimes 1 + z^{nj} \otimes h_j)^i = \sum_{t=0}^{i} \binom{i}{t} q^{nj} h_j^{i-t} z^{njt} \otimes h_j^t \in \bigoplus_{t=0}^{i} H \otimes H_{0, nj,t}.
\]

Note that the expansion of \( \Delta(h_j^i) \) contains no term from \( H \otimes H_{0, ni} \), because \( 0 < ni < nj \). Since \( \Delta(h_i^j) = \lambda \Delta(h_j^i) \), it follows that \( \binom{i}{j} q^{nj} = 0 \), from which we conclude that \( q \) must be a root of unity.

When \( S \neq \mathbb{Z}_{\geq 0} \) in Lemma 7.4, it must have at least two minimal generators. This leads us to the Hopf algebras \( B(n, p_0, \ldots, p_s, q) \), after the following lemma.

**Lemma 7.5.** Let \( \xi \in k^\times \). If there is an integer \( a \geq 2 \) such that \( \binom{a}{r}_{\xi} = 0 \) for \( 0 < r < a \), then \( \xi \) is a primitive \( a \)-th root of unity.

**Proof.** The assumption \( \binom{a}{r}_{\xi} = 0 \) implies that \( \xi^{a-1} + \xi^{a-2} + \cdots + 1 = 0 \), and so \( \xi^a = 1 \). Hence, \( \xi \) is a primitive \( b \)-th root of unity, for some positive divisor \( b \) of \( a \). If \( a = bc \), then \( \binom{a}{b}_{\xi} = \binom{c}{1}_{\xi} \neq 0 \) (see [15, 2.6(iii)], for instance), which is not possible for \( b < a \) by our hypotheses. Therefore \( b = a \).

\( \square \)
Theorem 7.6. Assume that $H$ is a Hopf algebra satisfying (H2) and having a Hopf quotient $\mathcal{H} = k[t^{\pm 1}]$ with $t$ grouplike and $H_0 \neq 0 H$. If $H$ is not commutative, then $H$ is isomorphic to one of the Hopf algebras $A(n, q)$ or $B(n, p_0, \ldots , p_s, q)$.

Proof. We adopt the notation of Lemma 7.4 and its proof. If $S = \mathbb{Z}_{\geq 0}$, the lemma shows that $H \cong A(n, q)$, and we are done. Assume now that $S \neq \mathbb{Z}_{\geq 0}$, and let $m_1, \ldots , m_s$ be the distinct minimal generators of $S$, arranged in descending order. We must have $s \geq 2$, and Lemma 7.4(d) shows that $q$ is a root of unity, say a primitive $\ell$-th root of unity, for some positive integer $\ell$. Hence, each of the elements $\xi_i := q^{m_i^n}$ is a primitive $p_i$-th root of unity, for some positive integer $p_i$.

As in the proof of Lemma 5.4(c), the graded ring of fractions of $H_0$ can be identified with a Laurent polynomial ring $k[y^{\pm 1}]$ where $y$ is a homogeneous element of degree $n$. Hence, we may choose $h_j = y^j$ for $j \in S$. Set $y_i = h_{m_i} = y^{m_i}$ for $i = 1, \ldots , s$, and recall from Lemma 7.4(b) that

$$\Delta(y_i) = y_i \otimes 1 + z^{m_i n} \otimes y_i$$

for such $i$.

Next, let $1 \leq i < j \leq s$. We claim that

(a) $p_i, p_j \geq 2$;
(b) $m_i p_i = m_j p_j$;
(c) $p_i$ and $p_j$ are relatively prime.

There are relatively prime positive integers $a$ and $b$ such that $m_i a = m_j b$, and $y_i^a = y_j^b$. Neither of $m_i$ or $m_j$ divides the other (since they are distinct minimal generators of $S$), so $a, b \geq 2$. Since $m_i a = m_j b = \text{lcm}(m_i, m_j)$, none of $m_i, 2m_i, \ldots , (a - 1)m_i$ can be divisible by $m_j$, and none of $m_j, 2m_j, \ldots , (b - 1)m_j$ can be divisible by $m_i$. Now

$$\sum_{r=0}^{a} \binom{a}{r} \xi_i y_i^{a-r} z^{m_i n r} \otimes y_i^r = \Delta(y_i^a) = \Delta(y_j^b) = \sum_{t=0}^{b} \binom{b}{t} \xi_j y_j^{b-t} z^{m_j n t} \otimes y_j^t.$$ 

Comparing components in $H \otimes H_{0, n l}$ for $l \geq 0$, we find that $\binom{a}{r} \xi_i = 0$ for $0 < r < a$ and $\binom{b}{t} \xi_j = 0$ for $0 < t < b$. By Lemma 7.5, $\xi_i$ is a primitive $a$-th root of unity and $\xi_j$ is a primitive $b$-th root of unity. Consequently, $a = p_i$ and $b = p_j$, from which (a), (b), (c) follow.

Now set $m = m_1 p_1 = \cdots = m_s p_s = \text{lcm}(m_1, \ldots , m_s)$. Since the $m_i$ were chosen in descending order, the $p_i$ must be in ascending order: $p_1 < p_2 < \cdots < p_s$. Further, since the $p_i$ are pairwise relatively prime, $p_j | m_i$ for all $i \neq j$. Since $m_1, \ldots , m_s$ generate $S$, their greatest common divisor is 1, and we conclude that $m = p_1 p_2 \cdots p_s$. Observe that $q^{m_i m n} = q^{m_i' n p_i} = 1$ and so $\ell | m_i m n$ for all $i$, whence $\ell | m n$. For any $i$, this yields $\ell | m_i n p_i$, whence $\ell | m_i' n p_i'$ for some positive integers $m_i' | m_i$, $n' | n$, and $p_i' | p_i$. In particular, $\ell | m_i' n p_i'$, and so $\xi_i^{p_i'} = q^{m_i' n p_i'} = 1$. Since $\xi_i$ is a primitive $p_i$-th root of unity, $p_i' = p_i$. Thus, all $p_i | \ell$, and hence $m | \ell$. At this stage, we have $\ell = m n_0$ for some positive integer $n_0 | n$.

Set $p_0 = n/n_0$, so that $\ell = (n/p_0) p_1 p_2 \cdots p_s$. We next show that $p_i$ is relatively prime to $p_0$ for each $i = 1, \ldots , s$. Write $\text{lcm}(p_0, p_i) = p_0 c_i$ for some positive integer $c_i$. Then $n_0 p_i$ divides $n_0 p_0 c_i = n c_i$, whence $\ell$ divides $n c_i \prod_{j \neq i} p_j^2 = m_i^2 n c_i$. This
implies \( \xi^i = q^{m_i^2 n_i} = 1 \) and so \( p_i | c_i \). It follows that \( p_0 \) and \( p_i \) are relatively prime, as claimed. Therefore \( n, p_0, p_1, \ldots, p_s, q \) satisfy the hypotheses (a), (b), (c) of Construction 1.2.

As \( k \)-algebras, \( H_0 = k[y_1, \ldots, y_s] \subset k[y] \) and \( H = H_0[z^{\pm 1} ; \sigma_r] \) (by Lemma 7.2), where \( \sigma_r \) is the restriction to \( H_0 \) of the \( k \)-algebra automorphism of \( k[y] \) such that \( y \mapsto qy \) and \( z \) is a grouplike element of \( H \). Further, \( y_1, \ldots, y_s \) are skew primitive, with \( \Delta(y_i) = y_i \otimes 1 + z^{m_i n_i} \otimes y_i \). Therefore \( H \cong B(n, p_0, \ldots, p_s, q) \). \( \square \)

8. Hopf quotient \( \overline{H} = k[t] \) in general

In this section, \( (H_\sharp) \) is not imposed.

Throughout the section, we assume that \( \overline{H} = k[t] \) with \( t \) primitive. In this case, the \( \overline{H} \)-comodule algebra structures \( \rho \) and \( \lambda \) on \( H \) are given by derivations, as follows.

**Lemma 8.1.** There exist locally nilpotent \( k \)-linear derivations \( \delta_r \) and \( \delta_l \) on \( H \) such that

\[
\rho(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n_r(h) \otimes t^n \quad \text{and} \quad \lambda(h) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \otimes \delta^n_l(h)
\]

for \( h \in H \). Moreover, \( \delta_r \) and \( \delta_l \) commute.

**Proof.** There exist \( k \)-linear maps \( d_0, d_1, \ldots \) on \( H \) such that

\[
\rho(h) = \sum_{n=0}^{\infty} d_n(h) \otimes t^n
\]

for \( h \in H \), where \( d_n(h) = 0 \) for \( n \gg 0 \). Since \( \epsilon(t) = 0 \), the counit axiom for \( \rho \) implies that \( h = d_0(h) \) for \( h \in H \), so that \( d_0 = \text{id}_H \). Since \( \rho \) is an algebra homomorphism,

\[
\sum_{n=0}^{\infty} d_n(hh') \otimes t^n = \sum_{i,j=0}^{\infty} d_i(h)d_j(h') \otimes t^{i+j}
\]

for \( h, h' \in H \). Comparing \((-) \otimes t \) terms in this equation, we find that

\[
d_i(hh') = d_0(h)d_i(h') + d_i(h)d_0(h')
\]

for \( h, h' \in H \). Thus, \( \delta_r := d_1 \) is a derivation on \( H \).

From the coassociativity of \( \rho \), we obtain

\[
\sum_{i, n=0}^{\infty} d_id_n(h) \otimes t^i \otimes t^n = (\rho \otimes \text{id})\rho(h) = (\text{id} \otimes \Delta)\rho(h)
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{m}{i} d_m(h) \otimes t^i \otimes t^{m-i}
\]

for \( h \in H \), whence \( d_id_n = \binom{n+i}{i} d_{n+i} \) for all \( i, n \). An easy induction then shows that \( d_n = (1/n! \delta^n_1) \) for all \( n \). This establishes the asserted formula for \( \rho \). In particular, it follows that \( \delta_r \) is locally nilpotent.

The asserted formula for \( \lambda \) is obtained symmetrically.

Since \( (\text{id} \otimes \rho)\lambda = (\lambda \otimes \text{id})\rho \) (Lemma 4.2(c)), we get

\[
\sum_{m, n=0}^{\infty} \frac{1}{m! n!} t^m \otimes \delta^n_r \delta^m_l(h) \otimes t^n = \sum_{m, n=0}^{\infty} \frac{1}{m! n!} t^m \otimes \delta^m_l \delta^n_r(h) \otimes t^n
\]
for $h \in H$. Comparing $t \otimes (-) \otimes t$ terms, we conclude that $\delta_r \delta_l = \delta_l \delta_r$. □

In particular, it follows from Lemma 8.1 that $H_0 = \ker \delta_r$ and $aH = \ker \delta_l$.

**Lemma 8.2.**

(a) $\Delta \delta_r = (\text{id} \otimes \delta_r) \Delta$ and $\Delta \delta_l = (\delta_l \otimes \text{id}) \Delta$.

(b) $\Delta(\ker \delta^n) \subseteq H \otimes \ker \delta^n$ and $\Delta(\ker \delta^n) \subseteq \ker \delta^n \otimes H$ for all $n \geq 0$.

(c) $\delta_r S = -S \delta_l$ and $\delta_l S = -S \delta_r$.

(d) $S(\ker \delta^n) \subseteq \ker \delta^n$ and $S(\ker \delta^n) \subseteq \ker \delta^n$ for all $n \geq 0$.

**Proof.**

(a) For $h \in H$, apply Lemma 4.2(b) to obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Delta \delta^n_r(h) \otimes t^n = (\Delta \otimes \text{id}) \rho(h) = (\text{id} \otimes \rho) \Delta(h) = \sum_{(h)} h_1 \otimes \rho(h_2)$$

$$= \sum_{(h)} \sum_{n=0}^{\infty} \frac{1}{n!} h_1 \otimes \delta^n_r(h_2) \otimes t^n.$$

It follows that $\Delta \delta_r(h) = \sum_{(h)} h_1 \otimes \delta_r(h_2) = (\text{id} \otimes \delta_r) \Delta(h)$, establishing the first identity. The second is symmetric.

(b) From (a), we also have $\Delta \delta^n = (\text{id} \otimes \delta^n) \Delta$ and $\Delta \delta^n = (\delta^n \otimes \text{id}) \Delta$ for all $n \geq 0$.

Part (b) follows.

(c) For $h \in H$, apply Lemma 4.2(d) to obtain

$$\sum_{m=0}^{\infty} \frac{1}{m!} \delta^n_r S(h) \otimes t^n = \rho S(h) = \tau(S \otimes S) \lambda(h) = \sum_{n=0}^{\infty} \frac{1}{n!} S \delta^n_{r}(h) \otimes (-t)^n.$$

Comparing $(-) \otimes t$ terms, we find that $\delta_r S(h) = -S \delta_l(h)$. This establishes the first formula, and the second is symmetric.

(d) From (c), we also have $\delta^n_r S = (-1)^n S \delta^n_l$ and $\delta^n_l S = (-1)^n S \delta^n_r$ for all $n \geq 0$.

Part (d) follows. □

We include the full hypotheses in the following result because of its independent interest.

**Theorem 8.3.** Let $H$ be a Hopf algebra over a field $k$ of characteristic zero, with a Hopf quotient $\overline{H} = H/K$ which is a polynomial ring $k[t]$ with $t$ primitive. Assume that the rings of coinvariants for the right and left $\overline{H}$-comodule algebra structures $\rho : H \to H \otimes \overline{H}$ and $\lambda : H \to \overline{H} \otimes H$ coincide.

(a) There exists an element $x \in H$ such that $\rho(x) = x \otimes 1 + 1 \otimes t$ and $x + K = t$.

(b) As a $k$-algebra, $H$ is a skew polynomial ring of the form $H = H_0[x; \partial]$ where $H_0$ is the ring of $\rho$-coinvariants in $H$ and $\partial$ is a $k$-linear derivation on $H_0$.

**Proof.** By Lemma 4.3(c), $H_0$ is a Hopf subalgebra of $H$.

Now $H = \bigcup_{n=0}^{\infty} \ker \delta^n_r$ because $\delta_r$ is locally nilpotent. If $\ker \delta^n_r = H_0 = \ker \delta_r$, then $\ker \delta^{n+1} = \ker \delta^n$ for all $n \geq 0$, from which it would follow that $\ker \delta^n_r = \ker \delta_r$ for all $n \geq 0$, and consequently $H = H_0$. However, there must be some $u \in H$ with $\pi(u) = t$, and $(\pi \otimes \text{id}) \rho(u) = (\pi \otimes \tau) \Delta(u) = \Delta(t) = t \otimes 1 + 1 \otimes t$, whence $\rho(u) \neq u \otimes 1$, that is, $u \notin H_0$. Thus, $\ker \delta^n_r = H_0$.

Now $I := \ker(\ker \delta^n_r)$ is a nonzero ideal of $H_0$, because $\delta_r$ gives an $H_0$-bimodule homomorphism from $\ker \delta^n_r$ to $H_0$. In view of Lemma 8.2(a)(b),

$$\Delta(I) = \Delta \delta_r(\ker \delta^n_r) = (\text{id} \otimes \delta_r) \Delta(\ker \delta^n_r) \subseteq (\text{id} \otimes \delta_r)(H \otimes \ker \delta^n_r) = H \otimes I.$$

Moreover, $I \subseteq H_0$ and so $\Delta(I) \subseteq H_0 \otimes H_0$. Therefore $\Delta(I) \subseteq H_0 \otimes I$. 
Choose \( v \in (\ker \delta^2_r) \setminus H_0 \), and write \( \Delta(v) = \sum_i a_i \otimes b_i \) for some \( a_i \in H \) and \( b_i \in \ker \delta^2_r \). Then \( \Delta \delta_r(v) = \sum_i a_i \otimes \delta_r(b_i) \) by Lemma 8.2(a), and the counit axiom yields \( 0 \neq \delta_r(v) = \sum_i a_i \epsilon \delta_r(b_i) \). Hence, \( \epsilon \delta_r(b_j) \neq 0 \) for some \( j \). Since \( b_j \) must lie in \((\ker \delta^2_r) \setminus H_0\), we may replace \( v \) by \( \epsilon \delta_r(b_j)^{-1} b_j \), that is, there is no loss of generality in assuming that \( \epsilon \delta_r(v) = 1 \).

Continuing with our previous notation, we write \( \Delta(v) = \sum_{i=1}^n a_i \otimes b_i \) for some \( a_i \in H \) and \( b_i \in \ker \delta^2_r \), but now assuming that the \( b_i \) are \( k \)-linearly independent. Moreover, we may assume that there is some \( m \leq n \) such that \( b_i \in H_0 \) for \( i < m \) while \( b_m, \ldots, b_n \) are linearly independent modulo \( H_0 \). Hence, \( \delta_r(b_m), \ldots, \delta_r(b_n) \) are linearly independent elements of \( I \). Now

\[
\Delta \delta_r(v) = \sum_{i=m}^n a_i \otimes \delta_r(b_i)
\]

(applying Lemma 8.2(a) once more). This sum lies in \( H_0 \otimes H_0 \), because \( \delta_r(v) \in H_0 \).

Since \( \delta_r(b_m), \ldots, \delta_r(b_n) \) are linearly independent elements of \( H_0 \), it follows that \( a_m, \ldots, a_n \in H_0 \), whence \( S(a_i) \in H_0 \) for \( i = m, \ldots, n \). Consequently, the antipode axiom implies that

\[
1_H = \epsilon \delta_r(v) 1_H = \sum_{i=m}^n S(a_i) \delta_r(b_i) \in I.
\]

Therefore, there exists \( x \in \ker \delta^2_r \) such that \( \delta_r(x) = 1 \). Since we may replace \( x \) by \( x - \epsilon(x) \), we may also assume that \( \epsilon(x) = 0 \). Now \( \rho(x) = x \otimes 1 + 1 \otimes t \), and so

\[
\Delta \pi(x) = (\pi \otimes \pi) \Delta(x) = (\pi \otimes \text{id}) \rho(x) = \pi(x) \otimes 1 + 1 \otimes t
\]

in \( \overline{H} \otimes \overline{H} \). Then \( \pi(x) = (\epsilon \otimes \text{id}) \Delta \pi(x) = t \), because \( \epsilon \pi(x) = \epsilon(x) = 0 \).

Note that because \( \rho(x^n) = \rho(x)^n = (x \otimes 1 + 1 \otimes t)^n \), we have \( \delta^n_r(x^n) = n! \) for all \( n \geq 0 \). In particular, \( x^n \in \ker \delta^{n+1}_r \). It follows that the \( H_0 \)-bimodule embedding \( (\ker \delta^{n+1}_r)/(\ker \delta^n_r) \rightarrow H_0 \) induced by \( \delta^n_r \) is an isomorphism, and so \( (\ker \delta^{n+1}_r)/(\ker \delta^n_r) \) is a free left or right \( H_0 \)-module of rank \( 1 \), with the coset of \( x^n \) as a basis element. Therefore \( H = \bigoplus_{n=0}^\infty H_0 x^n = \bigoplus_{n=0}^\infty H_0 H_0 \) as left and right \( H_0 \)-modules.

Finally, since the bimodule isomorphism \( (\ker \delta^2_r)/H_0 \rightarrow H_0 \) induced by \( \delta_r \) sends the coset of \( x \) to the central element \( 1 \), we conclude that \( xh - hx \in H_0 \) for all \( h \in H_0 \). Therefore the inner derivation \([x, -]\) restricts to a \( k \)-linear derivation \( \partial \) on \( H_0 \), and \( H = H_0[x; \partial] \) as a \( k \)-algebra. \( \Box \)

9. \( \overline{H} = k[t] \) under \( (H^2) \)

Assume \( (H^2) \), and that \( \overline{H} = k[t] \) with \( t \) primitive.

**Lemma 9.1.** If \( A \) is a \( k \)-algebra supporting a locally nilpotent \( k \)-linear derivation \( \delta \) with \( \dim_k \ker \delta < \infty \), then \( \text{GKdim} \ A \leq 1 \).

**Remark.** No restriction on the base field \( k \) is needed for this lemma.

**Proof.** Set \( d = \dim_k \ker \delta \), and note that for all \( n \geq 2 \), the map \( \delta^{n-1} \) induces an embedding \( (\ker \delta^n)/(\ker \delta^{n-1}) \rightarrow \ker \delta \). Hence, \( \dim_k \ker \delta^n \leq nd \) for all \( n \geq 0 \).

Now consider an arbitrary finite dimensional subspace \( V \subseteq A \). Since \( \delta \) is locally nilpotent, \( V \subseteq \ker \delta^m \) for some \( m > 0 \). Leibniz’ Rule implies that \( (\ker \delta^r)/(\ker \delta^s) \subseteq \ker \delta^{r+s} \) for all \( r, s > 0 \), whence \( V^n \subseteq \ker \delta^{nm} \) for all \( n > 0 \). Consequently,
\[ \dim_k V^n \leq nmd \text{ for all } n > 0, \text{ proving that these dimensions grow at most linearly with } n. \]

**Lemma 9.2.** If \( u \in H \backslash_{0} H \), then 1, \( u, u^2, \ldots \) are left (or right) linearly independent over \( _0H \).

**Proof.** By assumption, there is some \( d > 0 \) such that \( \delta^d_l(u) \neq 0 \) while \( \delta^{d+1}_l(u) = 0 \). Then for any positive integer \( n \), we have

\[ \lambda(u^n) = \lambda(u)^n = \left( \sum_{i=0}^{d} \frac{1}{i!} t^i \otimes \delta^i_l(u) \right)^n = \sum_{j=0}^{nd} t^j \otimes h_j \]

for some \( h_j \in H \) with \( h_{nd} = \left( \frac{1}{d!} \delta^d_l(u) \right)^n \neq 0 \). Consequently, \( \delta^{nd}_l(u^n) = (nd)!h_{nd} \neq 0 \) while \( \delta^{nd+1}_l(u^n) = 0 \).

Now if \( a_0 + a_1 u + \cdots + a_n u^n = 0 \) for some \( a_i \in _0H \), an application of \( \delta^{nd}_l \) yields \( a_n \delta^{nd}_l(u^n) = 0 \) and hence \( a_n = 0 \). Continuing inductively yields all \( a_i = 0 \), verifying that the \( u^i \) are left linearly independent over \( _0H \). Right linear independence is proved the same way. \( \square \)

**Lemma 9.3.** \( \text{GKdim } H_0 = \text{GKdim } _0H = 1 \). As a consequence, \( H_0 \) and \( _0H \) are commutative.

**Proof.** Since \( \text{GKdim } H > 1 \), Lemma 9.1 implies that \( H_0 \) and \( _0H \) are infinite dimensional over \( k \). Since they are domains, and \( k \) is algebraically closed, the only finite dimensional subalgebra of either \( H_0 \) or \( _0H \) is \( k \). Thus, \( H_0 \) and \( _0H \) have \( \text{GK-dimension at least } 1 \).

Choose an element \( u \in H \) such that \( \pi(u) = t \). As in the proof of Theorem 8.3, we see that \( u \notin H_0 \), and similarly \( u \notin _0H \). By Lemma 9.2, the powers \( u^i \) are left linearly independent over \( _0H \). The argument of Lemma 5.4(b) therefore shows that \( \text{GKdim } _0H = 1 \), and similarly for \( H_0 \). Lemma 4.5 says \( H_0 \) and \( _0H \) are commutative. \( \square \)

**Proposition 9.4.** Retain the hypotheses as above. Then \( H_0 = _0H \).

**Proof.** Suppose that \( H_0 \nsubseteq _0H = \ker \delta_l \). Since \( \delta_l \) commutes with \( \delta_r \), it leaves \( H_0 = \ker \delta_r \) invariant. Thus, \( \delta_l \) restricts to a nonzero locally nilpotent derivation \( \delta \) on \( H_0 \). Set \( H_{00} = H_0 \cap _0H = \ker \delta \), a subalgebra of \( H_0 \).

Choose \( u \in H_0 \backslash _0H \). By Lemma 9.2, the powers \( u^i \) are left linearly independent over \( _0H \) and hence over \( H_{00} \). The argument of Lemma 5.4(b) then shows that \( \text{GKdim } H_0 > \text{GKdim } H_{00} \), whence \( \text{GKdim } H_{00} = 0 \). Since \( H_{00} \) is a domain and \( k \) is algebraically closed, it follows that \( H_{00} = k \).

Since \( \delta \) is locally nilpotent, some \( \delta^{d}_l(u) \) will be a nonzero element of \( H_{00} \). Hence, by taking a suitable scalar multiple of \( \delta^{d-1}_l(u) \), we can obtain an element \( y \in H_0 \) such that \( \delta(y) = 1 \). This implies \( \lambda(y) = 1 \otimes y + t \otimes 1 \), whereas \( \rho(y) = y \otimes 1 \) because \( y \in H_0 \). Set \( \overline{y} = \pi(y) \in \overline{H} \) and compute \( \Delta(\overline{y}) \) in the following two ways:

\[ \Delta(\overline{y}) = (\pi \otimes \pi) \Delta(y) = (\pi \otimes \text{id}) \rho(y) = \overline{y} \otimes 1 \]
\[ \Delta(\overline{y}) = (\pi \otimes \pi) \Delta(y) = (\text{id} \otimes \pi) \lambda(y) = 1 \otimes \overline{y} + t \otimes 1. \]

The first equation implies that \( \overline{y} = (\epsilon \otimes \text{id}) \Delta(\overline{y}) = \epsilon(\overline{y})1_{\overline{H}} \). We now have

\[ \epsilon(\overline{y})(1 \otimes 1) = \Delta(\overline{y}) = \epsilon(\overline{y})(1 \otimes 1) + t \otimes 1 \]

and consequently \( t \otimes 1 = 0 \), which is impossible.
Therefore $H_0 \subseteq oH$. The reverse inclusion is obtained symmetrically.  

9.5. By Proposition 9.4 and Theorem 8.3, $H = H_0[x; \vartheta]$ (as a $k$-algebra) for some $k$-linear derivation $\vartheta$ on $H_0$, where $x$ is an element of $H$ such that $\rho(x) = x \otimes 1 + 1 \otimes t$ and $\pi(x) = t$. 

Since $K$ is an ideal of $H$, its contraction $H_0 \cap K$ must be a $\partial$-ideal of $H_0$. Consequently, $(H_0 \cap K)H = H(H_0 \cap K) = \bigoplus_{n=0}^{\infty}(H_0 \cap K)x^n$ is an ideal of $H$, contained in $K$. By Lemma 4.3(d), $H_0 = k1 + (H_0 \cap K)$. Hence, 

$$H = k[x] \oplus (H_0 \cap K)H.$$

Since $\pi$ maps $k[x]$ isomorphically onto $k[t] = \overline{H}$, the first two equalities below follow. The other two are symmetric (or follow from the fact that $H_0 = oH$).

$$(E9.5.1) \quad K = (H_0 \cap K)H = H(H_0 \cap K) = (oH \cap K)H = H_0(H \cap K).$$

For our main result, we need to know that affineness descends from $H$ to $H_0$ in the present circumstances. This requires the following lemma.

**Lemma 9.6.** Let $A$ be a commutative domain over $k$ with transcendence degree 1, and $\delta$ a $k$-linear derivation on $A$. Assume there exist $a_1, \ldots, a_r \in A$ such that $A$ is generated by $\{\delta^i(a_j) \mid i \geq 0, j = 1, \ldots, r\}$. Then $A$ is affine over $k$.

**Remark.** Here $k$ need not be algebraically closed, but we require char $k = 0$.

**Proof.** It is known that any $k$-subalgebra of a commutative affine domain with transcendence degree 1 over $k$ is itself affine (cf. [24, Theorem A] or [1, Corollary 1.4]). Hence, it suffices to show that $A$ is contained in some affine $k$-subalgebra of its quotient field $F$.

Extend $\delta$ to $F$ via the quotient rule. Set

$$A_0 = k[a_1, \ldots, a_r] \subseteq A_1 = A_0[\delta(a_1), \ldots, \delta(a_r)] \subseteq A,$$

and let $F_i = \text{Fract} A_i$ for $i = 0, 1$, so that $F_0 \subseteq F_1 \subseteq F$. Note that $\delta(A_0) \subseteq A_1$, whence $\delta(F_0) \subseteq F_1$. Because of characteristic zero, any element of $F$ which is algebraic over $k$ must lie in $\ker \delta$. Hence, if all of the $a_j$ were algebraic over $k$, we would have $\delta = 0$ and $A = A_0$ finite dimensional over $k$, contradicting the assumption of transcendence degree 1. Thus, at least one $a_j$ is transcendental over $k$. It follows that $F_0$ has transcendence degree 1 over $k$, and therefore $F$ must be algebraic over $F_0$.

Consider a nonzero element $u \in F_1$, and let $f(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_0 \in F_0[z]$ be its minimal polynomial over $F_0$. Applying $\delta$ to the equation $f(u) = 0$, we obtain

$$[\delta(\alpha_{n-1})u^{n-1} + \cdots + \delta(a_1)u + \delta(a_0)] + [nu^{n-1} + (n-1)\alpha_{n-1}u^{n-2} + \cdots + \alpha_1] \delta(u) = 0,$$

from which we see that $\delta(u) \in F_1$. Thus, $\delta(F_1) \subseteq F_1$.

In particular, each $\delta^2(a_j) \in F_1$, and so there is a nonzero element $v \in A_1$ such that $\delta^2(a_j) \in A_1v^{-1}$ for all $j$. It follows that the algebra $A_1[v^{-1}]$ is stable under $\delta$. Therefore $A$ is contained in the affine $k$-algebra $A_1[v^{-1}]$, completing the proof.  

**Proposition 9.7.** Retain the hypotheses as above, and assume that $H$ is affine or noetherian. Then either $H_0 = k[y^{\pm 1}]$ with $y$ grouplike or $H_0 = k[y]$ with $y$ primitive.
Proof. If $H$ is commutative, then it is affine, and the result follows from Proposition 2.3, with an appropriate choice of the Hopf ideal $K$. In case (a) of the proposition, $H = k[x, y]$ with $x$ and $y$ primitive. Take $K = \langle y \rangle$, and observe that $H_0 = k[y]$ in this case. Case (b) does not occur, since $k\Gamma$ has no Hopf quotient isomorphic to $k[t]$. In case (c), we again take $K = \langle y \rangle$ and observe that $H_0 = k[y]$.

We now assume that $H$ is not commutative. Since $H_0$ is a commutative domain of GK-dimension 1, it will suffice to show that $H_0$ is affine, by Proposition 2.1. Write $H = H_0[x; \partial]$ as in §9.5, and note that because of our noncommutativity assumption, $\partial \neq 0$. If $H$ is noetherian, then so is $H_0$, whence $H_0$ is affine by [22]. Now suppose that $H$ is affine.

By hypothesis, $H$ can be generated by finitely many elements $h_1, \ldots, h_m$. Each $h_j$ is a finite sum of terms $a_{ji}x^i$ for $a_{ji} \in H_0$ and $i \geq 0$. Hence, there is a finite sequence $a_1, \ldots, a_r$ of elements of $H_0$ such that $H$ can be generated by $x, a_1, \ldots, a_r$.

Consider the $k$-subalgebra

$$A = k[\partial^i(a_j) \mid i \geq 0, j = 1, \ldots, r] \subseteq H_0,$$

and note that $\partial(A) \subseteq A$. Now $\bigoplus_{n=0}^{\infty} A x^n$ is a subalgebra of $H$. Since it contains $x$ and the $a_j$, it must equal $H$. From $\bigoplus_{n=0}^{\infty} A x^n = \bigoplus_{n=0}^{\infty} H_0 x^n$, we conclude that $A = H_0$. Therefore $H_0$ is affine by Lemma 9.6.

10. $\overline{H} = k[t]$ and $H_0 = k[y]$

Assume $(H_2)$ throughout this section, and additionally that $\overline{H} = k[t]$ and $H_0 = k[y]$ with $t$ and $y$ primitive. We also assume that $H$ is not commutative.

10.1. Lemma 4.3(d) implies that $H_0 \cap K = H_0 \cap \ker \epsilon = yH_0$, and so by (E9.5.1) we get $K = yH = HY$. By §9.5, we also have $H = H_0[x; \partial]$ where $\pi(x) = t$ and $\rho(x) = x \otimes 1 + 1 \otimes t$; in particular, $\epsilon(x) = 0$ and $\delta_0(x) = 1$. For any $\alpha \in k^\times$, we have $\pi(\alpha x) = \alpha t$ and $\rho(\alpha x) = \alpha x \otimes 1 + 1 \otimes \alpha t$, while $H = H_0[\alpha x; \alpha \partial]$ and $\overline{H} = k[\alpha t]$ with $\alpha t$ primitive. Thus, we may replace $x$ by any nonzero scalar multiple of itself, as long as we replace $t$ and $\partial$ by corresponding multiples of themselves. Similarly, we may add any element of $H_0 \cap K$ to $x$.

Now $xy - yx = \partial(y) = f$ for some nonzero $f \in H_0$. Note that $f(0) = \epsilon(f) = \epsilon([x, y]) = 0$, so that $f$ is divisible by $y$. Moreover, $\partial = f \frac{d}{dy}$, and so $fH_0$ is a $\partial$-ideal of $H_0$. Hence, $fH = Hf$ is an ideal of $H$, and $H/fH$ is commutative. Since $f = [x, y]$, it follows that $fH = [H, H]$. By Lemma 3.7, $fH$ is thus a Hopf ideal of $H$. In particular,

$$\Delta(f) \in fH \otimes H + H \otimes fH = \bigoplus_{i, j = 0}^{\infty} [fH_0 x^i \otimes H_0 x^j + H_0 x^i \otimes fH_0 x^j].$$

On the other hand, $\Delta(f) \in \Delta(H_0) \subseteq H_0 \otimes H_0$, from which we conclude that $\Delta(f) \in fH_0 \otimes H_0 + H_0 \otimes fH_0$.

Write $f = \alpha_1 y + \cdots + \alpha_r y^r$ for some $\alpha_i \in k$ with $\alpha_r \neq 0$, and note that the cosets of the tensors $y^i \otimes y^j$ for $0 \leq i, j \leq r - 1$ form a basis for the quotient $(H_0 \otimes H_0)/(fH_0 \otimes H_0 + H_0 \otimes fH_0)$. Computing modulo $fH_0 \otimes H_0 + H_0 \otimes fH_0$, we find that
we obtain
\[
0 \equiv \Delta(f) = \sum_{i=1}^{r} \alpha_i (y \otimes 1 + 1 \otimes y)^i
\]
\[
= \sum_{i,j \geq 0 \atop 0 < i + j \leq r} \binom{i + j}{i} \alpha_{i+j} y^i \otimes y^j
\]
\[
= \sum_{r > i + j \geq 0 \atop 0 < i + j \leq r} \binom{i + j}{i} \alpha_{i+j} y^i \otimes y^j - \sum_{m=1}^{r-1} \alpha_m y^m \otimes 1 - \sum_{n=1}^{r-1} 1 \otimes \alpha_n y^n.
\]
If \( r \geq 2 \), the last line above has only one term involving \( y \otimes y^{r-1} \), and this term has the coefficient \( \binom{r}{1} \alpha_r \neq 0 \), which is impossible. Hence, \( r = 1 \), and \( f = \alpha_1 y \) with \( \alpha_1 \neq 0 \).

At this point, we replace \( x \) with \( \alpha_1^{-1} x \), so that \( xy - yx = y \), that is,
\[
(E10.1.1) \quad \partial = y \frac{d}{dy}.
\]

10.2. Write \( \Delta(x) = x \otimes 1 + 1 \otimes x + u \) for some \( u \in H \otimes H \), and observe that
\[
(id \otimes \Delta)(\Delta(x)) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 \otimes 1 \otimes 1 \otimes t + (id \otimes \rho)(u)
\]
\[
(\Delta \otimes id)\rho(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + u \otimes 1 + 1 \otimes 1 \otimes t.
\]
In view of Lemma 4.2(b), we obtain \((id \otimes \rho)(u) = u \otimes 1\), from which it follows that \( u \in H \otimes H_0 \). We also have
\[
(id \otimes \Delta)\Delta(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes u + (id \otimes \Delta)(u)
\]
\[
(\Delta \otimes id)\Delta(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + u \otimes 1 + 1 \otimes 1 \otimes x + (\Delta \otimes id)(u),
\]
and so coassociativity of \( \Delta \) implies that
\[
(E10.2.1) \quad 1 \otimes u + (id \otimes \Delta)(u) = u \otimes 1 + (\Delta \otimes id)(u).
\]
The counit axioms imply that
\[
x = x + m(id \otimes \varepsilon)(u) \quad \text{and} \quad x = x + m(\varepsilon \otimes id)(u),
\]
and consequently
\[
(E10.2.2) \quad m(id \otimes \varepsilon)(u) = m(\varepsilon \otimes id)(u) = 0.
\]

Next, observe that
\[
y \otimes 1 + 1 \otimes y = \Delta(y) = \Delta([x, y]) = [\Delta(x), \Delta(y)]
\]
\[
= y \otimes 1 + 1 \otimes y + [u, \Delta(y)],
\]
whence \([u, \Delta(y)] = 0\). Since \( u \) lies in \( H \otimes H_0 \), it commutes with \( 1 \otimes y \), and thus it must also commute with \( y \otimes 1 \). If we write \( u = \sum u_j \otimes y^j \) with the \( u_j \in H \), then we find that each \( u_j \) must commute with \( y \). As is easily checked, the centralizer of \( y \) in \( H \) is just \( H_0 \). Therefore \( u \in H_0 \otimes H_0 \). This allows us to write
\[
u = \sum_{i,j \geq 0} \beta_{ij} y^i \otimes y^j
\]
for some \( \beta_{ij} \in k \). Equation (E10.2.2) becomes
\[
\sum_{i \geq 0} \beta_{i0} y^i = \sum_{j \geq 0} \beta_{0j} y^j = 0,
\]
whence $\beta_{ii} = \beta_{ij} = 0$ for all $i, j$. Thus, $u = \sum_{i,j \geq 1} \beta_{ij} y^i \otimes y^j$.

At this point, (E10.2.1) becomes
\[
\sum_{i,j \geq 1} \beta_{ij} y^i \otimes y^j + \sum_{i,j \geq 1} \beta_{ij} y^i \otimes (y \otimes 1 + 1 \otimes y)^j = \sum_{i,j \geq 1} \beta_{ij} (y \otimes 1 + 1 \otimes y)^i \otimes y^j,
\]
and consequently
\[
(E10.2.3) \quad \sum_{i,j \geq 1} \sum_{l=0}^{j-1} \left( \frac{i}{l} \right) \beta_{ij} y^i \otimes y^j \otimes y^{j-l} = \sum_{i,j \geq 1} \sum_{m=0}^{i-1} \left( \frac{i}{m} \right) \beta_{ij} y^{i-m} \otimes y^m \otimes y^j.
\]
Comparing terms in (E10.2.3) involving $y^{i-1} \otimes y \otimes y^{j-1}$, we see that $j \beta_{i-1,j} = i \beta_{i,j-1}$ for all $i, j \geq 2$. It follows that
\[
\beta_{ij} = \frac{j+1}{i} \beta_{i-1,j+1} = \frac{(j+1)(j+2)}{i(i-1)} \beta_{i-2,j+2} = \cdots = \frac{(j+1)(j+i-1)}{i!} \beta_{i-j+1} = \frac{1}{i+j} \left( \frac{i+j}{i} \right) \beta_{i,j+i-1}
\]
for all $i, j \geq 2$, and similarly $\beta_{ij} = \frac{1}{i+j} \left( \frac{i+j}{j} \right) \beta_{i,j+1}$. Set $\gamma_s = (1/s) \beta_{1,s-1}$ for all $s \geq 2$, so that $\beta_{ij} = \left( \frac{i+j}{i} \right) \gamma_{i+j}$ for all $i, j \geq 1$. We now have
\[
\sum_{i,j \geq 1} \gamma_s \left( (y \otimes 1 + 1 \otimes y)^s - y^s \otimes 1 - 1 \otimes y^s \right) = g(y \otimes 1 + 1 \otimes y) - g \otimes 1 - 1 \otimes g,
\]
where $g = \sum_{s \geq 1} \gamma_s y^s \in H_0$. Note that since $g$ has zero constant term, $g \in K$.

The element $x' = x - g \in H$ satisfies $\pi(x') = t$ and $\rho(x') = x' \otimes 1 + 1 \otimes t$, as well as $x' y - yx' = y$. In view of (E10.2.4)
\[
\Delta(x') = x' \otimes 1 + 1 \otimes x + u - g(y \otimes 1 + 1 \otimes y) = x' \otimes 1 + 1 \otimes x' - u.
\]
Thus, we may replace $x$ by $x'$, so that
\[
\Delta(x) = x \otimes 1 + 1 \otimes x.
\]
Therefore $x$ is primitive.

**Theorem 10.3.** Assume $(H')$, and suppose that $H = k[t]$ and $H_0 = k[y]$ with $t$ and $y$ primitive. If $H$ is not commutative, then $H \cong U(g)$ where $g$ is a 2-dimensional nonabelian Lie algebra over $k$.

**Proof.** The work of Subsections 10.1 and 10.2 shows that there is a primitive element $x \in H$ such that $H = k[y][x; y^{\pm 1}]$. Thus, $H \cong U(g)$ where $g = kx + ky$ is a 2-dimensional nonabelian Lie algebra over $k$. □
11. $\overline{H} = k[t]$ and $H_0 = k[y^{\pm 1}]$

In this section, we assume $\langle H \rangle$, and also that $\overline{H} = k[t]$ and $H_0 = k[y^{\pm 1}]$ with $t$ primitive and $y$ grouplike. As in the previous section, we also assume that $H$ is not commutative.

11.1. Lemma 4.3(d) implies that $H_0 \cap K = H_0 \cap \ker \epsilon = (y - 1)H_0$, and so this time (E9.5.1) tells us that $K = (y - 1)H = H(y - 1)$. In particular, $\pi(y) = 1$. From §9.5, we have $H = H_0[x; \partial]$ for $x \in H$ with $\pi(x) = t$ and $\rho(x) = x \otimes 1 + 1 \otimes t$. As remarked in the beginning of the last section, we may replace $x$ by a nonzero scalar multiple of itself, and we may add an element of $H_0 \cap K$ to $x$. Since $\pi(y) = 1$, we may also replace $x$ by a power of $y$ times $x$. Because $\epsilon(t) = 0$, we always have $\epsilon(x) = 0$.

Now $xy - yx = f$ for some nonzero $f \in H_0$, and $f(1) = \epsilon(f) = 0$. Similarly to the last section, we see that $fH = [H, H]$ is a Hopf ideal of $H$, and then that $\Delta(f) \in fH_0 \otimes H_0 + H_0 \otimes fH_0$.

Write $f = y^r(\alpha_0 + \alpha_1 y + \cdots + \alpha_s y^s)$ for some $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$, and some $\alpha_i \in k$ with $\alpha_0, \alpha_s \neq 0$. (We must have $s > 0$ because $f(1) = 0$.) The cosets of the tensors $y^i \otimes y^j$ for $0 \leq i, j \leq s - 1$ form a basis for $(H_0 \otimes H_0)/(fH_0 \otimes H_0 + H_0 \otimes fH_0)$. Computing modulo $fH_0 \otimes H_0 + H_0 \otimes fH_0$, we obtain

$$0 \equiv (y^{-r} \otimes y^{-r})\Delta(f) = \sum_{i=0}^{s} \alpha_i y^i \otimes y^i \equiv \sum_{i=0}^{s-1} \alpha_i y^i \otimes y^i + \alpha_s^{-1} \sum_{l,m=0}^{s-1} \alpha_l y^l \otimes \alpha_m y^m.$$ 

Looking at $1 \otimes 1$ terms, we see that $\alpha_0 + \alpha_s^{-1} \alpha_0^2 = 0$, whence $\alpha_0 = -\alpha_s$. Looking at $y^l \otimes 1$ terms for $0 < l < s$, we see that $\alpha_s^{-1} \alpha_l \alpha_0 = 0$, whence $\alpha_l = 0$. Hence, $f = \alpha_s y^s(y^s - 1)$.

At this point, we replace $x$ by $\alpha_s^{-1} y^{1-r} x$, which replaces $f$ by $y^s - y$, where $n = s + 1 \geq 2$. Thus,

$$(E11.1.1) \quad \partial = (y^n - y) \frac{d}{dy}, \quad n \in \mathbb{Z}_{\geq 2}.$$ 

11.2. Write $\Delta(x) = x \otimes y^{n-1} + 1 \otimes x + u$ for some $u \in H \otimes H$, and observe that

$$(\text{id} \otimes \rho)\Delta(x) = x \otimes y^{n-1} \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes t + (\text{id} \otimes \rho)(u)$$

$$(\Delta \otimes \text{id})\rho(x) = x \otimes y^{n-1} \otimes 1 + 1 \otimes x \otimes 1 + u \otimes 1 + 1 \otimes 1 \otimes t.$$ 

In view of Lemma 4.2(b), we obtain $(\text{id} \otimes \rho)(u) = u \otimes 1$, from which it follows that $u \in H \otimes H_0$. We also have

$$(\text{id} \otimes \Delta)\Delta(x) = x \otimes y^{n-1} \otimes y^{n-1} + 1 \otimes x \otimes y^{n-1} + 1 \otimes 1 \otimes x + 1 \otimes u + (\text{id} \otimes \Delta)(u)$$

$$((\Delta \otimes \text{id})\Delta(x) = x \otimes y^{n-1} \otimes y^{n-1} + 1 \otimes x \otimes y^{n-1} + u \otimes y^{n-1} + 1 \otimes 1 \otimes x + (\Delta \otimes \text{id})(u),$$

and so coassociativity of $\Delta$ implies that

$$(E11.2.1) \quad 1 \otimes u + (\text{id} \otimes \Delta)(u) = u \otimes y^{n-1} + (\Delta \otimes \text{id})(u).$$

The counit axioms imply that

$$x = x + m(\text{id} \otimes \epsilon)(u) \quad \text{and} \quad x = x + m(\epsilon \otimes \text{id})(u),$$
and consequently
\[ m(id \otimes \varepsilon)(u) = m(\varepsilon \otimes id)(u) = 0. \]

Next, observe that
\[ y^n \otimes y^n - y \otimes y = \Delta(y^n - y) = \Delta([x, y]) = [\Delta(x), \Delta(y)] \]
\[ = [x \otimes y^{n-1} + 1 \otimes x + u, y \otimes y] \]
\[ = (y^n - y) \otimes y^n + y \otimes (y^n - y) + [u, y \otimes y], \]
whence \([u, y \otimes y] = 0 \). Since \( u \) lies in \( H \otimes H_0 \), it commutes with \( 1 \otimes y \), and thus it must also commute with \( y \otimes 1 \). As in §10.2, this forces \( u \in H_0 \otimes H_0 \), and so
\[ u = \sum_{i,j} \beta_{ij} y^i \otimes y^j \]
for some \( \beta_{ij} \in k \). Equation (E11.2.2) becomes
\[ \sum_{i,j} \beta_{ij} y^i = \sum_{i,j} \beta_{ij} y^j = 0, \]
whence
\[ \sum_j \beta_{ij} = 0 \quad (\text{all } i) \quad \text{and} \quad \sum_i \beta_{ij} = 0 \quad (\text{all } j). \]

At this point, (E11.2.1) becomes
\[ \sum_{i,j} \beta_{ij} \otimes y^i \otimes y^j + \sum_{i,j} \beta_{ij} y^i \otimes y^j \otimes y^j \]
\[ = \sum_{i,j} \beta_{ij} y^i \otimes y^j \otimes y^{n-1} + \sum_{i,j} \beta_{ij} y^i \otimes y^i \otimes y^j. \]

Comparing terms in (E11.2.4) involving \( 1 \otimes 1 \otimes 1 \), we see that \( \beta_{00} = 0 \), while comparing terms involving \( 1 \otimes y^{n-1} \otimes y^{n-1} \) yields \( \beta_{n-1,n-1} = 0 \). Turning to other terms \( 1 \otimes y^i \otimes y^j \), we obtain \( \beta_{jj} + \beta_{0j} = 0 \) for all \( j \neq 0, n - 1 \), while inspection of terms involving \( 1 \otimes y^i \otimes y^{n-1} \) yields \( \beta_{i,n-1} = \beta_{0i} \) for all \( i \neq 0, n - 1 \). Finally, looking at other terms \( 1 \otimes y^i \otimes y^j \), we conclude that \( \beta_{ij} = 0 \) whenever \( i \neq 0, j \neq n - 1 \).

Set \( \gamma_s = \beta_{s,n-1} \) for all \( s \). The only coefficients \( \beta_{ij} \) which could be nonzero are the following:
\[ \beta_{0j} = \begin{cases} 
\gamma_j & (j \neq 0, n - 1) \\
\gamma_0 & (j = n - 1) 
\end{cases} \]
\[ \beta_{i,n-1} = \gamma_i & (i \neq n - 1) \]
\[ \beta_{jj} = -\gamma_j & (j \neq 0, n - 1). \]

Moreover, (E11.2.3) implies that \( \sum_s \gamma_s = 0 \). Now
\[ u = \sum_{j \neq 0,n-1} \gamma_j \otimes y^j + \gamma_0 \otimes y^{n-1} + \sum_{i \neq 0,n-1} \gamma_i y^i \otimes y^{n-1} \]
\[ = \sum_{j \neq 0,n-1} \gamma_j y^i \otimes y^j \]
\[ = \gamma_0 \otimes y^{n-1} + 1 \otimes g + g \otimes y^{n-1} - g(y \otimes y), \]
where \( g = \sum_{j \neq 0,n-1} \gamma_j y^j \in H_0 \). Note that \( g(1) + \gamma_0 = \sum_j \gamma_j = 0 \), whence \( g + \gamma_0 \in H_0 \cap K \).
The element $x' = x + g + \gamma_0 \in H$ satisfies $\pi(x') = t$ and $\rho(x') = x' \otimes 1 + 1 \otimes t$, as well as $x'y - xy' = y^n - y$. In view of (E11.2.5),
\[
\Delta(x') = x \otimes y^{n-1} + 1 \otimes x + u + g(y \otimes y) + \gamma_0 \otimes 1 = x' \otimes y^{n-1} + 1 \otimes x'.
\]
Thus, we may replace $x$ by $x'$, so that
\[
\Delta(x) = x \otimes y^{n-1} + 1 \otimes x.
\]
Therefore $x$ is skew primitive.

The work in Subsections 11.1 and 11.2 yields the following theorem.

**Theorem 11.3.** Assume $(H_1)$, and suppose that $H = k[t]$ and $H_0 = k[y^{\pm 1}]$ with $t$ primitive and $y$ grouplike. If $H$ is not commutative, then $H \cong C(n)$ for some integer $n \geq 2$. \hfill $\Box$

### 12. Proof of Theorem 0.1

Observe first that the algebras listed in Theorem 0.1(I)–(V) are affine and noetherian. They are pairwise non-isomorphic, up to $A(0, q) \cong A(0, q^{-1})$, by Proposition 1.6.

Now assume that $H$ is a Hopf algebra satisfying $(H_2)$, and that $H$ is either affine or noetherian. If $H$ is commutative, then it is necessarily affine [22], and Proposition 2.3 implies that $H$ is isomorphic to a Hopf algebra of type (Ia), (IIa), or (III) (with $q = 1$). It remains to consider the case that $H$ is not commutative.

By Theorem 3.9, $H$ has a Hopf quotient $\overline{H}$ isomorphic to either $k[t^{\pm 1}]$ with $t$ grouplike or $k[t]$ with $t$ primitive. In the first case, Theorems 6.3 and 7.6 show that $H$ is isomorphic to a Hopf algebra of type (Ib), (III) (with $q \neq 1$), or (IV). In the second, Propositions 9.4 and 9.7 imply that $H_0 = qH$ and either $H_0 = k[y^{\pm 1}]$ with $y$ grouplike or $H_0 = k[y]$ with $y$ primitive. Theorems 10.3 and 11.3 then show that $H$ is isomorphic to a Hopf algebra of type (IIIb) or (V). \hfill $\Box$

### 13. General properties: Proof of Proposition 0.2

The Hopf algebras classified in Theorem 0.1 enjoy a number of common properties, which we summarized in Proposition 0.2. The *PI-degree*, denoted by $p. i. \deg(A)$, of an algebra $A$ satisfying a polynomial identity is defined to the minimal integer $n$ such that $A$ can be embedded in an $n \times n$-matrix algebra $M_n(C)$ over a commutative ring $C$. (If $A$ does not satisfy a polynomial identity, then $p. i. \deg(A) = \infty$.) Definitions of other relevant terms can be found, for instance, in [10, 14, 16, 23, 20].

**Proof of Proposition 0.2.** (a) As an algebra, $H$ can be written as either a skew-Laurent ring $R[x^{\pm 1}; \sigma]$ or a differential operator ring $R[x; \delta]$, where $R$ is a commutative noetherian domain of Krull dimension 1. We claim that in the first case, $R$ always has a height 1 prime ideal whose $\sigma$-orbit is finite, and that in the second, $R$ has a height 1 prime ideal which is $\delta$-stable. It then follows from [20, Theorem 6.9.13] that $\text{Kdim } H = 2$.

In case (I) (following the numbering of Theorem 0.1), we can write $H = R[x^{\pm 1}; \sigma]$ with $R = k[y^{\pm 1}]$ and either $\sigma(y) = y$ or $\sigma(y) = y^{-1}$. In either case, the $\sigma$-orbit of the height 1 prime $(y - 1)$ contains at most two points. For case (III), we have $H = R[x^{\pm 1}; \sigma]$ with $R = k[y]$ and $\sigma(y) = qy$. Here, the height 1 prime $(y)$ is $\sigma$-stable. Case (IV) has the form $H = R[x^{\pm 1}; \sigma]$ with $R = k[y_1, \ldots, y_s]$ and $\sigma(y_i) = q^{\gamma_i}y_i$ for all $i$. In this case, $(y_1, \ldots, y_s)$ is a $\sigma$-stable height 1 prime of $R$. 
Turning to case (II), we have \( H = R[x; \delta] \) with \( R = k[y] \) and either \( \delta(y) = 0 \) or \( \delta(y) = y \). In either case, \( \langle y \rangle \) is a \( \delta \)-stable height 1 prime. In case (V), \( H = R[x; \delta] \) with \( R = k[y^{\pm 1}] \) and \( \delta(y) = y^n - y \). Here, \( \langle y - \lambda \rangle \) is a \( \delta \)-stable height 1 prime for any \((n-1)\text{st} \) root of unity \( \lambda \in k^\times \).

This establishes the above claim, and thus \( \text{Kdim} \, H = 2 \). Moreover, when \( \text{gldim} \, R = 1 \), it follows from [20, Theorem 7.10.3] that \( \text{gldim} \, H = 2 \). This covers cases (I), (II), (III), (V). In case (IV), \( \text{gldim} \, H = \infty \), as already noted in the proof of Proposition 1.6.

(b) By [25, Theorems 0.1, 0.2], every affine noetherian PI Hopf algebra is Auslander-Gorenstein and GK-Cohen-Macaulay. In particular, this holds for cases (I) and (IV), and the commutative subcase of (IIa). In these cases, the GK-Cohen-Macaulay condition implies that the trivial module \( H \) is Auslander-regular, because \( \text{gldim} \, H = 2 \). Moreover, when \( \text{gldim} \, R = 1 \), it follows that \( \text{injdim} \, H = 2 \). In the remaining cases, \( \text{gldim} \, H = 2 \), and we get \( \text{injdim} \, H = 2 \) because \( H \) is noetherian.

In the noncommutative subcase of (IIb), \( H = k[y] \langle x; yd/dy \rangle \). Since \( k[y] \) is Auslander-regular, GK-Cohen-Macaulay, and connected graded, the desired properties pass to \( H \) by [10, Theorem I.15.3, Lemma I.15.4(a)].

For case (III), view \( H \) as the localization of \( A = k[x] \langle y; \sigma \rangle \) with respect to the regular normal element \( x \), where \( \sigma(x) = qx \). As in the previous paragraph, \( A \) is Auslander-regular and GK-Cohen-Macaulay by [10, Theorem I.15.3, Lemma I.15.4(a)]. These properties pass to \( H \) by [10, Theorem II.9.11(b)].

In case (V), finally, observe that \( H \cong A/\langle yz - 1 \rangle \) where

\[
A = k[y, z][x; (y^n - y)(yz\partial/\partial y - z^2\partial/\partial z)].
\]

As above, \( A \) is Auslander-regular and GK-Cohen-Macaulay by [10, Theorem I.15.3, Lemma I.15.4(a)]. Since \( yz - 1 \) is a central regular element in \( A \), [10, Lemma I.15.4(b)] implies that \( H \) is Auslander-Gorenstein and GK-Cohen-Macaulay. (Actually, \( H \) is Auslander-regular, because \( \text{gldim} \, H = 2 \).)

(c) Normal separation holds trivially in the commutative case, and it holds for rings module-finite over their centers by [16, Proposition 9.1]. This covers cases (I) and (IV), the commutative subcase of (IIa), and the subcase of (III) when \( q \) is a root of unity. The solvable enveloping algebra in case (IIb) is well known to have normal separation [16, p. 217, second paragraph].

Next, consider case (III), with \( q \) not a root of unity. Then \( y \) is normal in \( H \), and the localization \( H[y^{-1}] \), a generic quantum torus, is a simple ring [16, Corollary 1.18]. It follows that all nonzero prime ideals of \( H \) contain \( y \). Since \( H/\langle y \rangle \) is commutative, normal separation follows.

Case (V) remains. By [16, Proposition 2.1], the localization \( k(y)[x; \delta] \) of \( H \) is simple, and hence all nonzero prime ideals of \( H \) have nonzero contractions to \( k[y^{\pm 1}] \). On the other hand, any prime of \( H \) contracts to a prime \( \delta \)-ideal of \( k[y^{\pm 1}] \), by [16, Theorem 3.22]. Since \( \delta(y) = y^n - y \), the nonzero prime \( \delta \)-ideals of \( k[y^{\pm 1}] \) are the ideals \( \langle y - \lambda \rangle \) for \( \lambda \in k^\times \) with \( \lambda^{n-1} = 1 \). For any such \( \lambda \), the commutator \( [x, y - \lambda] = y^n - y \) is a multiple of \( y - \lambda \), from which we see that \( y - \lambda \) is normal in \( H \). Moreover, \( H/\langle y - \lambda \rangle \) is commutative. As in the previous case, we conclude that \( \text{Spec} \, H \) has normal separation.

(d) This follows from part (c) by [16, Theorem 12.17].

(e) The algebra \( H \) is affine as well as noetherian by Theorem 0.1, Auslander-Gorenstein and GK-Cohen-Macaulay with finite GK-dimension by parts (a),(b).
above, and Spec $H$ has normal separation by part (c). Therefore part (e) follows from [14, Theorem 1.6].

(f) In all of our cases, $H$ can be viewed as a constructible algebra in the sense of [20, §9.4.12], and therefore $H$ satisfies the Nullstellensatz [20, Theorem 9.4.21]. It follows that locally closed prime ideals of $H$ are primitive, and primitive ideals of $H$ are rational [10, Lemma II.7.15]. On the other hand, when $H$ is a PI-algebra, Posner’s theorem implies that rational primes of $H$ are maximal, and therefore locally closed. Thus, we obtain the Dixmier-Moeglin equivalence in cases (I) and (IV), and in case (III) when $q$ is a root of unity. (We have given the proof in this manner because we could not locate a reference in the literature for the relatively well known fact that affine noetherian PI-algebras satisfy this equivalence.)

In Dixmier’s development of the equivalence for enveloping algebras, the solvable case is covered in [13, Theorem 4.5.7]. This gives us the equivalence in case (II).

Finally, consider case (V). Our work in part (c) above shows that all nonzero prime factors of $H$ are commutative. Thus, all nonzero rational prime ideals of $H$ are maximal, and thus locally closed. It only remains to note that the zero ideal is locally closed, because any nonzero prime contains one of the finitely many primes $\langle y - \lambda \rangle$ for $\lambda \in k^\times$ with $\lambda^{n-1} = 1$.

(g) In all cases of Theorem 0.1, $H$ is generated as a $k$-algebra by grouplike and skew primitive elements. It thus follows from [23, Lemma 5.5.1] that $H$ is pointed.

(h) By Proposition 3.4(b) Ext$^1_H(H, H)$ is either 1-dimensional or 2-dimensional. If dim Ext$^1_H(H, H)$ is 2 which is the GK-dimension of $H$, then Proposition 3.6 says that $H$ is commutative. Therefore Ext$^1_H(H, H)$ is 1-dimensional if and only if $H$ is not commutative. □

Remark 13.1. One might have expected these Hopf algebras to satisfy other properties such as unique factorization. However, any unique factorization ring in the sense of Chatters and Jordan [12] is a maximal order [12, Theorem 2.4], and the Hopf algebras $B$ of Construction 1.2 are not – simply observe that $B$ is contained in the larger, and equivalent, order $k[y][x^\pm 1; \sigma]$. Thus, $B$ is not a unique factorization ring.

Remark 13.2. Our original expectation was that for the Hopf algebras $H$ in Theorem 0.1, the integral order io($H$) [19, Definition 2.2] would equal the PI-degree of $H$. It can be shown that io($H$) = p.i. deg($H$) in cases (I), (II), (III), and (V). In case (IV), however, it turns out that p.i. deg($H$) = $\ell$ while io($H$) = $\ell$/gcd($d$, $\ell$) where

$$d := \ell + m(s - 1) - \sum_{i=1}^s m_i$$

(in the notation of Construction 1.2). For example, if $H = B(7, 1, 3, 5, q)$, then p.i. deg($H$) = $\ell = 105$ while $d = 112$ and so io($H$) = 15.

In any case, io($H$) divides p.i. deg($H$).
Acknowledgement

We thank K.A. Brown and the referee for helpful comments, suggestions, and discussions.

References


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