A DIXMIER-MOEGLIN EQUIVALENCE FOR
POISSON ALGEBRAS WITH TORUS ACTIONS

K. R. GOODEARL

ABSTRACT. A Poisson analog of the Dixmier-Moeglin equivalence is established for any affine Poisson algebra R on which an algebraic torus H acts rationally, by Poisson automorphisms, such that R has only finitely many prime Poisson H-stable ideals. In this setting, an additional characterization of the Poisson primitive ideals of R is obtained – they are precisely the prime Poisson ideals maximal in their H-strata (where two prime Poisson ideals are in the same H-stratum if the intersections of their H-orbits coincide). Further, the Zariski topology on the space of Poisson primitive ideals of R agrees with the quotient topology induced by the natural surjection from the maximal ideal space of R onto the Poisson primitive ideal space. These theorems apply to many Poisson algebras arising from quantum groups.

The full structure of a Poisson algebra is not necessary for the results of this paper, which are developed in the setting of a commutative algebra equipped with a set of derivations.

INTRODUCTION

Motivated by existing and conjectured roles of Poisson structures in the theory of quantum groups, we address some problems in the ideal theory of Poisson algebras. Recall, for example, that Hodges and Levasseur [**15**, **16**] and Joseph [**18**] have constructed bijections between the primitive ideal space of the quantized coordinate ring $\mathcal{O}_q(G)$ of a semisimple Lie group G and the set of symplectic leaves in G corresponding to a Poisson structure which arises from the quantization process. In this "standard" case, the symplectic leaves in G are locally closed subvarieties, and they correspond to the Poisson primitive ideals in the classical coordinate ring $\mathcal{O}(G)$. Moreover, it is conjectured that the primitive ideal space of $\mathcal{O}_q(G)$ and the Poisson primitive ideal space of $\mathcal{O}(G)$, with their respective Zariski topologies, are homeomorphic. Results and conjectures such as these draw a focus on the ideal theory of Poisson algebras. Our main concern in this paper is the problem of identifying the Poisson primitive ideals in a Poisson algebra. To that end, we establish a Poisson version of the famous Dixmier-Moeglin equivalence, which applies to Poisson algebras equipped with suitable torus actions, a hypothesis satisfied in many examples of interest. We also look at the Zariski topology on the Poisson primitive spectrum of a

Poisson algebra R, and prove that it coincides with a natural quotient topology from the maximal ideal space of R in the case that the Poisson Dixmier-Moeglin equivalence holds.

In order to provide further detail, a few definitions are in order. A *Poisson algebra* is a commutative algebra R over a field k (usually assumed to have characteristic zero) together with an antisymmetric bilinear map $\{-,-\} : R \times R \rightarrow R$ satisfying two key properties: the Jacobi identity, so that R together with $\{-,-\}$ forms a Lie algebra, and the Leibniz rule, meaning that the maps $\{a,-\}$ (for $a \in R$) are derivations on R. The ideals I of R for which $\{R, I\} \subseteq I$ are called *Poisson ideals*, and within each ideal J of R there is a largest Poisson ideal, which we call (following [**3**]) the *Poisson core* of J. The *Poisson primitive ideals* of R, finally, are the Poisson cores of the maximal ideals.

Recall that a noetherian algebra A is said to satisfy the *Dixmier-Moeglin equivalence* provided the following types of prime ideals in A coincide: the primitive ideals; the *locally closed* prime ideals, meaning those which constitute locally closed points in the prime spectrum of A; and the *rational* prime ideals, meaning those prime ideals P in A such that the center of the Goldie quotient ring of A/P is algebraic over the base field. This equivalence was first established by Dixmier [**6**] and Moeglin [**20**] for the enveloping algebra $U(\mathfrak{g})$ of any finite dimensional complex Lie algebra $\mathfrak{g}$. Among other settings where the equivalence has been established is a class of algebras equipped with torus actions studied by Letzter and the author [**12**]. Our present work uses many ideas from [**12**], but we follow the route laid out in [**1**, Chapters II.1–II.3, II.7–II.8] (see [**1**, Theorem II.8.4] for the equivalence result).

Taking the natural Poisson analogs of the above ideas, one says that a Poisson algebra R satisfies the *Poisson Dixmier-Moeglin equivalence* provided the Poisson primitive ideals of R are precisely the locally closed points of the prime Poisson spectrum, and also precisely those prime Poisson ideals P such that the Poisson center of the quotient field of R/P is algebraic over the base field. Some examples in which this equivalence holds have been given by Oh [**21**, Theorem 2.4, Proposition 2.13], and Brown and Gordon have shown that it holds for affine Poisson algebras with only finitely many Poisson primitive ideals [**3**, Lemma 3.4]. Our primary goal here is to establish the Poisson Dixmier-Moeglin equivalence for Poisson algebras R for which there is an algebraic torus H, acting rationally on R by Poisson automorphisms, such that R has only finitely many prime Poisson ideals stable under H. As in [**12**], we obtain an additional equivalence in our main theorem, based on a stratification of the prime Poisson spectrum of R arising from the action of H. Namely, the Poisson primitive ideals of R are (under the given hypotheses) exactly those prime Poisson ideals which are maximal in their H-strata. (See below for precise definitions.)

In any Poisson algebra R, the process of taking Poisson cores defines a canonical surjection from the maximal ideal space, $\max R$, onto the Poisson primitive spectrum, P.prim R. We show that if R satisfies the Poisson Dixmier-Moeglin equivalence, then the Zariski topology on P.prim R coincides with the quotient topology induced by the above surjection. In particular, if R is the coordinate ring of an affine variety V over an algebraically closed field, P.prim R becomes a topological quotient of V. Combining this result with our main theorem, we thus obtain a large class of Poisson algebras in which the Poisson primitive spectrum is a topological quotient of the maximal ideal space. The methods are

taken from joint work with Letzter on the quotient topology problem for primitive spectra of quantized algebras [**13**].

We give a minimal sketch of Poisson structures and symplectic leaves in Section 4. For readers interested in further background on these and related subjects, we mention the following small sample of the available literature: [**1**, Chapter III.5], [**4**, Chapter 3], [**8**, Lecture 1], [**19**, Chapter 1]. For some algebraic approaches, see [**9**], [**10**].

The Lie algebra structure provided by a Poisson bracket plays no role in our proofs; in fact, all that is needed is the set of derivations $\{a, -\}$. Consequently, our results can be stated and proved in the context of a commutative algebra equipped with a set of derivations. We derive everything at this level of generality, in the first three sections of the paper. In the final section, we specialize to the case of Poisson algebras, and discuss various examples, in some of which the Poisson primitive ideals correspond to the symplectic leaves in a complex affine Poisson variety.

Let us fix a base field:

Throughout the paper, k will denote a field of characteristic zero.

1. Ideals stable under derivations

As noted in the Introduction, our basic object of study will be a commutative algebra equipped with a set of derivations. Readers who wish to concentrate on Poisson algebras could do so by first comparing the beginnings of this section and Section 4, and then substituting the prefix "Poisson" for "Δ-" in what follows.

The following terminology and notation will be convenient. Many of these concepts are standard, and the basic properties recorded in Lemma 1.1 mostly hold without any assumption of commutativity, but for consistency we shall impose commutativity throughout.

Definitions. A *commutative differential k-algebra* is a pair (R, Δ) where R is a commutative k-algebra and Δ a set of k-linear derivations on R. We make no assumptions about any structure on Δ – in particular, it need not be a Lie subalgebra nor even a linear subspace of $\mathrm{Der}_k(R)$. The *Δ-center* of R is the set

$$Z_\Delta(R) = \{r \in R \mid \delta(r) = 0 \text{ for all } \delta \in \Delta\},$$

a k-subalgebra of R.

A *Δ-ideal* of R is any ideal I of R that is stable under Δ, that is, $\delta(I) \subseteq I$ for all $\delta \in \Delta$. The *Δ-core* of an arbitrary ideal J in R is the largest Δ-ideal contained in J, which we denote $(J : \Delta)$. It may be described as follows [**7**, Lemma 3.3.2]:

$$(J : \Delta) = \{r \in J \mid \delta_1 \delta_2 \cdots \delta_n(r) \in J \text{ for all } \delta_1, \ldots, \delta_n \in \Delta\}.$$

Recall that the concept of a Δ-prime ideal is obtained by replacing arbitrary ideals by Δ-ideals in the definition of a prime ideal: A *Δ-prime ideal* of R is any proper Δ-ideal Q

of R such that whenever I and J are Δ-ideals of R with $IJ \subseteq Q$, either $I \subseteq Q$ or $J \subseteq Q$. In particular, $(P : \Delta)$ is Δ-prime for all prime ideals P.

Adapting terminology from the theory of Poisson algebras, we define a Δ-*primitive ideal* of R to be any Δ-ideal of the form $(M : \Delta)$ where M is a maximal ideal of R.

The Δ-*prime spectrum* of R is the set Δ-spec R of all Δ-prime ideals, equipped with the natural Zariski topology. Similarly, the Δ-*primitive spectrum* is the subset Δ-prim R consisting of all Δ-primitive ideals, also equipped with the Zariski topology.

Lemma 1.1. *Let (R, Δ) be a commutative differential k-algebra.*
  (a) $(P : \Delta)$ *is prime for all prime ideals P of R.*
  (b) *Every Δ-primitive ideal of R is prime.*
  (c) *Every prime ideal minimal over a Δ-ideal of R is a Δ-ideal.*
  (d) *If R is noetherian, every Δ-prime ideal of R is prime.*
  (e) *If R is affine over k, every Δ-prime ideal of R is an intersection of Δ-primitive ideals.*

Proof. (a) [**7**, Lemma 3.3.2].
  (b)(c) These are immediate from (a).
  (d) Let Q be a Δ-prime of R. There exist prime ideals $Q_1, \ldots, Q_n$ minimal over Q such that $Q_1 Q_2 \cdots Q_n \subseteq Q$. The Q_i are Δ-ideals by (c), so the Δ-primeness of Q implies that some $Q_i = Q$.
  (e) Let Q be a Δ-prime of R; then Q is prime by (d). By the Nullstellensatz, Q is an intersection of maximal ideals M_i, and so $Q = \bigcap_i (M_i : \Delta)$. $\square$

In view of Lemma 1.1(a) and the remarks above,

$$\Delta\text{-prim}\, R \subseteq \Delta\text{-spec}\, R \subseteq \text{spec}\, R$$

for any commutative noetherian differential k-algebra (R, Δ). Just as primitive ideals are more difficult to identify purely ideal-theoretically than prime ideals, Δ-primitive ideals are less accessible than Δ-prime ideals. A natural approach is to try to find the Δ-prime ideals first (even though there are more of them), and then to develop criteria to tell which Δ-prime ideals are Δ-primitive. Topological criteria (involving the space Δ-spec R) and algebraic criteria (involving quotients of R modulo Δ-prime ideals) are both useful. The key properties are given in terms of the following concepts.

Definitions. A *locally closed point* in a topological space X is any point which is (relatively) closed in some neighborhood. Note that a point $x \in X$ is locally closed if and only if $\{x\}$ is the intersection of an open and a closed set; hence, x is locally closed if and only if $\{x\}$ is (relatively) open in its closure.

A Δ-prime ideal P in a commutative noetherian differential k-algebra (R, Δ) is Δ-*rational* provided the field $Z_\Delta(\text{Fract}\, R/P)$ is algebraic over k, where $\text{Fract}\, R/P$ denotes the quotient field of R/P (recall from Lemma 1.1(d) that R/P is a domain).

The basic relations between these concepts and Δ-primitivity were given in the Poisson case by Oh [**21**, Propositions 1.7, 1.10]. Since the proofs are slightly simpler in our case, and there is a change of notation in the generalization to (R, Δ), we provide a sketch for the reader's convenience.

Proposition 1.2. *Let (R, Δ) be a commutative differential k-algebra, and assume that R is affine over k. Then every locally closed point in Δ-spec R is Δ-primitive, and every Δ-primitive ideal of R is Δ-rational.*

Proof. If P is a locally closed point in Δ-spec R, there exist ideals I and J in R such that

$$\{P\} = \{Q \in \Delta\text{-spec } R \mid Q \supseteq I\} \cap \{Q \in \Delta\text{-spec } R \mid Q \not\supseteq J\},$$

from which we see that

$$\bigcap \{Q \in \Delta\text{-spec } R \mid Q \supsetneq P\} \supseteq P + J \supsetneq P.$$

By Lemma 1.1(e), P is an intersection of Δ-primitive ideals, each of which is Δ-prime. One of these must equal P, proving that P is Δ-primitive.

Now let P be an arbitrary Δ-primitive ideal of R, and write $P = (M : \Delta)$ for some maximal ideal M of R. Since R/M is finite dimensional over k, it suffices to show that the Δ-center of the field $F = \operatorname{Fract} R/P$ embeds in R/M. After replacing R by R/P, there is no loss of generality in assuming that $P = 0$.

We claim that $Z_\Delta(F)$ is contained in the localization R_M. Given a fraction $ab^{-1} \in Z_\Delta(F)$, we have

$$0 = \delta(ab^{-1}) = \big(\delta(a)b - a\delta(b)\big)b^{-2}$$

and so $a\delta(b) = \delta(a)b$, for any $\delta \in \Delta$. Hence, $ab^{-1} = \delta(a)\delta(b)^{-1}$. Repeating this argument, we see that

$$ab^{-1} = \delta_1(a)\delta_1(b)^{-1} = \delta_2\delta_1(a)\delta_2\delta_1(b)^{-1} = \cdots = \big[\delta_n \cdots \delta_2\delta_1(a)\big]\big[\delta_n \cdots \delta_2\delta_1(b)\big]^{-1}$$

for any $\delta_1, \ldots, \delta_n \in \Delta$. Since $b \neq 0$ while $(M : \Delta) = P = 0$, there exist $\delta_1, \ldots, \delta_n \in \Delta$ such that $\delta_n \cdots \delta_2\delta_1(b) \notin M$, and the claim is proved.

Therefore we obtain a k-algebra homomorphism

$$\phi : Z_\Delta(F) \xrightarrow{\subseteq} R_M \xrightarrow{\text{quo}} R_M/MR_M \xrightarrow{\cong} R/M.$$

Since $Z_\Delta(F)$ is a field, ϕ is an embedding, as desired. $\square$

The topological relationship between the spaces of prime and Δ-prime ideals in a commutative noetherian differential algebra is given by the following result, which is a special case of [**13**, Proposition 1.7(c)]. We provide a proof in order to avoid setting up the machinery of [**13**]. Recall that a *topological quotient* of a topological space X is a space Y together with a surjection $\pi : X \to Y$ such that the topology on Y is the quotient topology induced by π, that is, the closed subsets of Y are exactly those subsets $W \subseteq Y$ for which $\pi^{-1}(W)$ is closed in X.

Theorem 1.3. *Let (R, Δ) be a commutative noetherian differential k-algebra. Then the rule $\pi(Q) = (Q : \Delta)$ defines a continuous retraction*

$$\pi : \operatorname{spec} R \to \Delta\text{-spec} R,$$

and Δ-spec R is a topological quotient of $\operatorname{spec} R$ via π.

Proof. By Lemma 1.1(d), Δ-spec $R \subseteq \operatorname{spec} R$. Thus, the given rule defines a set-theoretic retraction of $\operatorname{spec} R$ onto Δ-spec R.

Let V be a closed subset of Δ-spec R, and write $V = \{P \in \Delta\text{-spec} R \mid P \supseteq I\}$ for some ideal I of R. Since we may replace I by the intersection of the ideals in V, there is no loss of generality in assuming that I is a Δ-ideal. Hence,

$$\pi^{-1}(V) = \{Q \in \operatorname{spec} R \mid (Q : \Delta) \supseteq I\} = \{Q \in \operatorname{spec} R \mid Q \supseteq I\},$$

a closed set in $\operatorname{spec} R$. This shows that π is continuous.

Now consider a set $W \subseteq \Delta$-spec R such that $\pi^{-1}(W)$ is closed in $\operatorname{spec} R$, say

$$\pi^{-1}(W) = \{Q \in \operatorname{spec} R \mid Q \supseteq I\}$$

for some ideal I of R. Since $\pi(P) = P$ for $P \in \Delta$-spec R, we have

$$W = \pi^{-1}(W) \cap \Delta\text{-spec} R = \{P \in \Delta\text{-spec} R \mid P \supseteq I\},$$

a closed set in Δ-spec R. Therefore the topology on Δ-spec R is indeed the quotient topology induced by π. $\square$

In the situation of Theorem 1.3, the map π sends $\max R$ to Δ-prim R. This restriction is continuous because π is, and it is surjective by definition of Δ-prim R. However, it need not be a topological quotient map. For example, let $R = k[x]_{(x^2-x)}$, the localization of a polynomial ring $k[x]$ at the semimaximal ideal $(x^2 - x)$, and let $\Delta = \{x\frac{d}{dx}\}$. Then Δ-prim R consists of the two ideals $(Rx : \Delta) = Rx$ and $(R(x-1) : \Delta) = 0$. The map $M \mapsto (M : \Delta)$ from $\max R$ to Δ-prim R is a continuous bijection, but Δ-prim R is not a topological quotient of $\max R$, since $\max R$ has the discrete topology while Δ-prim R does not. A more satisfactory example would be one affine over k, but this is an open problem:

Question 1.4. *Let (R, Δ) be a commutative differential k-algebra, with R affine over k. Is the Zariski topology on Δ-prim R equal to the quotient topology induced from the continuous surjection $\max R \to \Delta$-prim R given by $M \mapsto (M : \Delta)$?*

In order for Δ-prim R to be a topological quotient of $\max R$, it suffices, by [**13**, Proposition 1.8], to have

$$(1.1) \qquad P = \bigcap \left[\pi^{-1}(\{P\}) \cap \max R \right] = \bigcap \{M \in \max R \mid (M : \Delta) = P\}$$

for all $P \in \Delta$-prim R. A sufficient condition for (1.1), in turn, would be a converse of Proposition 1.2, as shown in the proof below.

Theorem 1.5. *Let (R, Δ) be a commutative differential k-algebra, and assume that R is affine over k. Then the rule $\pi_m(Q) = (Q : \Delta)$ defines a continuous surjection*

$$\pi_m : \max R \to \Delta\text{-prim}\, R.$$

If, in addition, every Δ-primitive ideal of R is locally closed in Δ-spec R, then Δ-prim R is a topological quotient of $\max R$ via π_m.

Proof. We have already noted that π_m is a continuous surjection. Now assume that every Δ-primitive ideal of R is locally closed in Δ-spec R.

We claim that (1.1) holds for any $P \in \Delta$-prim R. By assumption, P is locally closed in Δ-spec R, and so we see (as in the proof of Proposition 1.2) that

$$\bigcap \{Q \in \Delta\text{-spec}\, R \mid Q \supsetneq P\} =: L \supsetneq P.$$

Separate the maximal ideals containing P as follows:

$$\mathcal{M}_1 = \{M \in \max R \mid M \supseteq L\} \qquad \text{and} \qquad \mathcal{M}_2 = \{M \in \max R \mid M \supseteq P \text{ but } M \not\supseteq L\};$$

then $P = (\bigcap \mathcal{M}_1) \cap (\bigcap \mathcal{M}_2)$ by the Nullstellensatz. Since P is prime and $\bigcap \mathcal{M}_1 \supseteq L \supsetneq P$, we must have $\bigcap \mathcal{M}_2 = P$. For $M \in \mathcal{M}_2$, the ideal $(M : \Delta)$ is a Δ-prime which contains P but not L, whence $(M : \Delta) = P$. This establishes (1.1).

Now consider a set $W \subseteq \Delta$-prim R such that $\pi_m^{-1}(W)$ is closed in $\max R$, say

$$\pi_m^{-1}(W) = \{M \in \max R \mid M \supseteq I\}$$

for some ideal I. If $P \in \Delta$-prim R and $P \supseteq I$, then $P = \pi_m(M)$ for some $M \in \max R$, whence $M \supseteq (M : \Delta) = P \supseteq I$ and so $M \in \pi_m^{-1}(W)$, yielding $P = \pi_m(M) \in W$. Conversely, if $P \in W$, then $\pi_m^{-1}(\{P\}) \subseteq \pi_m^{-1}(W)$ and so every ideal in $\pi_m^{-1}(\{P\})$ contains I. By (1.1), the intersection of the ideals in $\pi_m^{-1}(\{P\})$ equals P, whence $P \supseteq I$. Thus

$$W = \{P \in \Delta\text{-prim}\, R \mid P \supseteq I\},$$

a closed set in Δ-prim R. Therefore the topology on Δ-prim R is indeed the quotient topology induced by π_m. $\square$

2. Graded fields

Our main results will involve commutative differential algebras graded by (free abelian) groups, and certain localizations of these algebras will be what are known as "graded fields" – this refers to the graded analog of the concept of a field (see below), rather than to the idea of a field equipped with a grading. Here we develop some properties of commutative differential graded fields. In particular, we show that in such an algebra the Δ-prime ideals correspond precisely to the prime ideals of the Δ-center.

Definitions. Let G be a group, which we assume to be abelian and written multiplicatively. Recall that a *G-graded k-algebra* is a k-algebra R equipped with a vector space direct sum decomposition $R = \bigoplus_{g \in G} R_g$ such that $1 \in R_1$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The algebra R is called *strongly G-graded* if $R_g R_h = R_{gh}$ for all $g, h \in G$. The subspaces R_g are called the *homogeneous components* of R (with respect to the given grading), and the elements of a given component R_g are said to be *homogeneous of degree g*. Finally, R is called a *graded field* provided R is a domain and all its nonzero homogeneous elements are invertible. In that case, R is necessarily strongly graded.

A linear map $f : R \to R$ is *homogeneous of degree d* for some $d \in G$ provided $f(R_g) \subseteq R_{gd}$ for all $g \in G$.

To avoid repeating lengthy hypotheses, we define a *G-graded differential k-algebra* to be a pair (R, Δ) such that

   (1) R is a G-graded k-algebra;
   (2) Δ is a linear subspace of $\operatorname{Der}_k(R)$;
   (3) $\Delta = \bigoplus_{d \in G} \Delta_d$ where each Δ_d consists of homogeneous derivations of degree d.

(This concept is quite different than that of a "differential graded" algebra.)

The simplest example of a G-graded field is the group algebra kG, equipped with its standard grading, for any torsionfree abelian group G. (The nonzero homogeneous elements of kG are always invertible, but kG is not a domain if G has torsion.) Versions of the results in this section have been proved for Poisson structures on kG, where G is free abelian of finite rank, by Oh and Park [**22**, Lemma 2.2, Theorem 2.3] and Vancliff [**25**, Lemma 1.2].

Lemma 2.1. *Let G be an abelian group and (R, Δ) a commutative G-graded differential k-algebra. Assume that R is a graded field.*

(a) The Δ-center $Z_\Delta(R)$ is a homogeneous subalgebra of R, strongly graded by the subgroup $G_Z = \{x \in G \mid Z_\Delta(R) \cap R_x \neq 0\}$ of G.

(b) As $Z_\Delta(R)$-modules, $Z_\Delta(R)$ is a direct summand of R.

(c) Suppose that G_Z is free abelian of finite rank, with basis $\{g_1, \ldots, g_n\}$. Choose a nonzero element $z_j \in Z_\Delta(R) \cap R_{g_j}$ for each j. Then $Z_\Delta(R)$ is a Laurent polynomial ring of the form

$$Z_\Delta(R) = Z_\Delta(R_1)[z_1^{\pm 1}, \ldots, z_n^{\pm 1}],$$

where the coefficient ring $Z_\Delta(R_1) = Z_\Delta(R) \cap R_1$ is a field.

Proof. (a) Since a homogeneous derivation δ on R maps the homogeneous components of an element $r \in R$ into distinct homogeneous components of R, we see that δ cannot vanish on r unless it vanishes on the components of r. Hence, $Z_\Delta(R)$, which equals the intersection of the kernels of the homogeneous derivations in Δ, must be a homogeneous subalgebra of R. Consequently, $Z_\Delta(R) = \bigoplus_{x \in G_Z} Z_\Delta(R) \cap R_x$ where G_Z is the subset of G defined above. Since R is a graded field, it is clear that G_Z must be a subgroup of G.

(b) and (c) are proved exactly as [**12**, Lemma 6.3] (cf. [**1**, Lemma II.3.7]). $\square$

Proposition 2.2. *Let G be an abelian group and (R, Δ) a commutative G-graded differential k-algebra. Assume that R is a graded field. Then contraction $(I \mapsto I \cap Z_\Delta(R))$ and extension $(J \mapsto RJ)$ provide inverse isomorphisms between the lattice of Δ-ideals of R and the lattice of ideals of $Z_\Delta(R)$.*

Proof. It suffices to prove that

  (a) $I = R(I \cap Z_\Delta(R))$ for any Δ-ideal I of R;
  (b) $J = (RJ) \cap Z_\Delta(R)$ for any ideal J of $Z_\Delta(R)$.

Statement (b) is clear from Lemma 2.1(b).

(a) Set $J = I \cap Z_\Delta(R)$, and suppose that $I \neq RJ$. Choose an element $r \in I \setminus RJ$ with support $\{x_1, \ldots, x_n\}$ of minimal cardinality, and write $r = r_1 + \cdots + r_n$ for some nonzero elements $r_i \in R_{x_i}$.

We claim that there cannot exist a nonzero element $s \in I$ whose support is contained in $\{x_2, \ldots, x_n\}$. If such an element does exist, then after possibly renumbering, we may assume that the support of s includes x_n. Write $s = s_2 + \cdots + s_n$, with each $s_i \in R_{x_i}$ and $s_n \neq 0$. The element $t = r_n^{-1}r - s_n^{-1}s$ is then an element in I with support contained in $\{x_n^{-1}x_1, \ldots, x_n^{-1}x_{n-1}\}$. By the minimality of n, both s and t must lie in RJ. But then $r \in RJ$, contradicting our assumptions. This establishes the claim.

Now set $r' = r_1^{-1}r$, which is an element of I with support $\{1, x_1^{-1}x_2, \ldots, x_1^{-1}x_n\}$ and identity component 1. Then $r' \in I \setminus RJ$, and in particular $r' \notin Z_\Delta(R)$. Hence, there is some $\delta \in \Delta$ such that $\delta(r') \neq 0$. Because of our hypotheses, we may assume that δ is homogeneous of some degree $d \in G$. Since $\delta(1) = 0$, the support of $\delta(r')$ is contained in $\{x_1^{-1}x_2 d, \ldots, x_1^{-1}x_n d\}$. There is some $i \geq 2$ such that $\delta(r_1^{-1}r_i) \neq 0$, and then $u = r_i^{-1}\delta(r_1^{-1}r_i)$ is a nonzero homogeneous element of degree $x_1^{-1}d$. But then $u^{-1}\delta(r')$ is a nonzero element of I with support contained in $\{x_2, \ldots, x_n\}$, contradicting the claim above.

Therefore $I = RJ$. $\square$

Corollary 2.3. *Let G be an abelian group and (R, Δ) a commutative G-graded differential k-algebra. Assume that R is a graded field. Then contraction and extension provide mutually inverse homeomorphisms between Δ-spec R and $\operatorname{spec} Z_\Delta(R)$. Moreover, if R is affine over k, contraction and extension provide mutually inverse homeomorphisms between Δ-prim R and $\max Z_\Delta(R)$.*

Proof. Suppose that contraction maps Δ-spec R to $\operatorname{spec} Z_\Delta(R)$, and that extension maps $\operatorname{spec} Z_\Delta(R)$ to Δ-spec R. Then Proposition 2.2 implies that these restricted maps are mutually inverse bijections. Further, since contraction and extension preserve inclusions, the proposition shows that the restricted maps are both continuous, hence homeomorphisms.

Consider $Q \in \Delta$-spec R. If I and J are ideals of $Z_\Delta(R)$ such that $IJ \subseteq Q \cap Z_\Delta(R)$, then RI and RJ are Δ-ideals of R such that $(RI)(RJ) \subseteq Q$. Hence, $RI \subseteq Q$ or $RJ \subseteq Q$, and so $I \subseteq Q \cap Z_\Delta(R)$ or $J \subseteq Q \cap Z_\Delta(R)$. This shows that $Q \cap Z_\Delta(R)$ lies in $\operatorname{spec} Z_\Delta(R)$.

Conversely, consider $P \in \operatorname{spec} Z_\Delta(R)$. If I and J are Δ-ideals of R such that $IJ \subseteq RP$, then by Proposition 2.2, $I = RI'$ and $J = RJ'$ where $I' = I \cap Z_\Delta(R)$ and $J' = J \cap Z_\Delta(R)$. Then $I'J' \subseteq RP \cap Z_\Delta(R) = P$, whence $I' \subseteq P$ or $J' \subseteq P$, and so $I \subseteq RP$ or $J \subseteq RP$. This shows that $RP \in \Delta$-spec R.

Therefore contraction and extension do map Δ-spec R and spec $Z_\Delta(R)$ into each other, as desired.

Now assume that R is affine over k. As above, it will suffice to show that contraction and extension map Δ-prim R and max $Z_\Delta(R)$ into each other.

If $Q \in \Delta$-prim R, then $Q = (M : \Delta)$ for some maximal ideal M of R. Since R/M is finite dimensional over k, the contraction $M \cap Z_\Delta(R)$ is a maximal ideal of $Z_\Delta(R)$. By Proposition 2.2, $Q' = R(M \cap Z_\Delta(R))$ is a maximal proper Δ-ideal of R, and $Q' \cap Z_\Delta(R) = M \cap Z_\Delta(R)$. Since $Q' \subseteq M$, we must have $Q' = (M : \Delta) = Q$. Thus $Q \cap Z_\Delta(R) = M \cap Z_\Delta(R)$, a member of max $Z_\Delta(R)$.

Finally, if $N \in \max Z_\Delta(R)$, then RN is a maximal proper Δ-ideal of R. Hence, $RN = (M : \Delta)$ for any maximal ideal $M \supseteq RN$, and so $RN \in \Delta$-prim R.

Therefore contraction and extension do map Δ-prim R and max $Z_\Delta(R)$ into each other, as desired. $\square$

In the situations of interest to us, gradings on algebras arise from rational actions of algebraic tori, as follows. For further detail, see [**1**, Chapter II.2].

Definitions. An *algebraic torus over* k is a group of the form $H = (k^\times)^r$, where r is a nonnegative integer, called the *rank* of H. (Strictly speaking, H should be called "the group of k-rational points of the algebraic group $(\overline{k}^\times)^r$.") The group H is also an algebraic group, based on its natural structure as an algebraic variety (specifically, a locally closed subset of the affine space k^r). A *character* of H is any group homomorphism $H \to k^\times$, and the *rational characters* of H are those which are also morphisms of algebraic varieties. The set $\widehat{H}$ of rational characters of H is a group under pointwise multiplication of functions. (Note: Because $\widehat{H}$ is abelian, some authors prefer to write it additively.) In fact, $\widehat{H}$ is free abelian of rank r; a basis is given by the r projection maps $H = (k^\times)^r \to k^\times$. Having written $\widehat{H}$ as a multiplicative group, questions of independence take an exponential form. Specifically, elements $f_1, \ldots, f_d \in \widehat{H}$ are $\mathbb{Z}$-*linearly independent* if and only if the only integers $m_1, \ldots, m_d$ for which $f_1^{m_1} f_2^{m_2} \cdots f_d^{m_d} = 1$ are $m_1 = \cdots = m_d = 0$.

Suppose that H acts on a k-algebra R by k-algebra automorphisms. A nonzero element $r \in R$ is an *H-eigenvector* provided $h(r) \in kr$ for all $h \in H$. In that case, there is a unique character f of H such that $h(r) = f(h)r$ for all $h \in H$, and f is called the *H-eigenvalue* of r. The collection of all H-eigenvectors in R with H-eigenvalue f, together with 0, is a linear subspace called the *H-eigenspace of R with H-eigenvalue f*. Note that the action of H on R induces an action on $\mathrm{Der}_k(R)$ by k-linear automorphisms, where $h.\delta = h \circ \delta \circ h^{-1}$ for $h \in H$ and $\delta \in \mathrm{Der}_k(R)$.

The action of H on R is *rational* provided

(1) The algebra R is the direct sum of its H-eigenspaces (i.e., the action of H on R is *semisimple*);
(2) The H-eigenvalues for the H-eigenspaces in R are all rational.

(This definition of a rational action is valid only for actions of algebraic tori; see [**1**, Definitions II.2.6] for the general concept, and [**1**, Theorem II.2.7] for the equivalence of

the two definitions.) When we have a rational action of H on R, we obtain a decomposition

$$R = \bigoplus_{f \in \widehat{H}} R_f$$

where R_f denotes the H-eigenspace of R with H-eigenvalue f. This decomposition turns R into an $\widehat{H}$-graded k-algebra (cf. [**1**, §II.2.10]).

Finally, suppose that (R, Δ) is a differential k-algebra. We shall say that H *acts rationally on* (R, Δ) provided

(1) We are given a rational action of H on R by k-algebra automorphisms;
(2) Δ is a linear subspace of $\mathrm{Der}_k(R)$, stable under the induced H-action;
(3) Δ is the direct sum of its H-eigenspaces, and the corresponding H-eigenvalues are all rational.

Observe that if $\delta \in \Delta$ is an H-eigenvector with H-eigenvalue d, then δ is homogeneous of degree d with respect to the $\widehat{H}$-grading on R. Namely, since $h\delta h^{-1} = d(h)\delta$ for $h \in H$, we have

$$h\big(\delta(r)\big) = d(h)\delta\big(h(r)\big) = d(h)\delta\big(f(h)r\big) = (df)(h)\delta(r)$$

for $f \in \widehat{H}$ and $r \in R_f$, whence $\delta(R_f) \subseteq R_{df}$. Thus, when H acts rationally on (R, Δ), the pair (R, Δ) becomes an $\widehat{H}$-graded differential k-algebra.

Proposition 2.4. *Let (R, Δ) be a commutative differential k-algebra and $H = (k^\times)^r$ an algebraic torus acting rationally on (R, Δ). Assume that R is a graded field (with respect to its $\widehat{H}$-grading).*

(a) The Δ-center $Z_\Delta(R)$ is a Laurent polynomial ring, in at most r indeterminates, over the fixed field $Z_\Delta(R)^H$, which coincides with the fixed field $Z_\Delta(\mathrm{Fract}\,R)^H$. The indeterminates can be chosen to be H-eigenvectors with $\mathbb{Z}$-linearly independent H-eigenvalues.

(b) Every Δ-ideal of R is generated by its intersection with $Z_\Delta(R)$.

(c) Contraction and extension provide mutually inverse homeomorphisms between the spaces Δ-$\mathrm{spec}\,R$ and $\mathrm{spec}\,Z_\Delta(R)$. If R is affine over k, then contraction and extension also provide mutually inverse homeomorphisms between Δ-$\mathrm{prim}\,R$ and $\max Z_\Delta(R)$.

Proof. Except for the equality $Z_\Delta(R)^H = Z_\Delta(\mathrm{Fract}\,R)^H$, these statements follow from Lemma 2.1, Proposition 2.2, and Corollary 2.3.

The inclusion $Z_\Delta(R)^H \subseteq Z_\Delta(\mathrm{Fract}\,R)^H$ is clear. Given an element $u \in Z_\Delta(\mathrm{Fract}\,R)^H$, observe that the set $I = \{r \in R \mid ru \in R\}$ is a nonzero H-stable ideal of R. Since H acts semisimply on R, it follows that I is spanned by H-eigenvectors, that is, I is a homogeneous ideal (with respect to the $\widehat{H}$-grading on R). Since R is a graded field, $I = R$, whence $u \in R$. Therefore $u \in Z_\Delta(R)^H$, as desired. $\square$

3. STRATIFICATION

We now investigate the general case of a commutative differential k-algebra equipped with a rational torus action. The results of the previous section will be applied to suitable localizations of factor algebras.

Definitions. Let H be a group acting on a ring R by automorphisms. An H-*ideal* of R is any ideal I of R that is stable under H, that is, $h(I) = I$ for all $h \in H$. (It is sufficient to check that $h(I) \subseteq I$ for all $h \in H$.) Given an arbitrary ideal I in R, let $(I : H)$ denote the largest H-ideal of R contained in I, that is, $(I : H) = \bigcap_{h \in H} h(I)$. An H-*prime ideal* of R is any proper H-ideal Q of R such that whenever I and J are H-ideals of R with $IJ \subseteq Q$, either $I \subseteq Q$ or $J \subseteq Q$. In particular, $(P : H)$ is H-prime for any prime ideal P.

Now suppose that we also have a set Δ of derivations on R. An (H, Δ)-*ideal* of R is any ideal of R that is stable under both H and Δ. We then define the notion of an (H, Δ)-*prime ideal* in parallel with H-prime or Δ-prime ideals, and we write (H, Δ)-spec R for the (H, Δ)-*spectrum* of R, that is, the set of all (H, Δ)-prime ideals of R, equipped with the natural Zariski topology.

Lemma 3.1. *Let (R, Δ) be a commutative noetherian differential k-algebra and H an algebraic torus acting rationally on (R, Δ).*

(a) If I is a Δ-ideal of R, then $(I : H)$ is an (H, Δ)-ideal. Similarly, if I is an H-ideal of R, then $(I : \Delta)$ is an (H, Δ)-ideal.

(b) $(P : H) \in (H, \Delta)$-spec R for all $P \in \Delta$-spec R.

(c) The following sets coincide:

    (1) (H, Δ)-spec R;

    (2) The set of all H-prime Δ-ideals in R;

    (3) The set of all Δ-prime H-ideals in R;

    (4) The set of all prime (H, Δ)-ideals in R.

Proof. (a) Assume first that I is a Δ-ideal of R. If $\delta \in \Delta$ is an H-eigenvector with H-eigenvalue d, then

$$h\delta(I : H) = d(h) \cdot \delta h(I : H) \subseteq \delta(I) \subseteq I$$

for all $h \in H$, whence $\delta(I : H) \subseteq (I : H)$. It follows that $(I : H)$ is stable under Δ, and so it is an (H, Δ)-ideal.

Now assume that I is an H-ideal of R, and consider $h \in H$. For any H-eigenvectors $\delta_1, \ldots, \delta_n \in \Delta$ with respective H-eigenvalues $d_1, \ldots, d_n$, we have

$$\delta_1 \delta_2 \cdots \delta_n h\big((I : \Delta)\big) = \big[(d_1 d_2 \cdots d_n)(h)\big]^{-1} h\delta_1 \delta_2 \cdots \delta_n\big((I : \Delta)\big) \subseteq h(I) = I,$$

from which it follows that $h\big((I : \Delta)\big) \subseteq (I : \Delta)$. Hence, $(I : \Delta)$ is stable under H, and so it is an (H, Δ)-ideal.

(b) If $P \in \Delta$-spec R, then P is prime by Lemma 1.1(d), and so $(P : H)$ is an H-prime ideal. By part (a), $(P : H)$ is an (H, Δ)-ideal, and we conclude from the H-primeness of $(P : H)$ that it must be (H, Δ)-prime.

(c) Since all Δ-prime ideals of R are prime (Lemma 1.1), as are all H-prime ideals (e.g., [**1**, Proposition II.2.9]), the sets (2), (3), and (4) coincide. Clearly $(4) \subseteq (1)$.

Now let Q be an (H, Δ)-prime ideal of R. Any prime ideal minimal over Q is a Δ-ideal by Lemma 1.1. Since R is noetherian, there are only finitely many prime ideals minimal over Q, and they are permuted by H, so their H-orbits are finite. Thus, [**1**, Proposition II.2.9]

implies that the prime ideals minimal over Q are H-ideals, and so they all lie in the set (4). Now there exist prime ideals $Q_1, \ldots, Q_n$ minimal over Q such that $Q_1 Q_2 \cdots Q_n \subseteq Q$. Since Q is (H, Δ)-prime, we conclude that some $Q_j = Q$. Therefore $(1) \subseteq (4)$. $\square$

Definitions. Let (R, Δ) be a commutative noetherian differential k-algebra and H an algebraic torus acting rationally on (R, Δ). For $J \in (H, \Delta)$-spec R, we define the J-*stratum* of Δ-spec R to be the set

$$\Delta\text{-spec}_J R = \{P \in \Delta\text{-spec} R \mid (P : H) = J\}.$$

In view of Lemma 3.1(b), we obtain a partition

$$\Delta\text{-spec} R = \bigsqcup_{J \in (H, \Delta)\text{-spec} R} \Delta\text{-spec}_J R,$$

which we refer to as the *H-stratification of Δ-spec R*. We define strata denoted $\Delta\text{-prim}_J R$ in Δ-prim R in the same way as $\Delta\text{-spec}_J R$, and obtain a corresponding H-stratification of that space:

$$\Delta\text{-prim} R = \bigsqcup_{J \in (H, \Delta)\text{-spec} R} \Delta\text{-prim}_J R.$$

For any $J \in (H, \Delta)$-spec R, let $\mathcal{E}_J$ denote the set of H-eigenvectors in R/J. Since R/J is a domain (by Lemma 3.1), $\mathcal{E}_J$ is multiplicatively closed, and the localization $R_J = (R/J)[\mathcal{E}_J^{-1}]$ is a subalgebra of $\text{Fract}(R/J)$. Note that the actions of H and Δ on R both extend naturally to R/J and R_J, and then to $\text{Fract} R/J$, so that we have commutative differential k-algebras $(R/J, \Delta)$ and (R_J, Δ), as well as $(\text{Fract} R/J, \Delta)$.

Theorem 3.2. *Let (R, Δ) be a commutative noetherian differential k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on (R, Δ). Let J be a prime (H, Δ)-ideal of R.*
  (a) *The algebra R_J is a graded field, with respect to its induced $\widehat{H}$-grading.*
  (b) *$\Delta\text{-spec}_J R$ is homeomorphic to $\Delta\text{-spec} R_J$ via localization and contraction.*
  (c) *$\Delta\text{-spec} R_J$ is homeomorphic to $\text{spec} Z_\Delta(R_J)$ via contraction and extension.*
  (d) *$Z_\Delta(R_J)$ is a Laurent polynomial ring, in at most r indeterminates, over the fixed field $Z_\Delta(R_J)^H = Z_\Delta(\text{Fract} R/J)^H$. The indeterminates can be chosen to be H-eigenvectors with $\mathbb{Z}$-linearly independent H-eigenvalues.*

Proof. (a) With respect to the $\widehat{H}$-grading, R_J is obtained from the domain R/J by inverting all nonzero homogeneous elements. Consequently, R_J is a graded field.
  (b) Standard localization theory yields that localization and contraction give mutually inverse homeomorphisms between the sets

$$X_J := \{P \in \text{spec} R \mid P \supseteq J \text{ and } (P/J) \cap \mathcal{E}_J = \varnothing\}$$

and $\text{spec} R_J$. It is clear that these maps restrict to mutually inverse homeomorphisms between $X_J \cap \Delta\text{-spec} R$ and $\Delta\text{-spec} R_J$, so it just remains to show that $X_J \cap \Delta\text{-spec} R = \Delta\text{-spec}_J R$.

If $P \in \Delta\text{-spec}_J R$, then $(P/J : H) = 0$ and so P/J contains no nonzero H-ideals of R/J. Hence, P/J contains no H-eigenvectors, that is, $P \in X_J$.

Conversely, if $P \in X_J \cap \Delta\text{-spec}\,R$, then $P \supseteq J$ but P/J contains no H-eigenvectors of R/J. Since all H-ideals of R/J are spanned by their H-eigenvectors, P/J contains no nonzero H-ideals, that is, $(P/J : H) = 0$. Thus, $(P : H) = J$ and $P \in \Delta\text{-spec}_J R$, as desired.

(c)(d) These follow from (a) together with Proposition 2.4. $\square$

We can now give the following Dixmier-Moeglin equivalence for commutative differential algebras equipped with rational torus actions.

Theorem 3.3. *Let (R, Δ) be a commutative differential k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on (R, Δ).*

Assume that R is affine over k, and that it has only finitely many prime (H, Δ)-ideals. Let J be one of them. For $P \in \Delta\text{-spec}_J R$, the following conditions are equivalent:

(a) *P is locally closed in $\Delta\text{-spec}\,R$.*
(b) *P is Δ-primitive.*
(c) *$Z_\Delta(\operatorname{Fract} R/P)$ is algebraic over k.*
(d) *P is maximal in $\Delta\text{-spec}_J R$.*

Proof. (a)$\Longrightarrow$(b)$\Longrightarrow$(c) by Proposition 1.2.

(d)$\Longrightarrow$(a): If J is maximal in (H, Δ)-spec R, then every Δ-prime containing J is in $\Delta\text{-spec}_J R$. In this case, P is maximal in $\Delta\text{-spec}\,R$, and thus is trivially locally closed in $\Delta\text{-spec}\,R$.

Now suppose that J is not maximal in (H, Δ)-spec R, and let $J_1, \ldots, J_n$ be the prime (H, Δ)-ideals of R that properly contain J. Then the ideal $\widehat{J} = J_1 \cap \cdots \cap J_n$ must properly contain J. But $J = (P : H)$, so this implies that $\widehat{J} \not\subseteq P$. Any Δ-prime ideal Q which properly contains P cannot lie in $\Delta\text{-spec}_J R$. Since $(Q : H) \supseteq (P : H) = J$, it follows that $(Q : H) = J_i$ for some i, and so $Q \supseteq J_i \supseteq \widehat{J}$. Thus, the Δ-prime ideals properly containing P all contain $\widehat{J}$, which shows that

$$\{P\} = \{Q \in \Delta\text{-spec}\,R \mid Q \supseteq P\} \cap \{Q \in \Delta\text{-spec}\,R \mid Q \not\supseteq \widehat{J}\}.$$

Therefore P is locally closed in $\Delta\text{-spec}\,R$.

(c)$\Longrightarrow$(d): For this part of the proof, we pass to R/J, and so we may assume that $J = 0$. Thus, R is now a domain.

By Theorem 3.2(b), P is disjoint from $\mathcal{E}_J$ and P induces a Δ-prime ideal PR_J in R_J. Note that R_J/PR_J is a localization of R/P, and so $\operatorname{Fract} R_J/PR_J = \operatorname{Fract} R/P$. Hence,

$$Z_\Delta(\operatorname{Fract} R_J/PR_J) = Z_\Delta(\operatorname{Fract} R/P),$$

which is algebraic over k by hypothesis.

Set $Q = PR_J \cap Z_\Delta(R_J)$, which is a prime ideal of $Z_\Delta(R_J)$ by Theorem 3.2(c). Further, the natural embedding

$$Z_\Delta(R_J)/Q \to R_J/PR_J \to \operatorname{Fract} R_J/PR_J$$

maps $Z_\Delta(R_J)/Q$ into $Z_\Delta(\operatorname{Fract} R_J/PR_J)$, and so $Z_\Delta(R_J)/Q$ must be algebraic over k. It follows that $Z_\Delta(R_J)/Q$ is a field, whence Q is a maximal ideal of $Z_\Delta(R_J)$. By Theorem 3.2(b)(c), PR_J is maximal in Δ-$\operatorname{spec} R_J$, and therefore P is maximal in Δ-$\operatorname{spec}_J R$. $\square$

Corollary 3.4. *Let (R, Δ) be a commutative differential k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on (R, Δ). Assume that R is affine over k, and that it has only finitely many prime (H, Δ)-ideals. Then the rule $\pi_m(Q) = (Q : \Delta)$ defines a continuous surjection*

$$\pi_m : \max R \to \Delta\text{-}\operatorname{prim} R,$$

and Δ-$\operatorname{prim} R$ is a topological quotient of $\max R$ via π_m.

Proof. Theorems 3.3 and 1.5. $\square$

Finally, we tighten up the picture for the case that k is algebraically closed.

Theorem 3.5. *Let (R, Δ) be a commutative differential k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on (R, Δ). Assume that k is algebraically closed, that R is affine over k, and that R has only finitely many prime (H, Δ)-ideals.*

(a) *For each prime (H, Δ)-ideal J of R, the algebra $Z_\Delta(R_J)$ is a Laurent polynomial ring of the form $k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, for some nonnegative integer $n = n(J) \leq r$. Consequently, the Δ-primitive ideals within Δ-$\operatorname{spec}_J R$ are precisely the inverse images in R of the ideals*

$$(R/J) \cap \big(R_J(z_1 - \alpha_1) + \cdots + R_J(z_n - \alpha_n) \big) \lhd R/J,$$

for arbitrary nonzero scalars $\alpha_1, \ldots, \alpha_n \in k^\times$.

(b) *The H-orbits within Δ-$\operatorname{prim} R$ coincide with the H-strata of Δ-$\operatorname{prim} R$. In particular, there are only finitely many H-orbits in Δ-$\operatorname{prim} R$.*

Proof. (a) In view of Theorems 3.2 and 3.3, the Δ-primitive ideals within Δ-$\operatorname{spec}_J R$ are precisely the inverse images in R of the ideals in R/J extended from maximal ideals of $Z_\Delta(R_J)$. Thus, we just need to show that $Z_\Delta(R_J)$ is a Laurent polynomial algebra over k in at most r variables. That will follow from Theorem 3.2(d) once we show that $Z_\Delta(R_J)^H = k$.

Since Δ-$\operatorname{spec}_J R$ is nonempty (it contains J), there exists a Δ-prime ideal P which is maximal in Δ-$\operatorname{spec}_J R$. By Theorem 3.3, the field $Z_\Delta(\operatorname{Fract} R/P)$ is algebraic over k, and thus equals k. Now since $Z_\Delta(R_J)^H$ is a field, the natural homomorphism

$$Z_\Delta(R_J)^H \overset{\subseteq}{\longrightarrow} Z_\Delta(R_J) \to Z_\Delta(R_J/PR_J) \overset{\subseteq}{\longrightarrow} Z_\Delta(\operatorname{Fract} R_J/PR_J) \cong Z_\Delta(\operatorname{Fract} R/P)$$

is an embedding. Therefore $Z_\Delta(R_J)^H = k$, as required.

(b) It is clear that the H-action on Δ-$\operatorname{prim} R$ preserves the H-strata. Hence, any H-orbit within Δ-$\operatorname{prim} R$ is contained in a stratum Δ-$\operatorname{prim}_J R$, for some prime (H, Δ)-ideal J of R, and it only remains to show that H acts transitively on Δ-$\operatorname{prim}_J R$.

Thus, let $P_1, P_2 \in \Delta$-$\operatorname{prim}_J R$. By Theorems 3.2 and 3.3, the P_i induce maximal Δ-ideals in R_J, which contract to maximal ideals $Q_i := P_i R_J \cap Z_\Delta(R_J)$ in $Z_\Delta(R_J)$. Now

by Theorem 3.2(d), we can choose the indeterminates z_i in part (a) to be H-eigenvectors with $\mathbb{Z}$-linearly independent H-eigenvalues f_i. Each

$$Q_i = \langle z_1 - \alpha_{i1}, \ldots, z_n - \alpha_{in} \rangle$$

for some $\alpha_{ij} \in k^\times$. Since $f_1, \ldots, f_n$ are $\mathbb{Z}$-linearly independent elements of $\widehat{H}$, there exists $h \in H$ such that $f_j(h) = \alpha_{1j}\alpha_{2j}^{-1}$ for all j (e.g., [**17**, Lemma 16.2C]). Then

$$h(z_j - \alpha_{1j}) = f_j(h)z_j - \alpha_{1j} = \alpha_{1j}\alpha_{2j}^{-1}(z_j - \alpha_{2j})$$

for all j, whence $h(Q_1) = Q_2$. As a result,

$$h(P_1)R_J \cap Z_\Delta(R_J) = h(Q_1) = Q_2 = P_2 R_J \cap Z_\Delta(R_J),$$

and therefore we conclude from Theorem 3.2 that $h(P_1) = P_2$, as desired. $\square$

4. The Poisson case

Definitions. A *Poisson k-algebra* is a pair $(R, \{-, -\})$ where R is a commutative k-algebra and $\{-, -\}$ is a *Poisson bracket* on R, that is, a bilinear map $\{-, -\} : R \times R \to R$ such that

   (a) The vector space R equipped with the binary operation $\{-, -\}$ is a Lie algebra over k, and
   (b) For each $a \in R$, the k-linear map $\{a, -\} : R \to R$ is a derivation (called the *Hamiltonian associated to a*).

We typically assume that a Poisson bracket has been given and is denoted by $\{-, -\}$, and hence will refer to R itself as a Poisson algebra. A *Poisson automorphism* of R is any k-algebra automorphism of R which preserves the Poisson bracket.

   All of the concepts defined at the beginning of Section 1 are considered for a Poisson algebra R relative to the set $\{R, -\}$ of Hamiltonians on R. Thus, the *Poisson center* of R is the $\{R, -\}$-center, which we shall denote $Z_P(R)$. (The elements of the Poisson center are sometimes called *Casimirs*, in which case $Z_P(R)$ is denoted Cas R.) A *Poisson ideal* of R is any $\{R, -\}$-ideal. Given an arbitrary ideal J in R, there is a largest Poisson ideal contained in J, which in the notation of Section 1 would be written $(J : \{R, -\})$. Following [**3**], we call $(J : \{R, -\})$ the *Poisson core of J*; we shall denote it P.core(J). The *Poisson primitive ideals* of R are the $\{R, -\}$-primitive ideals, that is, the Poisson cores of the maximal ideals of R (in [**21**], these are called *symplectic ideals*). A *Poisson-prime* ideal of R is any $\{R, -\}$-prime ideal. If R is noetherian, then by Lemma 1.1(d), the Poisson-prime ideals in R are precisely the ideals which are both Poisson ideals and prime ideals; in that case, the hyphen in the term "Poisson-prime" becomes unnecessary. Finally, we write P.spec R and P.prim R for $\{R, -\}$-spec R and $\{R, -\}$-prim R, respectively.

   In order to apply the results of previous sections to the Poisson setting, one observation is needed: If R is a Poisson algebra and H is an algebraic torus acting rationally on R by Poisson automorphisms, then H acts rationally on $(R, \{R, -\})$. To see this, observe

that given $h \in H$ and $a \in R$, we have $h(\{a, h^{-1}(b)\}) = \{h(a), b\}$ for all $b \in R$, whence $h.\{a, -\} = h \circ \{a, -\} \circ h^{-1} = \{h(a), -\}$. In particular, if a is an H-eigenvector, then so is $\{a, -\}$, with the same eigenvalue. As every element of R is a sum of H-eigenvectors with rational H-eigenvalues, the same holds in $\{R, -\}$. Therefore H acts rationally on $(R, \{R, -\})$, as claimed.

Theorem 4.1. *Let R be a noetherian Poisson k-algebra.*
(a) *The rule $\pi(Q) = \mathrm{P.core}(Q)$ defines a continuous retraction*

$$\pi : \mathrm{spec}\, R \to \mathrm{P.spec}\, R,$$

and $\mathrm{P.spec}\, R$ is a topological quotient of $\mathrm{spec}\, R$ via π.
(b) *Now suppose that R is an affine k-algebra. Let $H = (k^\times)^r$ be an algebraic torus acting rationally on R by Poisson automorphisms, and assume that R has only finitely many prime Poisson H-ideals. Then π restricts to a continuous surjection*

$$\pi_m : \mathrm{max}\, R \to \mathrm{P.prim}\, R,$$

and $\mathrm{P.prim}\, R$ is a topological quotient of $\mathrm{max}\, R$ via π_m.

Proof. Theorem 1.3 and Corollary 3.4. $\square$

As in Section 3, when we have a group H acting on a Poisson algebra R by Poisson automorphisms, the Poisson spectrum $\mathrm{P.spec}\, R$ obtains an *H-stratification*, namely the partition

$$\mathrm{P.spec}\, R = \bigsqcup_{\text{prime Poisson } H\text{-ideals } J} \mathrm{P.spec}_J\, R,$$

where the *H-strata* $\mathrm{P.spec}_J\, R$ are given by

$$\mathrm{P.spec}_J\, R = \{P \in \mathrm{P.spec}\, R \mid (P : H) = J\}.$$

There is a parallel H-stratification of $\mathrm{P.prim}\, R$. Given a prime Poison H-ideal J in R, we again denote the localization of R/J with respect to the multiplicative set of its H-eigenvectors by R_J, and we equip R_J and R/J, as well as $\mathrm{Fract}\, R/J$, with the induced Poisson structures. In Poisson notation, the stratification theorem takes the following form:

Theorem 4.2. *Let R be a noetherian Poisson k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on R by Poisson automorphisms. Let J be a prime Poisson H-ideal of R.*
(a) *The algebra R_J is a graded field, with respect to its induced $\widehat{H}$-grading.*
(b) *$\mathrm{P.spec}_J\, R$ is homeomorphic to $\mathrm{P.spec}\, R_J$ via localization and contraction.*
(c) *$\mathrm{P.spec}\, R_J$ is homeomorphic to $\mathrm{spec}\, Z_P(R_J)$ via contraction and extension.*
(d) *$Z_P(R_J)$ is a Laurent polynomial ring, in at most r indeterminates, over the fixed field $Z_P(R_J)^H = Z_P(\mathrm{Fract}\, R/J)^H$. The indeterminates can be chosen to be H-eigenvectors with $\mathbb{Z}$-linearly independent H-eigenvalues.*

Proof. Theorem 3.2. $\square$

Our main theorem, given next, provides a Dixmier-Moeglin equivalence for Poisson algebras equipped with rational torus actions.

Theorem 4.3. *Let R be an affine Poisson k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on R by Poisson automorphisms. Assume that R has only finitely many prime Poisson H-ideals, and let J be one of them. For $P \in \text{P.spec}_J R$, the following conditions are equivalent:*

(a) *P is locally closed in $\text{P.spec}\, R$.*
(b) *P is Poisson primitive.*
(c) *$Z_P(\text{Fract}\, R/P)$ is algebraic over k.*
(d) *P is maximal in $\text{P.spec}_J R$.*

Proof. Theorem 3.3. $\square$

Theorem 4.3 sets up a general framework which covers various previous examples of the Poisson Dixmier-Moeglin equivalence, such as those of Oh [**21**, Theorem 2.4, Proposition 2.13]. See below for further detail about these examples. Note that if R is an affine Poisson algebra with only finitely many Poisson primitive ideals, then, by Lemma 1.1(e), R has only finitely many prime Poisson ideals. Thus, the case of Theorem 4.3 in which $H = \langle 1 \rangle$ covers [**3**, Lemma 3.4].

Theorem 4.4. *Let R be an affine Poisson k-algebra, and $H = (k^\times)^r$ an algebraic torus acting rationally on R by Poisson automorphisms. Assume that k is algebraically closed, and that R has only finitely many prime Poisson H-ideals.*

(a) For each prime Poisson H-ideal J of R, the algebra $Z_P(R_J)$ is a Laurent polynomial ring of the form $k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, for some nonnegative integer $n = n(J) \leq r$. Consequently, the Poisson primitive ideals within $\text{P.spec}_J R$ are precisely the inverse images in R of the ideals

$$(R/J) \cap \big(R_J(z_1 - \alpha_1) + \cdots + R_J(z_n - \alpha_n)\big) \lhd R/J,$$

for arbitrary nonzero scalars $\alpha_1, \ldots, \alpha_n \in k^\times$.

(b) The H-orbits within $\text{P.prim}\, R$ coincide with the H-strata of $\text{P.prim}\, R$. In particular, there are only finitely many H-orbits in $\text{P.prim}\, R$.

Proof. Theorem 3.5. $\square$

In the setting of Theorem 4.4, part (a) provides an explicit recipe for writing down all the Poisson primitive ideals of R, provided one can find all the prime Poisson H-ideals J in R and one can compute indeterminates for the Laurent polynomial rings $Z_P(R_J)$. The following examples exhibit some uses of this recipe.

Example 4.5. Let $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial algebra over k, and let $\pi = (\pi_{ij})$ be an $n \times n$ antisymmetric matrix over k. (At this point, k does not need to be algebraically closed.) There is a unique Poisson bracket on R such that

$$\{x_i, x_j\} = \pi_{ij} x_i x_j$$

for all i, j; a complete formula for this bracket is

(4.1) $$\{f, g\} = \sum_{i,j=1}^n \pi_{ij} x_i x_j \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

for $f, g \in R$. The torus $H = (k^\times)^n$ acts on R by k-algebra automorphisms such that

(4.2) $$(\alpha_1, \ldots, \alpha_n).x_i = \alpha_i x_i$$

for $(\alpha_1, \ldots, \alpha_n) \in H$ and $i = 1, \ldots, n$. Observe that this is a rational action by Poisson automorphisms. It is easily checked that 0 is the only prime H-ideal of R, and thus the only prime Poisson H-ideal. Hence, Theorem 4.3 implies that R satisfies the Poisson Dixmier-Moeglin equivalence, recovering a result of Oh [**21**, Theorem 2.4]. Further, Theorem 4.1 shows that P.prim R is a topological quotient of max R, via the map $M \mapsto \text{P.core}(M)$. This result is implicit in the work of Oh, Park, and Shin [**23**].

Since P.spec R consists of a single H-stratum, Theorem 4.3 also implies that the Poisson primitive ideals of R are precisely the maximal Poisson ideals. We can get a handle on these with Theorem 4.2. Observe that the H-eigenvectors in R are just the monomials $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, which are already invertible in R. Thus, the algebra R_0 of Theorem 4.2 (corresponding to the prime Poisson H-ideal 0) is just R itself. Consequently, the theorem shows that the maximal Poisson ideals of R are precisely the ideals extended from maximal ideals of the Poisson center $Z_P(R)$. By [**22**, Lemma 2.2] or [**25**, Lemma 1.2(a)],

$$Z_P(R) = k\text{-span of } \{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mid m_i \in \mathbb{Z} \text{ and } \sum_{i=1}^n m_i \pi_{ij} = 0 \text{ for all } j\}.$$

Thus, $Z_P(R)$ can be a Laurent polynomial algebra in any number of indeterminates from 0 to n, with suitable choices of the matrix π.

Now specialize to the case where k is algebraically closed and $\pi_{ij} = 1$ for all $i < j$. If n is even, one computes that $Z_P(R) = k$, while if n is odd, one gets $Z_P(R) = k[z^{\pm 1}]$ where $z = x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_{n-1}^{-1} x_n$. (E.g., apply the row reduction steps given in [**1**, Example I.14.3(1)] to π.) Therefore, we conclude that

$$\text{P.prim } R = \{0\} \qquad\qquad\qquad\qquad (n \text{ even})$$
$$\text{P.prim } R = \{\langle x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_{n-1}^{-1} x_n - \lambda \rangle \mid \lambda \in k^\times\} \qquad (n \text{ odd}).$$

Example 4.6. Now take $R = k[x_1, \ldots, x_n]$ to be a polynomial algebra, and again choose an antisymmetric matrix $\pi = (\pi_{ij}) \in M_n(k)$. The Poisson bracket on $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ described in (4.1) restricts to a Poisson bracket on R. Also, the action of the torus $H = (k^\times)^n$ given by (4.2) restricts to a rational action on R by Poisson automorphisms. It is easily checked that the prime H-ideals in R are the ideals

$$J(X) = \langle x_i \mid i \in X \rangle \qquad \text{for } X \subseteq \{1, \ldots, n\},$$

and they are all Poisson ideals. Thus, R has precisely 2^n prime Poisson H-ideals. Theorem 4.3 therefore implies that R satisfies the Poisson Dixmier-Moeglin equivalence, and Theorem 4.1 implies that P.prim R is a topological quotient of max R. The latter result is implicit in [**23**].

The localizations R_J of Theorem 4.2 can be identified with Poisson subalgebras of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ as follows:

$$R_{J(X)} = k[x_i^{\pm 1} \mid i \in \{1, \ldots, n\} \setminus X].$$

Theorems 4.2 and 4.3 imply that the Poisson primitive ideals of R can be obtained by pulling back the ideals extended from the maximal ideals of the Poisson centers $Z_P(R_{J(X)})$. These Poisson centers can be computed as in Example 4.5.

Now specialize to the case where k is algebraically closed, $n = 3$, and $\pi_{ij} = 1$ for all $i < j$. Taking account of Example 4.5, we find that the Poisson primitive ideals of $R = k[x_1, x_2, x_3]$ in this case can be listed as follows:

$$\langle x_1, x_2, x_3 \rangle \qquad\qquad \langle x_1 - \alpha, x_2, x_3 \rangle \quad (\alpha \in k^\times)$$
$$\langle x_1 \rangle \qquad\qquad \langle x_1, x_2 - \beta, x_3 \rangle \quad (\beta \in k^\times)$$
$$\langle x_2 \rangle \qquad\qquad \langle x_1, x_2, x_3 - \gamma \rangle \quad (\gamma \in k^\times)$$
$$\langle x_3 \rangle \qquad\qquad \langle x_1 x_3 - \lambda x_2 \rangle \quad (\lambda \in k^\times).$$

Example 4.7. On the polynomial algebra $R = k[a, b, c, d]$, there is a Poisson bracket arising from what is called the "semiclassical limit of quantum 2×2 matrices", meaning that the standard quantized coordinate ring of 2×2 matrices is a "quantization" of R (see [**1**, §III.5.4], for instance, for a discussion of this quantization concept). This Poisson bracket is given by the following data:

$$\{a, b\} = ab \qquad\qquad \{b, d\} = bd$$
$$\{a, c\} = ac \qquad\qquad \{c, d\} = cd$$
$$\{a, d\} = 2bc \qquad\qquad \{b, c\} = 0$$

(e.g., [**24**, Example 2.1], [**5**, p. 255], or [**21**, §2.9], where in the latter two papers the bracket has been scaled by 2). The torus $H = (k^\times)^4$ acts on R by k-algebra automorphisms such that

$$(\alpha_1, \alpha_2, \beta_1, \beta_2).a = \alpha_1 \beta_1 a \qquad\qquad (\alpha_1, \alpha_2, \beta_1, \beta_2).b = \alpha_1 \beta_2 b$$
$$(\alpha_1, \alpha_2, \beta_1, \beta_2).c = \alpha_2 \beta_1 c \qquad\qquad (\alpha_1, \alpha_2, \beta_1, \beta_2).d = \alpha_2 \beta_2 d$$

for $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in H$. Observe that this is a rational action by Poisson automorphisms.

It is known that R has precisely 14 prime Poisson H-ideals:

$$\langle a, b, c, d \rangle$$
$$\langle a, b, d \rangle \qquad \langle a, b, c \rangle \qquad \langle b, c, d \rangle \qquad \langle a, c, d \rangle$$
$$\langle a, b \rangle \qquad \langle b, d \rangle \qquad \langle b, c \rangle \qquad \langle a, c \rangle \qquad \langle c, d \rangle$$
$$\langle b \rangle \qquad \langle ad - bc \rangle \qquad \langle c \rangle$$
$$\langle 0 \rangle.$$

(For instance, this can be calculated as in [**5**, pp. 255-258]. When k is algebraically closed, it also follows from [**5**, Theorem 9] or [**21**, p. 2179].) Consequently, Theorem 4.3 implies that R satisfies the Poisson Dixmier-Moeglin equivalence, recovering a result of Oh [**21**, Proposition 2.13].

If k is algebraically closed, one can use the information above, together with Theorem 4.4 and calculations of appropriate Poisson centers of localizations, to find all the Poisson primitive ideals of R. The calculations are essentially the same as those employed by Cho and Oh to the same end (see [**5**, Theorem 9] and [**21**, p. 2179, display before Theorem 3.4]). $\diamondsuit$

We conclude by sketching some examples in which Poisson primitive ideals correspond to symplectic leaves. Although the concept of a symplectic leaf arises in differential geometry, we shall immediately restrict attention to situations in which it can be described in terms of algebraic geometry.

First, a *Poisson manifold* is a smooth complex manifold M together with a Poisson bracket on the algebra $C^\infty(M)$ of smooth functions on M. The derivations $\{a, -\}$ on $C^\infty(M)$ define *Hamiltonian vector fields* on M, and smooth paths in M whose tangent vectors come from Hamiltonian vector fields are called *Hamiltonian paths*. One can then define the symplectic leaves of M to be the connected components relative to the relation "connected by a piecewise Hamiltonian path". (Differential geometers typically prefer an equivalent definition in terms of symplectic submanifolds – e.g., see [**4**, §5.1] or [**1**, §III.5.2].) Now any smooth complex affine variety V has a natural smooth manifold structure, and a Poisson bracket on the coordinate ring $\mathcal{O}(V)$ extends uniquely to $C^\infty(V)$, thus making V into a Poisson manifold. However, the Hamiltonian paths in V need not be algebraic curves, since they are typically defined by exponential functions, and so the symplectic leaves in V are not necessarily algebro-geometric objects. However, when these symplectic leaves are algebraic, in the sense that they are locally closed subvarieties, they correspond to the Poisson primitive ideals of $\mathcal{O}(V)$, by a result of Brown and Gordon [**3**, Proposition 3.6(2)]. Namely, if $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}(V)$ corresponding to a point $x \in V$, then the symplectic leaf containing x coincides with the set

$$(4.3) \qquad \mathcal{C}(x) := \{y \in V \mid \mathrm{P.core}(\mathfrak{m}_y) = \mathrm{P.core}(\mathfrak{m}_x)\},$$

which is called the *symplectic core of* $\mathfrak{m}_x$ in [**3**, §3.3]. Moreover, under these hypotheses, half of the Poisson Dixmier-Moeglin equivalence already holds, in that the Poisson primitive ideals in $\mathcal{O}(V)$ are precisely the locally closed points of $\mathrm{P.spec}\,\mathcal{O}(V)$ [**3**, Proposition 3.6(2)].

Let us define an *affine Poisson variety* (over $\mathbb{C}$) to be a smooth affine complex variety V together with a Poisson bracket on the coordinate ring $\mathcal{O}(V)$. (In order to define Poisson structures on projective varieties, Poisson brackets need to be rewritten in terms of bivector fields; for instance, see [**8**, §1.4.1] or [**19**, Section 1.1].) The following proposition, based on the ideas of Brown and Gordon [**3**], describes a geometric setting in which our main results apply. For any subset $X \subseteq V$, let $I(X)$ denote the defining ideal of the closure $\overline{X}$, that is, $I(X)$ is the set of those functions in $\mathcal{O}(V)$ which vanish on X (equivalently, on $\overline{X}$).

Proposition 4.8. *Let V be an affine Poisson variety over $\mathbb{C}$, and H an algebraic group acting morphically on V via automorphisms of Poisson varieties. There is an induced*

action of H on $\mathcal{O}(V)$ by Poisson automorphisms. Assume that V has only finitely many H-orbits of symplectic leaves.

(a) *There are only finitely many prime Poisson H-ideals in $\mathcal{O}(V)$.*

(b) *Now assume that all the symplectic leaves in V are locally closed subvarieties, and that $H = (\mathbb{C}^{\times})^r$ is a complex algebraic torus. Then the rule*

$$\big(\, H\text{-orbit } \mathcal{L} \text{ of symplectic leaves} \,\big) \longmapsto I\big(\, \bigcup\{L \in \mathcal{L}\} \,\big)$$

defines a bijection between the set of H-orbits of symplectic leaves in V and the set of prime Poisson H-ideals in $\mathcal{O}(V)$.

Proof. Recall the symplectic cores $\mathcal{C}(x)$ defined in (4.3). There is a bijection between the set of symplectic cores in V and the set $\text{P.prim}\,\mathcal{O}(V)$, given by the rule $\mathcal{C}(x) \mapsto \text{P.core}(\mathfrak{m}_x)$, and this induces a bijection between the set of H-orbits of symplectic cores in V and the set of H-orbits in $\text{P.prim}\,\mathcal{O}(V)$.

(a) Since each symplectic core is a union of symplectic leaves [**3**, Proposition 3.6(1)], it follows from our hypotheses that there are only finitely many H-orbits of symplectic cores in V, and thus only finitely many H-orbits in $\text{P.prim}\,\mathcal{O}(V)$. Now for any $P \in \text{P.prim}\,\mathcal{O}(V)$, the ideal $(P : H)$, which we shall call the H-*core* of P, equals the intersection of the H-orbit $\{h(P) \mid h \in H\}$. Hence, there are only finitely many H-cores of Poisson primitive ideals in $\mathcal{O}(V)$.

Let Q be any prime Poisson H-ideal in $\mathcal{O}(V)$. By Lemma 1.1(e), Q is an intersection of Poisson primitive ideals P_α, and since Q is stable under H, it is also the intersection of the H-cores of the P_α. Since there are only finitely many H-cores of Poisson primitive ideals in $\mathcal{O}(V)$, there can only be finitely many intersections of such ideals, and therefore part (a) is proved.

(b) As noted above, under the present hypotheses, the symplectic leaves in V are precisely the sets $\mathcal{C}(x)$ [**3**, Proposition 3.6(2)]. Moreover, by [**3**, Lemma 3.5], $\text{P.core}(\mathfrak{m}_x) = I(\mathcal{C}(x))$ for all $x \in V$. Due to part (a) and the assumption that H is a torus, Theorem 4.4(b) shows that the H-orbits in $\text{P.prim}\,\mathcal{O}(V)$ coincide with the H-strata.

Let H-symp V denote the set of H-orbits of symplectic leaves in V, and set

$$\theta(\mathcal{L}) := I\big(\, \bigcup\{L \in \mathcal{L}\} \,\big)$$

for $\mathcal{L} \in H$-symp V. Since $\mathcal{L}$ is the H-orbit of a symplectic leaf $\mathcal{C}(x)$, for some $x \in V$, we have

$$(4.4) \qquad \theta(\mathcal{L}) = \bigcap_{h \in H} I\big(\mathcal{C}(h.x)\big) = \bigcap_{h \in H} \text{P.core}(\mathfrak{m}_{h.x}) = (\text{P.core}(\mathfrak{m}_x) : H),$$

and thus, by Lemma 3.1, $\theta(\mathcal{L})$ is a prime Poisson H-ideal in $\mathcal{O}(V)$. To prove (b), it remains to establish the following:

(*) For each prime Poisson H-ideal Q in $\mathcal{O}(V)$, there is a unique $\mathcal{L} \in H$-symp V such that $\theta(\mathcal{L}) = Q$.

As in the proof of (a), there are only finitely many H-cores of Poisson primitive ideals in $\mathcal{O}(V)$, say $(P_1 : H), \ldots, (P_m : H)$, for some $P_1, \ldots, P_m$ in P.prim $\mathcal{O}(V)$. As also noted there, any prime Poisson H-ideal Q in $\mathcal{O}(V)$ is an intersection of H-cores of Poisson primitive ideals, say $Q = (P_{i_1} : H) \cap \cdots \cap (P_{i_t} : H)$. Since Q is prime, it follows that $Q = (P_j : H)$ for some $j \in \{i_1, \ldots, i_t\}$. Now $P_j = \text{P.core}(\mathfrak{m}_x)$ for some $x \in V$, and thus $Q = \theta(\mathcal{L})$ by (4.4), where $\mathcal{L}$ is the H-orbit of $\mathcal{C}(x)$.

Finally, suppose that also $Q = \theta(\mathcal{L}') = (\text{P.core}(\mathfrak{m}_y) : H)$, where $\mathcal{L}'$ is the H-orbit of some symplectic leaf $\mathcal{C}(y)$. Then P.core$(\mathfrak{m}_x)$ and P.core$(\mathfrak{m}_y)$ lie in the same H-stratum P.prim$_Q \mathcal{O}(V)$, and thus in the same H-orbit. As noted at the beginning of the proof, this implies that $\mathcal{C}(x)$ and $\mathcal{C}(y)$ lie in the same H-orbit of symplectic leaves of V, that is, $\mathcal{L}' = \mathcal{L}$. Therefore $\mathcal{L}$ is unique, and (*) is verified. $\square$

Example 4.9. Let G be a connected semisimple complex Lie group, with opposite Borel subgroups $B^{\pm}$ and corresponding Cartan subgroup $H = B^+ \cap B^-$, and let H act on G by left translation. There is a "standard" H-invariant Poisson structure on G (e.g., [**15**, Section A.1] or [**19**, Section 5.3]), and there are only finitely many H-orbits of symplectic leaves in G [**15**, Theorem A.2.1]. Hence, if we put the corresponding Poisson bracket on $\mathcal{O}(G)$, Proposition 4.8(a) tells us that $\mathcal{O}(G)$ has only finitely many prime Poisson H-ideals. Therefore, $\mathcal{O}(G)$ satisfies the Poisson Dixmier-Moeglin equivalence, by Theorem 4.3. It also follows from [**15**, §§A.1, A.2] that the symplectic leaves in G are locally closed subvarieties, and that the unions of H-orbits of symplectic leaves coincide with the double Bruhat cells $B^+ w_+ B^+ \cap B^- w_- B^-$, for elements $w_{\pm}$ in the Weyl group of G. Hence, Proposition 4.8(b) implies that the prime Poisson H-ideals in $\mathcal{O}(G)$ are the defining ideals of the closures of the double Bruhat cells in G.

As a specific example, let $G = SL_n(\mathbb{C})$, and take B^+, B^-, H to be the respective subgroups of upper triangular, lower triangular, diagonal matrices in G. For $i, j = 1, \ldots, n$, let $X_{ij} \in \mathcal{O}(G)$ denote the function that takes matrices to their i, j-entries. The Poisson bracket on $\mathcal{O}(G)$ is determined by the folllowing data:

$$(4.5) \qquad \{X_{ij}, X_{lm}\} = \begin{cases} X_{ij} X_{lm} & (i = l, \ j < m) \\ X_{ij} X_{lm} & (i < l, \ j = m) \\ 0 & (i < l, \ j > m) \\ 2 X_{im} X_{lj} & (i < l, \ j < m). \end{cases}$$

Note that when $n = 2$ and $\mathcal{O}(SL_2(\mathbb{C}))$ is identified with $k[a, b, c, d]/\langle ad - bc - 1 \rangle$, the Poisson bracket above is induced from the one discussed in Example 4.7. Since the Weyl group of G can be identified with the subgroup of signed permutation matrices in G, the double Bruhat cells in G are easy to identify. Results of Fulton [**11**, p. 390] allow one to characterize these cells in terms of vanishing and nonvanishing of certain minors, as in [**2**, Example 4.4]. $\diamond$

Example 4.10. Let G be a connected reductive complex Lie group, with opposite Borel subgroups $B^{\pm}$ and corrresponding Cartan subgroup $H = B^+ \cap B^-$, and let H act on G by left translation. Let P_J^+ be a standard parabolic subgroup containing B^+. There is a

standard Poisson structure on the flag variety G/P_J^+ (induced from the standard Poisson structure on G) [**14**, Proposition 1.3], which restricts to the open B^--orbit $B^-.P_J^+ \subseteq G/P_J^+$ (that is, the orbit of the coset $P_J^+ = eP_J^+$ under left translation by B^-). These Poisson structures are invariant under the induced left actions by H. According to [**2**, Theorem 1.9] and [**14**, Theorem 1.5], the symplectic leaves in G/P_J^+ are locally closed subvarieties, and there are only finitely many H-orbits of symplectic leaves. Both properties are inherited by the affine Poisson variety $B^-.P_J^+$. Therefore, Proposition 4.8(a) shows that $\mathcal{O}(B^-.P_J^+)$ has only finitely many prime Poisson H-ideals, and Theorem 4.3 implies that $\mathcal{O}(B^-.P_J^+)$ satisfies the Poisson Dixmier-Moeglin equivalence.

Some explicit examples of the above Poisson structures are given in [**14**, §§5.4–5.7], and the main object of [**2**] is of this type. For the latter, choose positive integers m and n, let $G = GL_{m+n}(\mathbb{C})$, and take

$$P_J^+ = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \ \middle| \ a \in GL_n(\mathbb{C}),\ b \in M_{n,m}(\mathbb{C}),\ c \in GL_m(\mathbb{C}) \right\}.$$

(As in Example 4.9, we take B^+, B^-, H to be the respective subgroups of upper triangular, lower triangular, diagonal matrices in G.) In the present case, $B^-.P_J^+$ is isomorphic, as a Poisson variety, to the matrix variety $M_{m,n}(\mathbb{C})$, equipped with its standard Poisson structure [**2**, Proposition 3.4]. (See [**2**, §1.5] for the standard Poisson structure on $M_{m,n}(\mathbb{C})$. If we take $X_{ij} \in \mathcal{O}(M_{m,n}(\mathbb{C}))$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, to be the usual matrix-entry functions, the data for the Poisson bracket on $\mathcal{O}(M_{m,n}(\mathbb{C}))$ are given by (4.5).) The H-orbits of symplectic leaves in $M_{m,n}(\mathbb{C})$ are described in three different ways in [**2**, Theorems 3.9, 4.2, 5.11]. Combining this information with Proposition 4.8(b) yields descriptions of the prime Poisson H-ideals in $\mathcal{O}(M_{m,n}(\mathbb{C}))$. ◇

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Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: goodearl@math.ucsb.edu