PERIODIC FLAT MODULES, AND FLAT MODULES FOR FINITE GROUPS

D. J. BENSON AND K. R. GOODEARL

ABSTRACT. The main theorem says that if R is a ring of coefficients and G a finite group, then a flat RG-module M which is projective as an R-module is necessarily projective as an RG-module. This is proved using the following theorem about flat modules over an arbitrary ring R. If a flat R-module M sits in a short exact sequence $0 \to M \to P \to M \to 0$ with P projective, then M is projective.

1. INTRODUCTION

In the representation theory of finite groups, a great deal of attention has been given to the problem of determining whether a module over the group ring is projective. For example, a well known theorem of Chouinard [9] states that a module is projective if and only if its restriction to each elementary abelian subgroup is projective. A theorem of Dade [10] states that over an algebraically closed field of characteristic p, a finitely generated module for an elementary abelian p-group is projective if and only if its restriction to each cyclic shifted subgroup is projective, where a cyclic shifted subgroup is a certain kind of cyclic subgroup of the group algebra. For an infinitely generated module, the statement is no longer valid, but in [6] it is proved that an infinitely generated module is projective if and only if its restriction to each cyclic shifted subgroup defined over each extension field is projective. These theorems have formed the basis for the development of the theory of varieties for modules [2, 5, 6, 7].

The main theorem of this paper is the following characterization of projectivity for flat modules over group rings. The proof can be found in Section 5.

Theorem 1.1. *Let R be a ring of coefficients (not necessarily commutative) and G a finite group. If M is a flat RG-module which is projective as an R-module, then M is projective as an RG-module.*

This is proved using the following interesting property of flat modules over an arbitrary ring. The following theorem is proved in Section 2.

Theorem 1.2. *Let R be a ring and M a flat R-module. If there exists a short exact sequence*

$$0 \to M \to P \to M \to 0$$

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with P a projective R-module, then M itself is projective.

Also in order to reach the main theorem, we need some theorems about reduction modulo primes, which seem to be of independent interest. The following results occupy Sections 3 and 4.

Theorem 1.3. *Let R be a ring of coefficients and G a finite group. An RG-module M is projective (respectively, flat) if and only if*

(i) *M is projective (respectively, flat) as an R-module, and*

(ii) *M/pM is projective (respectively, flat) as an $(R/pR)G$-module for each prime number p dividing $|G|$.*

On the basis of this, it is tempting to suppose that it might be true that an RG-module M is projective if and only if it is projective as an R-module, and $K \otimes_R M$ is projective as a KG-module for every field K containing $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$. In Section 6, we give an example to show that this is false in general, even if R is a discrete valuation ring. In Section 7 we show it is true if R is a Dedekind domain whose field of fractions has characteristic prime to $|G|$, such as a ring of algebraic integers, or $\mathfrak{p}$-local integers, or $\mathfrak{p}$-adic integers.

2. Periodic flat modules

The purpose of this section is to prove that periodic flat modules do not exist, over any ring. Throughout this section, all modules are modules over a ring R which has an identity element, but which is not necessarily commutative.

Lemma 2.1 (Villamayor). *Let M be a submodule of a projective R-module P. The following conditions are equivalent:*

(a) *P/M is flat.*

(b) *For each $x \in M$, there exists $f \in \mathrm{Hom}_R(P, M)$ such that $f(x) = x$.*

(c) *For each $x_1, \dots, x_n \in M$, there exists $f \in \mathrm{Hom}_R(P, M)$ such that $f(x_i) = x_i$ for $i = 1, \dots, n$.*

Proof. See Rotman [12], Theorem 3.57. □

Proposition 2.2. *Let M be a countably generated submodule of a projective R-module P. If P/M is flat, then M is projective.*

Proof. Let $x_1, x_2, \dots$ be a countable sequence of generators for M. By induction and Lemma 2.1, there exist $f_1, f_2, \dots$ in $\mathrm{Hom}_R(P, M)$ such that

$$(1 - f_n)(1 - f_{n-1}) \cdots (1 - f_1)(x_i) = 0$$

for $n \geq i$. It follows that the union of the kernels of the homomorphisms

$$g_n = (1 - f_n)(1 - f_{n-1}) \cdots (1 - f_1)|_M$$

equals M. Hence, there is a homomorphism $g : M \to P^{(\omega)} = P \oplus P \oplus \cdots$ given by the rule

$$g(x) = (x, g_1(x), g_2(x), \dots).$$

Let $f : P^{(\omega)} \to M$ be the homomorphism given by

$$f(a_1, a_2, \dots) = f_1(a_1) + f_2(a_2) + \cdots .$$

For all n, we have $g_n = (1 - f_n)g_{n-1}$ (where $g_0 = 1$), and so $f_n g_{n-1} = g_{n-1} - g_n$. Hence, for $x \in M$ we see that

$$\begin{aligned} fg(x) &= f_1(x) + f_2 g_1(x) + f_3 g_2(x) + \dots \\ &= x - g_1(x) + g_1(x) - g_2(x) + g_2(x) - g_3(x) + \cdots = x. \end{aligned}$$

Thus $fg = \mathrm{id}_M$, and therefore M is projective. □

In particular, Proposition 2.2 implies the well known fact that countably related flat modules have projective dimension at most one.

Lemma 2.3. *Let M be a submodule of a projective R-module P, and suppose that P/M is flat. Then any countably generated submodule of M is contained in a countably generated submodule K of M such that P/K is flat.*

Proof. Since it is harmless to replace M and P by $M \oplus 0$ and $P \oplus Q$ for any projective module Q, there is no loss of generality in assuming that P is free, say with basis X.

Given a countably generated submodule $K_0 \subseteq M$, choose a countable subset $Y \subseteq X$ such that K_0 is contained in the submodule P_0 of P generated by Y. Then P_0 is a direct summand of P. Let $\pi : P \to P_0$ be a projection, so that π is an idempotent endomorphism of P with countably generated image containing K_0.

We claim that M has a countably generated submodule $K_1 \supseteq K_0$ such that for each $x \in K_0$, there exists $f \in \mathrm{Hom}_R(P, K_1)$ with $f(x) = x$.

Let $x_1, x_2, \dots$ be a countable sequence of generators for K_0. By Lemma 2.1, there exist $f_1, f_2, \dots$ in $\mathrm{Hom}_R(P, M)$ such that $f_n(x_i) = x_i$ for $n \geq i$. Since $\pi(x_i) = x_i$ for all i, we may replace f_n by $f_n\pi$. Now each $f_n P$ is countably generated, and so the module

$$K_1 = K_0 + \sum_{n=1}^{\infty} f_n P$$

is a countably generated submodule of M. Any $x \in K_0$ lies in the submodule generated by $x_1, \dots, x_n$ for some n, whence $f_n(x) = x$. Since $f_n P \subseteq K_1$, this establishes the claim.

Now repeat this process ω times, obtaining countably generated submodules $K_0 \subseteq K_1 \subseteq \dots$ of M such that for each $x \in K_t$, there is some $f \in \mathrm{Hom}_R(P, K_{t+1})$ with $f(x) = x$. Then $K = \bigcup_{t=0}^{\infty} K_t$ is a countably generated submodule of M, and P/K is flat by Lemma 2.1. □

Lemma 2.4. *Let $0 \to M \xrightarrow{\subseteq} \bigoplus_{i \in I} P_i \xrightarrow{g} M \to 0$ be a short exact sequence with M flat and the P_i countably generated projective. Given any countable subset $J_0 \subseteq I$ and any countably generated submodule $K_0 \subseteq M$, there exists a countable subset $J \subseteq I$ such that $J_0 \subseteq J$, the submodule $P_J = \bigoplus_{j \in J} P_j$ contains K_0, the image gP_J equals $M \cap P_J$, and gP_J is projective.*

Proof. Set $P_J = \bigoplus_{j \in J} P_j$ and $M_J = M \cap P_J$ for all subsets $J \subseteq I$.

We first show that given J_0 and K_0, there exists a countable subset $J \subseteq I$, containing J_0, such that $K_0 \subseteq gP_J$ and gP_J is projective.

In view of Lemma 2.3, after enlarging K_0 we may assume that P_I/K_0 is flat. Then, after enlarging J_0, we may also assume that $K_0 \subseteq gP_{J_0}$. Now apply Lemma 2.3 again to obtain a countably generated submodule $K_1 \subseteq M$ such that $gP_{J_0} \subseteq K_1$ and P_I/K_1 is flat. Continuing in this manner, we obtain countable subsets $J_0 \subseteq J_1 \subseteq \dots$ in I and countably generated submodules $K_0 \subseteq K_1 \subseteq \dots$ in M such that $K_i \subseteq gP_{J_i} \subseteq K_{i+1}$ and P_I/K_i is flat for all i. Then $J = \bigcup_{i=0}^{\infty} J_i$ is a countable subset of I, and $K_0 \subseteq gP_J = \bigcup_{i=0}^{\infty} K_i$. Since the P_I/K_i are flat, so is P_I/gP_J, and thus gP_J is projective by Proposition 2.2. This establishes the first claim.

We next observe that any countable subset J_0 of I is contained in a countable subset J such that M_J is countably generated. To see this, use the first claim to find a countable subset J of I such that $J_0 \subseteq J$ and gP_J is projective. Hence, the short exact sequence

$$0 \to M_J \to P_J \to gP_J \to 0$$

splits, and so M_J is countably generated.

Now return to arbitrary J_0 and K_0 as in the hypotheses of the lemma. After enlarging J_0, we may assume that $K_0 \subseteq P_{J_0}$. By the second claim above, there exists a countable subset $J_1 \subseteq I$ such that $J_0 \subseteq J_1$ and M_{J_1} is countably generated. By the first claim, there exists a countable subset $J_2 \subseteq J$, containing J_1, such that $M_{J_1} \subseteq gP_{J_2}$ and gP_{J_2} is projective. Since gP_{J_2} is countably generated, it is contained in P_{J_3} for some countable subset $J_3 \subseteq I$, and thus $gP_{J_2} \subseteq M_{J_3}$. It is harmless to enlarge J_3 enough to contain J_2, and the second claim allows us, after a further enlargement, to assume that M_{J_3} is countably generated.

Continuing in this manner, we obtain countable subsets $J_0 \subseteq J_1 \subseteq \dots$ in I such that

$$M_{J_{2i-1}} \subseteq gP_{J_{2i}} \subseteq M_{J_{2i+1}}$$

and $gP_{J_{2i}}$ is projective for all i. Then $J = \bigcup_{i=1}^{\infty} J_i$ is a countable subset of I such that $K_0 \subseteq P_J$ and $gP_J = M_J$. Since M_J is the ascending union of the projectives $gP_{J_{2i}}$, it must be flat. In view of the short exact sequence

$$0 \to M_J \to P_J \to M_J \to 0,$$

we conclude from Proposition 2.2 that M_J is projective. $\square$

Theorem 2.5. *If $0 \to M \to P \to M \to 0$ is a short exact sequence of R-modules with M flat and P projective, then M must be projective.*

Proof. We may assume that the homomorphism $M \to P$ is an inclusion map, and we let g denote the homomorphism $P \to M$.

By a theorem of Kaplansky [11], the projective module P is a direct sum of countably generated submodules, say $P = \bigoplus_{i \in I} P_i$. Set $P_J = \bigoplus_{j \in J} P_j$ and $M_J = M \cap P_J$ for all subsets $J \subseteq I$.

Consider the set $\mathcal{P}$ of all pairs (J, L) where J is a subset of I such that $gP_J = M_J$ and L is a submodule of P_J such that $P_J = M_J \oplus L$. In particular, $(\emptyset, 0) \in \mathcal{P}$. Order $\mathcal{P}$ by componentwise inclusion:

$$(J_1, L_1) \leq (J_2, L_2) \quad \iff \quad J_1 \subseteq J_2 \text{ and } L_1 \subseteq L_2.$$

If $\{(J_\alpha, L_\alpha)\}$ is any nonempty chain in $\mathcal{P}$, then $(\bigcup_\alpha J_\alpha, \bigcup_\alpha L_\alpha)$ clearly belongs to $\mathcal{P}$. Therefore Zorn's Lemma gives us a maximal element of $\mathcal{P}$, say (J, L).

If $J = I$, then $P = M \oplus L$ and we are done. Hence, assume that $J \neq I$.

Since $gP_J = M_J$, it follows from the Nine Lemma or the Snake Lemma that there is a short exact sequence

$$0 \to M/M_J \xrightarrow{f} P/P_J \xrightarrow{\overline{g}} M/M_J \to 0$$

where f is induced by the inclusion map $M \to P$ and $\overline{g}$ is induced by g. Then f maps M/M_J isomorphically onto the submodule $\overline{M} = (M + P_J)/P_J \subseteq P/P_J$, and we obtain a short exact sequence

$$0 \to \overline{M} \xrightarrow{\subseteq} P/P_J \xrightarrow{h} \overline{M} \to 0$$

where $h = f\overline{g}$. Note that $P/P_J \cong P_{I \setminus J}$; in particular, P/P_J is projective, and is a direct sum of countably generated submodules $(P_i + P_J)/P_J$ for $i \in I \setminus J$. Since M_J is a direct summand of P_J, it is also a direct summand of P. Then from $M_J \subseteq M \subseteq P$ it follows that M_J is a direct summand of M. Hence, M/M_J, and thus $\overline{M}$, is flat.

Now apply Lemma 2.4 to the situation above, with initial data corresponding to a nonempty subset of $I \setminus J$. The lemma then gives us a subset $K \subseteq I$, properly containing J, such that $h(P_K/P_J) = \overline{M} \cap (P_K/P_J)$ and $h(P_K/P_J)$ is projective. Since

$$h(P_K/P_J) = f(gP_K/M_J) = (gP_K + P_J)/P_J$$
$$\overline{M} \cap (P_K/P_J) = [(M + P_J) \cap P_K]/P_J = (M_K + P_J)/P_J,$$

we obtain $gP_K + P_J = M_K + P_J$. Intersecting with M and using the modular law, we see that $gP_K = M_K$.

Since the kernel of $h|_{P_K/P_J}$ is the submodule $\overline{M} \cap (P_K/P_J) = (M_K + P_J)/P_J$, we find that $h(P_K/P_J) \cong P_K/(M_K + P_J)$. The latter quotient is thus projective, and so $P_K = (M_K + P_J) \oplus T$

for some T. Since $P_J = M_J \oplus L$ and

$$M_K \cap L = M_K \cap P_J \cap L = M_J \cap L = 0,$$

we have $M_K + P_J = M_K \oplus L$ and so $P_K = M_K \oplus L \oplus T$. But then $(K, L \oplus T) \in \mathcal{P}$, contradicting the maximality of (J, L).

Therefore $J = I$ and $P = M \oplus L$. $\square$

3. Reduction modulo p and projectivity

In this section, we prove that projectivity for modules over a group ring is determined by the reduction modulo each prime dividing the group order. The proof works in the context of a group and a subgroup of finite index, so this is the context in which we present it. We are mostly interested in the case $H = \{1\}$ of the following theorem.

Theorem 3.1. *Let R be a ring of coefficients and H a subgroup of finite index in a group G. An RG-module M is projective if and only if*

(i) *M is projective as an RH-module, and*

(ii) *M/pM is projective as an $(R/pR)G$-module for each prime number p dividing $[G : H]$.*

Proof. It is easy to see that a projective RG-module satisfies (i) and (ii). Conversely, we suppose that M is an RG-module satisfying (i) and (ii), and we shall prove that M is projective. Since M is projective as an RH-module, it suffices to prove that M is projective relative to H. By D. G. Higman's criterion (see for example Proposition 3.6.4 of [4]), this amounts to showing that the identity endomorphism can be written as

$$1_M = \mathrm{Tr}_{H,G}(\alpha)$$

for some $\alpha \in \mathrm{End}_{RH}(M)$, where $\mathrm{Tr}_{H,G}(\alpha)$ is the RG-endomorphism defined by

$$\mathrm{Tr}_{H,G}(\alpha)(m) = \sum_{g \in G/H} g(\alpha(g^{-1}(m))).$$

The sum runs over a set of left coset representatives of H in G.

Let $[G : H] = \prod_i p_i^{\gamma_i}$ where the p_i are distinct primes. Since M is projective as an RH-module, for each p_i, the natural map

$$\mathrm{End}_{RH}(M) \to \mathrm{End}_{(R/p_iR)H}(M/p_iM)$$

given by reduction modulo p_i is surjective, and the kernel of this map equals $p_i\mathrm{End}_{RH}(M)$. Since M/p_iM is a projective $(R/p_iR)G$-module, its identity endomorphism can be written as $\mathrm{Tr}_{H,G}(\overline{\alpha}_i)$

for some $\overline{\alpha}_i \in \mathrm{End}_{(R/p_iR)H}(M/p_iM)$. In view of the observations above, this implies that there exist elements α_i, $\beta_i \in \mathrm{End}_{RH}(M)$ such that

$$1_M = \mathrm{Tr}_{H,G}(\alpha_i) + p_i\beta_i.$$

Note that $p_i\beta_i = 1_M - \mathrm{Tr}_{H,G}(\alpha_i)$ is an RG-homomorphism. Multiplying the above expressions, we get

$$1_M = \prod_i (\mathrm{Tr}_{H,G}(\alpha_i) + p_i\beta_i)^{\gamma_i} = \mathrm{Tr}_{H,G}(\alpha') + [G:H]\beta'$$

for some $\alpha', \beta' \in \mathrm{End}_{RH}(M)$. (Here, we have used the fact that if $a \in \mathrm{End}_{RH}(M)$ and $b \in \mathrm{End}_{RG}(M)$, then $\mathrm{Tr}_{H,G}(a)b = \mathrm{Tr}_{H,G}(ab)$ and $b\mathrm{Tr}_{H,G}(a) = \mathrm{Tr}_{H,G}(ba)$.) Now the map $[G:H]\beta' = 1_M - \mathrm{Tr}_{H,G}(\alpha')$ is an RG-homomorphism, so applying $\mathrm{Tr}_{H,G}$ we get

$$[G:H]\mathrm{Tr}_{H,G}(\beta') = [G:H]^2\beta',$$

and hence $\delta = \mathrm{Tr}_{H,G}(\beta') - [G:H]\beta'$ satisfies $[G:H]\delta = 0$. Hence,

$$\delta = \delta.1_M = \delta(\mathrm{Tr}_{H,G}(\alpha') + [G:H]\beta') = \mathrm{Tr}_{H,G}(\delta\alpha'),$$

and so

$$[G:H]\beta' = \mathrm{Tr}_{H,G}(\beta') - \delta = \mathrm{Tr}_{H,G}(\beta' - \delta\alpha').$$

Finally,

$$1_M = \mathrm{Tr}_{H,G}(\alpha') + [G:H]\beta' = \mathrm{Tr}_{H,G}(\alpha' + \beta' - \delta\alpha'). \qquad \square$$

4. Reduction modulo p and flatness

In this section, we prove that flatness for modules over a group ring is determined by the reduction modulo each prime dividing the group order. Again, we prove the theorem in the generality of a subgroup of finite index.

Lemma 4.1. *Let R be a ring of coefficients and H a subgroup of finite index in a group G. If M is a left RG-module and N a right RG-module with M flat as a left RH-module, then $[G:H]$ annihilates $\mathrm{Tor}_n^{RG}(N,M)$ for all $n > 0$.*

Proof. Let $g_1, \ldots, g_l$ be a set of left coset representatives of H in G, and let

$$0 \to M' \xrightarrow{\phi} F \to M \to 0$$

be a short exact sequence with F a free left RG-module. Then we have the long exact sequence

$$0 \to \mathrm{Tor}_1^{RG}(N,M) \to N \otimes_{RG} M' \xrightarrow{\phi_*} N \otimes_{RG} F \to N \otimes_{RG} M \to 0.$$

Consider an element $\sum_{i=1}^k n_i \otimes m_i'$ in the kernel of ϕ_*. By Lemma 2.1, there is an RH-homomorphism $\rho : F \to M'$ such that $\rho\phi(g_j^{-1}m_i') = g_j^{-1}m_i'$ for $1 \leq j \leq l$ and $1 \leq i \leq k$. Let $\theta = \mathrm{Tr}_{H,G}(\rho)$. Then $\theta : F \to M'$ is an RG-homomorphism such that $\theta\phi(m_i') = [G : H]m_i'$ for $1 \leq i \leq k$. So we have

$$0 = \theta_*\phi_*\left(\sum_{i=1}^k n_i \otimes m_i'\right) = \sum_{i=1}^k n_i \otimes \theta\phi(m_i') = [G : H]\sum_{i=1}^k n_i \otimes m_i'.$$

This proves that $[G : H]$ annihilates $\mathrm{Tor}_1^{RG}(N, M)$. Since M' is also flat as an RH-module and

$$\mathrm{Tor}_n^{RG}(N, M') \cong \mathrm{Tor}_{n+1}^{RG}(N, M)$$

for $n \geq 1$, an induction on n (i.e., dimension shifting) shows that

$$[G : H]\mathrm{Tor}_n^{RG}(N, M) = 0$$

for $n \geq 1$. □

Lemma 4.2. *Let S be a ring, $p \in S$ a central element, and M a left S-module. Assume that M/pM is flat as a left S/pS-module, and that $\mathrm{Tor}_n^S(S/pS, M) = 0$ for all $n > 0$. If N is any right S-module, then $\mathrm{Tor}_n^S(N, M)$ is p-torsionfree for all $n > 0$ and p-divisible for all $n > 1$.*

Proof. For any right S/pS-module L, we have

$$\mathrm{Tor}_n^S(L, M) \cong \mathrm{Tor}_n^{S/pS}(L, M/pM) = 0$$

for all $n > 0$ by [8], Proposition VI.4.1.2. In particular,

$$\mathrm{Tor}_n^S(N/pN, M) = \mathrm{Tor}_n^S(N[p], M) = 0$$

for all $n > 0$, where $N[p] = \mathrm{ann}_N(p)$. Consider the short exact sequences

$$0 \to pN \xrightarrow{\phi} N \to N/pN \to 0$$

$$0 \to N[p] \to N \xrightarrow{\pi} pN \to 0$$

where ϕ is the inclusion map and π is multiplication by p. From the long exact sequences for $\mathrm{Tor}_*^S(-, M)$, we see that $\mathrm{Tor}_n^S(\phi, M)$ is an isomorphism for all $n > 0$, while $\mathrm{Tor}_n^S(\pi, M)$ is an isomorphism for all $n > 1$ and a monomorphism for $n = 1$.

Therefore $\mathrm{Tor}_n^S(\phi\pi, M)$ is an isomorphism for all $n > 1$ and a monomorphism for $n = 1$. Since $\phi\pi : N \to N$ is just multiplication by p, so is $\mathrm{Tor}_n^S(\phi\pi, M)$ (e.g., [12], Theorem 8.13), and the lemma follows. □

Theorem 4.3. *Let R be a ring of coefficients and H a subgroup of finite index in a group G. An RG-module M is flat if and only if*

(i) *M is flat as an RH-module, and*

(ii) *M/pM is flat as an $(R/pR)G$-module for each prime p dividing $[G : H]$.*

Proof. That flatness of M as an RG-module implies (i) and (ii) is well known and easy to see. Condition (i) holds because $(-) \otimes_{RH} M$ is naturally equivalent to the composition of the exact functors $(-) \otimes_{RH} RG$ and $(-) \otimes_{RG} M$. Condition (ii) holds because $(-) \otimes_{(R/pR)G} (M/pM)$ is naturally equivalent to the restruction of $(-) \otimes_{RG} M$ to the category of right $(R/pR)G$-modules.

For the converse, we begin by observing that (i) implies that M is flat as an R-module. Since RG is a flat R-module, it now follows from [8], Proposition VI.4.1.1 that

$$\mathrm{Tor}_n^{RG}(RG/pRG, M) = \mathrm{Tor}_n^{RG}((R/pR) \otimes_R RG, M) \cong \mathrm{Tor}_n^{R}(R/pR, M) = 0$$

for all $n > 0$ and any prime p.

Now let N be an arbitrary right RG-module. By (ii) and Lemma 4.2, $\mathrm{Tor}_n^{RG}(N, M)$ is p-torsion-free for all $n > 0$ and any prime p dividing $[G : H]$. On the other hand, Lemma 4.1 shows that $[G : H]$ annihilates $\mathrm{Tor}_n^{RG}(N, M)$, and therefore $\mathrm{Tor}_n^{RG}(N, M) = 0$ for all $n > 0$. □

5. R-projective and RG-flat imply RG-projective

In this section, we prove the main theorem of the paper. This is a matter of putting together what we have proved so far with the results of [3].

Theorem 5.1. *Let R be a ring of coefficients and G a finite group. If M is a flat RG-module which is projective as an R-module, then M is projective as an RG-module.*

Proof. Suppose that M is a flat RG-module which is projective as an R-module, but not as an RG-module. Then by Theorem 4.3 (with $H = \{1\}$), for each prime p dividing $|G|$, M/pM is a flat $(R/pR)G$-module. By Theorem 3.1 (again with $H = \{1\}$), there exists a prime p dividing $|G|$ such that M/pM is not a projective $(R/pR)G$-module. By Chouinard's theorem [9], we can find an elementary abelian p-subgroup $E \cong (\mathbb{Z}/p)^r$ of G such that M/pM is flat but not projective as an $(R/pR)E$-module. Choose a chain of subgroups

$$1 = E_0 < E_1 < \cdots < E_{r-1} < E_r = E$$

so that each E_{i-1} has index p in E_i. Then there is a unique index j in the range $0 \le j \le r-1$ so that M/pM is projective as an $(R/pR)E_j$-module and flat but not projective as an $(R/pR)E_{j+1}$-module. Set $R' = (R/pR)E_j$. Then we can regard the group ring $(R/pR)E_{j+1}$ as $R'(\mathbb{Z}/p)$. We are now in a position to be able to apply theorem 2.4 of [3] to deduce the existence of a flat R'-module M' which is not projective, but which sits in a short exact sequence $0 \to M' \to P \to M' \to 0$ with P projective.[1] This contradicts Theorem 2.5, and so the theorem is proved. □

[1] Theorem 2.4 of [3] is stated under the hypothesis that the coefficient ring is commutative, but this hypothesis is not used in the proof.

6. Passage to fields of coefficients

Let R be a commutative ring of coefficients and G a finite group. On the basis of what we have proved, it is tempting to suppose that it might be true that an RG-module M is projective if and only if

(i) M is projective as an R-module, and

(ii) $K \otimes_R M$ is projective as a KG-module for every field K containing $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$.

In this section, we give an example to show that this is false, and in the next section we show that it is true if R is a Dedekind domain of characteristic zero.

Let k be an algebraically closed field of characteristic 2 and let $G = \mathbb{Z}/2 = \langle g \mid g^2 = 1\rangle$. Let $R = k[[t]]$ be the ring of formal power series over k in a variable t. Then $X = k[t^{-1}]$ can be considered as an R-module in an obvious way. In fact, X is the injective hull of k as an R-module. Since R has global dimension one, there is an R-module projective resolution of X of the form

$$0 \to Q_1 \xrightarrow{\rho} Q_0 \to X \to 0.$$

Let M be the RG-module whose underlying R-module is $Q_0 \oplus Q_1$, with the group element g acting as the matrix $\left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right)$. Then M is not even flat, let alone projective, since the image of $1 - g$ does not coincide with the kernel of $1 + g$ (see Theorem 2.1 of [3]). Since Q_0 and Q_1 are projective as R-modules, so is M, and so (i) is satisfied. To check condition (ii), we note that any candidate for the field K is an extension of either $k = R/(t)$ or of $k((t))$, the field of fractions of R, so it suffices to examine these two fields. Now $k \otimes_R X = 0$ because every element of X is in the image of multiplication by t, so $k \otimes_R M$ is a projective kG-module. (Here we use the criterion of [3], Theorem 2.1 again.) Similarly, $k((t)) \otimes_R X = 0$ because every element is killed by a suitably high power of t, so $k((t)) \otimes_R M$ is a projective $k((t))G$-module.

7. Dedekind domains of coefficients

In this section, we examine what happens if R is a Dedekind domain of coefficients whose characteristic (i.e., the characteristic of the field of fractions of R) is prime to the order of the group. In this case, if n is a nonzero integer prime to the characteristic of R, then the principal ideal nR is nonzero, and so it can be written as a finite product of maximal ideals (cf. Atiyah and Macdonald [1]). All that we actually need is that nR contain a finite product of maximal ideals, and for this to hold it suffices that R be a commutative noetherian domain of Krull dimension one whose characteristic is prime to n.

Theorem 7.1. *Let H be a subgroup of finite index in a group G and R a commutative noetherian ring of coefficients such that the principal ideal generated by $[G : H]$ (times the identity element) contains a finite product of maximal ideals that contain $[G : H]$. Then an RG-module M is projective if and only if*

(i) *M is projective as an RH-module, and*

(ii) *$(R/\mathfrak{m}) \otimes_R M$ is projective as an $(R/\mathfrak{m})G$-module for each maximal ideal $\mathfrak{m} \subseteq R$ containing $[G : H]$.*

Proof. We modify the proof of Theorem 3.1 as follows.

Let M be an R-module satisfying (i) and (ii), and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be maximal ideals of R, containing $[G : H]$, such that $\prod_i \mathfrak{m}_i$ is contained in the ideal $[G : H]R$. Since R is noetherian, each $\mathfrak{m}_i$ is finitely generated. Thus, since $M/\mathfrak{m}_i M$ is a projective $(R/\mathfrak{m}_i)G$-module, it follows as in the previous proof that

$$1_M = \mathrm{Tr}_{H,G}(\alpha_i) + \beta_i$$

for some $\alpha_i \in \mathrm{End}_{RH}(M)$ and some $\beta_i \in \mathfrak{m}_i\mathrm{End}_{RH}(M)$. Multiplying these expressions, and noting that β_i is an RG-homomorphism, we get

$$1_M = \mathrm{Tr}_{H,G}(\alpha') + \prod_i \beta_i$$

for some $\alpha' \in \mathrm{End}_{RH}(M)$. Since R is commutative and $\prod_i \mathfrak{m}_i \subseteq [G : H]R$, we get $\prod_i \beta_i = [G : H]\beta'$ for some $\beta' \in \mathrm{End}_{RH}(M)$.

The remainder of the proof now proceeds as before. □

Theorem 7.2. *Let R be a commutative noetherian domain of coefficients of Krull dimension one, and G a finite group whose order is relatively prime to the characteristic of R. Then an RG-module M is projective if and only if*

(i) *M is projective as an R-module, and*

(ii) *$M/\mathfrak{m}M$ is projective as an $(R/\mathfrak{m})G$-module for each maximal ideal $\mathfrak{m} \subseteq R$ containing $|G|$.*

Proof. The ideal $I = |G|R$ is nonzero by hypothesis. Well known results then imply that I contains a finite product of maximal ideals that contain I. For instance, R/I has Krull dimension zero and so is artinian (e.g., [1], Theorem 8.5), whence R/I has only finitely many prime ideals, all of which are maximal (e.g., [1], Propositions 8.1 and 8.3). Thus the prime radical (nilradical) of R/I contains a finite product of maximal ideals. Since this radical is nilpotent (e.g., [1], Proposition 8.4), the desired conditions on I hold. Therefore the theorem follows from Theorem 7.1. □

Remark This theorem seems to be well known for *finitely generated* $\mathbb{Z}G$-modules, but does not seem to be well known in the generality described here.

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Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: `djb@byrd.math.uga.edu`

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
E-mail address: `goodearl@math.ucsb.edu`