

1 Separation of variables - review

To solve the boundary value problems for the heat and wave equations on the finite interval $(0, l)$, we used the method of separation of variables. This method relies on the idea of representing the solution to the general boundary value problem as a linear combination of separated solutions $u(x, t) = X(x)T(t)$. For these separated solutions the PDEs reduce to pairs of ODEs. Indeed, plugging the separated solution into the heat and wave equations, and separating the variables gives the following.

$$\text{Wave equation, } u_{tt} - c^2 u_{xx} = 0.$$

$$\text{Heat equation, } u_t - k u_{xx} = 0.$$

$$T''(t)X(x) - c^2 T(t)X''(x) = 0$$

$$T'(t)X(x) - kT(t)X''(x) = 0$$

$$-\frac{X''}{X} = -\frac{T''}{c^2 T} = \lambda$$

$$-\frac{X''}{X} = -\frac{T'}{kT} = \lambda$$

$$X'' = -\lambda X \quad \text{and} \quad T'' = -\lambda c^2 T.$$

$$X'' = -\lambda X \quad \text{and} \quad T' = -\lambda k T.$$

The different boundary conditions applied to the separated solution imply the following.

$$\text{Dirichlet: } u(0, t) = u(l, t) = 0 \quad \Rightarrow \quad X(0)T(t) = X(l)T(t), \forall t \quad \Rightarrow \quad X(0) = X(l) = 0.$$

$$\text{Neumann: } u_x(0, t) = u_x(l, t) = 0 \quad \Rightarrow \quad X'(0)T(t) = X'(l)T(t), \forall t \quad \Rightarrow \quad X'(0) = X'(l) = 0.$$

Using these conditions for the $X(x)$ component of the separated solution, we arrive at the following eigenvalue problems

$$\text{Dirichlet: } \begin{cases} X'' = -\lambda X, \\ X(0) = X(l) = 0. \end{cases}$$

$$\text{Neumann: } \begin{cases} X'' = -\lambda X, \\ X'(0) = X'(l) = 0. \end{cases}$$

The values of λ for which these problems have a nontrivial solution ($X \not\equiv 0$) are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions. Notice that for each of the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ the equation $X'' = -\lambda X$ has qualitatively different solutions due to the sign of the discriminant of the characteristic quadratic equation. After considering the three cases separately, and eliminating those which lead to only trivial solutions, we arrive at the following eigenvalues and eigenfunctions for the respective boundary conditions.

Dirichlet:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}$$

for $n = 1, 2, \dots$

Neumann:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos \frac{n\pi x}{l}$$

for $n = 0, 1, 2, \dots$

Notice that the eigenvalues for the Dirichlet and Neumann boundary conditions are exactly the same, with the only extra eigenvalue $\lambda_0 = 0$ in the Neumann case. The eigenfunctions corresponding to these eigenvalues are sine in the Dirichlet case, and cosine in the Neumann case. This corresponds to the necessity of taking odd extensions of the Dirichlet data, and even extension of the Neumann data, which we saw in the reflection method.

Using the above eigenvalues λ_n , we can solve the T equations corresponding to the wave and heat equations, arriving at the following solutions.

Wave:

$$T_n(t) = A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l},$$

for $n = 1, 2, \dots$

$$T_0(t) = \frac{A_0}{2} + \frac{B_0}{2}t.$$

Heat:

$$T_n(t) = A_n e^{-(n\pi/l)^2 kt},$$

for $n = 1, 2, \dots$

$$T_0(t) = \frac{A_0}{2}.$$

Then the solutions to the boundary value problems can be written as linear combinations of the separated solutions $u_n(x, t) = X_n(x)T_n(t)$, leading to the series solutions for the wave and heat boundary value problems. These solutions will solve the corresponding problems, if the initial conditions are also satisfied, which will be the case provided the initial data can be expanded into corresponding Fourier sine and cosine series. We next list each of these solutions for the wave and heat boundary value problems.

Wave Dirichlet:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), \\ u(0, t) = u(l, t) = 0. \end{cases}$$

The series solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l},$$

provided the initial data can be expanded into the Fourier sine series

$$\begin{aligned} \phi(x) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}, \\ \psi(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \end{aligned}$$

Wave Neumann:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), \\ u_x(0, t) = u_x(l, t) = 0. \end{cases}$$

The series solution has the form

$$u(x, t) = \frac{A_0}{2} + \frac{B_0}{2}t + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \cos \frac{n\pi x}{l},$$

provided the initial data can be expanded into the Fourier cosine series

$$\begin{aligned} \phi(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}, \\ \psi(x) &= \frac{B_0}{2} + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos \frac{n\pi x}{l}. \end{aligned}$$

Heat Dirichlet:

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), \\ u(0, t) = u(l, t) = 0. \end{cases}$$

The series solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l},$$

provided the initial data can be expanded into the Fourier sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

Heat Neumann:

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), \\ u_x(0, t) = u_x(l, t) = 0. \end{cases}$$

The series solution has the form

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos \frac{n\pi x}{l},$$

provided the initial data can be expanded into the Fourier cosine series

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$

Having these series solutions, all there remains to do to find the complete solutions to the boundary value problems is to compute the coefficients in the Fourier expansions of the initial data. This will be the subject of our subsequent lectures, in which we will also address questions of convergence of such series.